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Probabilistic Methods for Switching Control in Discrete Time

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Control systems with a finitely many of control settings, i. e. dynamical polysystems, are considered. It is assumed that a polysystem functions in continuous time and switchings of control occur in some discrete instants of time. The control goal is a transition of a polysystem from an initial state to a final state. Controllability of the polysystems is studied. Probabilistic methods are applied. Some probability characteristics of dynamical polysystems are defined. It is shown that under the rank condition, the switching controls always exist and the estimates of control times can be find by numerical methods. The rate of convergence of estimates of control times is established. The scale of the rates of convergence is described.

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1. Introduction

In this paper, we continue to investigate a class of continuous-time dynamical polysystems considered in [1–4]. These polysystem consist of a finite number of switched dynamical systems. It is assumed that the switchings occur in some discrete instants of time. We explore controllability of dynamical polysystems and find the control times which provide a sufficient accuracy of target condition, i. e. provide ϵ - controllability either with a finite ϵ or with an arbitrarily small ϵ. In the last case, if the values of ϵ decrease then the control times become arbitrarily large. Therefore, for these control systems, there is a problem to characterize a rate of convergence of estimates of control times. This characterizing can be obtained by analysis of the distribution function of values of control times. In paper [3], we investigated a class of polysystems having the uniform distribution function of control times. In this case, the relative rate of convergence is inversely proportional to the rate of growth of control time. In the present paper, we consider arbitrary distribution functions. In this case, we apply the inverse transformation method and reduce the problem to the special case of uniform distribution function. It is shown that for polysystems with an arbitrary distribution function, the rate of convergence of estimates of control times can be arbitrarily slow and arbitrarily fast.

Enough information on theory of dynamical systems and other topics can be found in [11–14].

2. Definition of a polysystem and related notions

Let us denote a state space with elements x by $\mathbb{X}$, a control set with elements u by $\mathbb{U}$, a one-dimensional space of time points t by $\mathbb{R}$.

For an arbitrary fixed $u \in \mathbb{U}$, consider a family of the maps (diffeomorphisms)

$$F_u^t : \mathbb{X} \to \mathbb{X}, \quad t \in \mathbb{R}$$

such that for any $x \in \mathbb{X}$

$$F_u^{t_1}(F_u^{t_2}(x)) = (F_u^{t_1+t_2})(x),$$

i. e., the family F_u^t defines a dynamical system. Let us to each $t \in \mathbb{R}$ assign some $u \in \mathbb{U}$, i. e.,

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give a function $\hat{u}$ by formula $u = \hat{u}(t)$. Then we say that the set $\{F_u^t, u \in \mathbb{U}\}$ and the function $\hat{u}$ define a dynamical control system.

Next, we consider a finite control set belonging to the set $\mathbb{U}$. Namely, for an arbitrary fixed integer $l \in \mathbb{N}$ and some $u_1, \dots, u_l \in \mathbb{U}$, consider the set of the one parameters families

$$\{F_{u_1}^t, \dots, F_{u_l}^t\}. \tag{2.1}$$

Let the values $t_j \in \mathbb{R}$, $t_{j-1} \le t_j$, $j = 1, \dots, l$, $l \ge n$ be time moments of control switching. Thus, to the time moment t_j, it is assigned the element F_{u_j} of the set given by formula (2.1), i. e. it is designated the correspondence

$$t_j \to u_j, \quad j = 1, \dots, l. \tag{2.2}$$

A restriction of a control system to a finite set $\{u_1, \dots, u_l\} =: \mathbb{U}_0 \subset \mathbb{U}$ is called a dynamical polysystem.

Assume that at $t_0 \in \mathbb{R}$ the polysystem starts from $x_0 \in \mathbb{X}$. Consider the space $\mathbb{R}_+^l$ of points τ in the form

$$\tau = (\tau_1, \dots, \tau_l) \in \mathbb{R}_+^l, \quad \tau_j = t_j - t_{j-1} \ge 0. \tag{2.3}$$

Thus, the dynamical system F_{u_j} functions during of time period of length τ_j. The value

$$|\tau| = |\tau_1| + \dots + |\tau_l| \tag{2.4}$$

of the norm of $\tau \in \mathbb{R}_+^l$ is a full control time. The state at the last time is as follows, i. e.,

$$(F_{u_l}^{\tau_l} \circ F_{u_{l-1}}^{\tau_{l-1}} \circ \dots \circ F_{u_1}^{\tau_1})(x_0) := F_{\boldsymbol{u}}^{\tau}(x_0) \tag{2.5}$$

where $\boldsymbol{u} = (u_1, \dots, u_l) \in \mathbb{U}^l$.

For maps F', F'', a symbol «$\circ$» means a superposition of these maps, i. e., for every x the condition $(F' \circ F'')(x) = F'(F''(x))$ is valid. Thus, the polysystem generates a l-parameter family of maps in the form (2.5).

Let $\boldsymbol{u}$ be fixed. Then a dynamical polysystem can be interpreted as a map in the form

$$F : \mathbb{R}^l \times \mathbb{X} \to \mathbb{X}, \quad F(\tau, x') = x'', \qquad \tau \in \mathbb{R}^l,\ x' \in \mathbb{X},\ x'' \in \mathbb{X}, \tag{2.6}$$

or as an action of the family $F^\tau, \tau \in \mathbb{R}^l$ on $\mathbb{X}$, i. e.,

$$F^\tau : \mathbb{X} \to \mathbb{X}, \quad F^\tau(x') = x'', \qquad x' \in \mathbb{X},\ x'' \in \mathbb{X}. \tag{2.7}$$

Definition 1 *The polysystem given by eq. (2.1) is (exactly) controllable from $x_0 \in \mathbb{X}$ to $x_* \in \mathbb{X}$ if there exists a time vector $\tau = (\tau_{i_1}, \dots, \tau_{i_l}) \in \mathbb{R}_+^l$ such that*

$$x_* - F^\tau(x_0) = 0. \tag{2.8}$$

The value τ depends on x_0, x_*. Let τ_0 be the smallest value τ for which eq. (2.8) is valid. Then we write

$$x_0 \xrightarrow{\tau} x_*.$$

Definition 2 *The polysystem given by eq. (2.1) is ϵ-controllable from $x_0 \in \mathbb{X}$ to $x_* \in \mathbb{X}$ if for $\epsilon > 0$ there exists a time vector $\tau = (\tau_{i_1}, \dots, \tau_{i_l}) \in \mathbb{R}_+^l$ such that*

$$|F^\tau(x_0) - x_*| \le \epsilon \tag{2.9}$$

where $\tau = \tau(\epsilon)$.

The full control time is equal to $|\tau(\epsilon)|$. A dependence of τ on ϵ can be ambiguous. We consider some concrete dependence of τ on ϵ. In this case, we give the following definition.

Definition 3 *The polysystem given by eq. (2.1) is approximately controllable from $x_0 \in \mathbb{X}$ to $x_* \in \mathbb{X}$ if*

$$\lim_{\epsilon \to +0} |F^{\tau(\epsilon)}(x_0) - x_*| = 0. \tag{2.10}$$

If the polysystem is only approximately controllable from the state x_0 to the state x_*, then the full control time

$$|\tau(\epsilon)| \to +\infty, \quad \epsilon \to +0.$$

In the sequel, we shall investigate the dependence $|\tau(\epsilon)|$ on ϵ for fixed boundary states x_0, x_*.

Next, we assume that the initial value x_0 is fixed and the final value of x_* is arbitrary. We consider the set X_*^0 of points x_* attainable from the point x_0 for τ_0. Assume that for $|\tau| \geq |\tau_0|$, the set X_* of points x_* attainable from the point x_0 for τ coincides with the set X_*^0, i. e., $X_*^0 = X_*$. Thus, we have a family of transformations

$$X_*^0 \xrightarrow{\tau} X_*, \tag{2.11}$$

depending on parameter τ.

Next, we introduce the notion of dynamical polysystem associated with the original polysystem, and its invariant measure. Using these notions, we describe the conditions of controllability polysystem in discrete time.

3. Definition of the associated dynamical polysystem

Consider eq. (2.8) which we rewritten in the form of eq. (2.6), i. e.,

$$F(\tau, x_0) = x_*, \quad \tau \in \mathbb{R}_+^l \tag{3.1}$$

where $F(\tau_0, x_0) = x_*$. Eq. (3.1) defines the orbit corresponding to the value τ_0 of the polysystem given by eq. (2.7).

Differentiating eq. (3.1) on the parameter τ, we obtain the matrix equality

$$\frac{\partial F}{\partial \tau} d\tau = 0. \tag{3.2}$$

Assume that for any τ and x_0 the condition

$$\operatorname{rank}(f(\tau)) = n \tag{3.3}$$

is valid where $\frac{\partial F}{\partial \tau}(\tau) =: f^*(\tau)$ and x_0 is omitted. Consequently, at any point $\tau \in \mathbb{R}_+^l$, eq. (3.2) gives m-dimensional subspace in the space $\mathbb{R}^l$ where $m = l - n$. This subspace is the tangent subspace at $\tau \in \mathbb{R}_+^l$ to the surface given by eq. (3.1). This surface can be given in the form

$$\tau = \hat{\tau}(\sigma, \tau_0), \quad \sigma = (\sigma_1, \ldots, \sigma_m) \in \Sigma. \tag{3.4}$$

We assume that $\Sigma = \mathbb{R}^m$ or $\Sigma = \mathbb{R}_+^m$ and $\tau_0 = \hat{\tau}(0, \tau_0)$. Since τ_0 depends on $x_* \in X_*^0$, there is a correspondence

$$X_*^0 \to \mathbb{R}_0^l, \quad \tilde{\tau}(x_*) = \tau_0$$

where the set $\mathbb{R}_0^l = \mathbb{R}_+^l \setminus \operatorname{Int}(\mathbb{R}_+^l)$.

For surface (3.4), at any point $\tau \in \mathbb{R}_+^l$, consider the n-dimensional subspace which is orthogonal to the original m- dimensional subspace. Suppose that the matrix $\frac{\partial F}{\partial \tau}$ is represented as a set of row vectors, i. e.,

$$f(\tau) = \operatorname{col}(f_1^*(\tau), \ldots, f_n^*(\tau)).$$

Let the set of column vectors

$$\operatorname{row}(g_1(\tau), \ldots, g_m(\tau)) =: g(\tau)$$

form a basis of the orthogonal subspace, i.e., for every $\tau \in \mathbb{R}_+^l$ and for any pair i, j the relation

$$f_i^*(\tau) g_j(\tau) = 0 \quad \Longleftrightarrow \quad f^*(\tau) g(\tau) = 0$$

is valid. Let $g_1(\tau), \ldots, g_m(\tau)$ be smooth on τ. We assume also that the vector fields are in involution. For simplicity, we assume that

$$[g_i(\tau), g_j(\tau)] = 0, \quad i, j = 1, \ldots, m. \tag{3.5}$$

The set $g(\tau)$ of vector fields $g_1(\tau), \ldots, g_m(\tau)$ generates a polysystem with the state space $\mathbb{R}^l$ and the multidimensional space $\mathbb{R}^m$ of time points σ. This polysystem can be given by differential equation in the form

$$\frac{d\tau}{d\sigma} = g(\tau), \quad \tau(0) = \tau_0 \tag{3.6}$$

where $\tau_0 \in \mathbb{R}_0^l$.

Factor the state space $\mathbb{R}_+^l$ of this polysystem on the integer lattice $\mathbb{Z}_+^l$ and get the torus $\mathbb{T}^l = \mathbb{R}_+^l / \mathbb{Z}_+^l$ which is the state space of the factor-polysystem.

The set g of vector fields $g_1, \ldots, g_m$ generates a set of one-parameter of groups $G_1^{\sigma_1}, \ldots, G_m^{\sigma_m}$ where $G_j^{\sigma_j}$ is an evolution operator for the vector field g_j, $j = 1, \ldots, m$. From eq. (3.5), it follows that

$$G_i^{\sigma_i} \circ G_j^{\sigma_j} = G_j^{\sigma_j} \circ G_i^{\sigma_i}, \quad i, j = 1, \ldots. \quad (3.7)$$

Consider the smallest m-parameter group G^{σ} containing these groups. In particular, this group contains elements (monomials) of the form

$$G^{\sigma} := G_m^{\sigma_m} \circ \cdots \circ G_1^{\sigma_1} \quad (3.8)$$

where $\sigma = (\sigma_1, \ldots, \sigma_m) \in \Sigma = \mathbb{R}_+^m$. Obviously,

$$G^{\sigma'} \circ G^{\sigma'} = G^{\sigma' + \sigma''}, \quad \sigma', \sigma'' \in \mathbb{R}_+^m.$$

The action of the family $\{G^{\sigma}, \sigma \in \Sigma\}$ gives the polysystem in the form

$$G^{\sigma} : \mathbb{T}^l \to \mathbb{T}^l, \quad G^{\sigma}(\tau_0) = \tau, \, G^0(\tau_0) = \tau_0, \quad (3.9)$$
$$\sigma \in \mathbb{R}_+^m, \, \tau_0 \in \mathbb{T}^l, \, \tau \in \mathbb{T}^l,$$

which is called the polysystem associated with the original polysystem (eq. (2.7)). The orbit of original polysystem (see eq. (3.1) or eq. (3.4)) corresponds to the orbit of polysystem (3.9) rewritten in the form

$$\{\tau | G^{\sigma}(\tau_0) = \tau, \, \sigma \in \Sigma\}.$$

The set $\mathbb{R}_+^l$ (eq. (2.3)) can be interpreted as the total space of points $\tau = (\tau_1, \ldots, \tau_l)$ (multi-dimensional time). For polysystem eq. (3.9), the set $\Sigma = \mathbb{R}_+^m$ can be interpreted as the space of the multi-dimensional time points $\sigma = (\sigma_1, \ldots, \sigma_m)$. Thus, for continuous time, polysystem (3.9) is a m-dimensional flow. For $m = 1$, we have an usual flow.

For discrete time space $\Sigma_0 = \mathbb{Z}_+^m$, polysystem (3.9) is a m-dimensional cascade

$$G^{\sigma} : \mathbb{T}^l \to \mathbb{T}^l, \quad G^{\sigma}(\tau_0) = \tau, \, G^0(\tau_0) = \tau_0, \quad (3.10)$$
$$\sigma \in \mathbb{Z}_+^m, \, \tau_0 \in \mathbb{T}^l, \, \tau \in \mathbb{T}^l,$$

which is embedded in flow (3.9).

Consider the polysystem given by the equation

$$\frac{d\tau}{d\varsigma} = f(\tau), \quad \tau(0) = \tau_0 \quad (3.11)$$

where $\varsigma \in \mathbb{R}_+^n$. This polysystem corresponds to family of transformations (2.11). This system can be factorized similar to system (3.6).

4. Definition of probability characteristics of a polysystem

For a description of the statistical properties of the trajectories of a polysystem (in particular for the evaluation of the rate of approximation of the trajectories to specific sets) we need the probability characteristics of the polysystem.

4.1. Probability invariant measures of a polysystem and distribution functions

Let $\mathfrak{A}$ be the Borel σ-algebra of sets on the torus $\mathbb{T}^l$. Assume that for the polysystem generated by family of maps (3.9), there is a set of Borel invariant probability measures P on σ-algebra $\mathfrak{A}$ of sets on the torus $\mathbb{T}^l$. For the flow given by eq. (3.9), the invariance of the measure means that the condition

$$P(A) = P((G^{\sigma_0})^{-1}(A)), \; A \in \mathfrak{A}, \, \sigma_0 \in \mathbb{R}_+^m \quad (4.1)$$

is valid. For the cascade given by eq. (3.10), there is the analogous condition.

Let a discrete dynamical system be defined as action of group of transformations of the kind (3.8) on a state space $\mathbb{T}^l$, i.e.,

$$G = \{G^{\sigma}, \, \sigma \in \mathbb{Z}_+^m\}.$$

According to the classical theorem of Krylov-Bogolyubov, an invariant measure exists for a one-parameter group of transformations. In our case, for an m-parameter group of transformations with property (3.7), an invariant measure also exists. An invariant measure can exist also in more general case. Thus, we assume that there is an invariant measure for group G of transformations.

4.2. Strictly ergodic measure and its density function

Furthermore, we assume that there is a unique ergodic measure which we denote by P_G. For simplicity, we assume that the invariant measure is given on the unit cube $\mathbb{I}^l$. For this measure, we will use the same notation P_G.

For P_G, let us define a distribution function

$$\mathcal{F}_G(x) = P_G(\tau_0|G^\sigma(\tau_0) < x),$$

which does not depend on σ. Then $f_G(x) = \frac{dF_G}{dx}(x)$ is the density function.

Let us define the distribution function of the module of $G^\sigma(\tau_0)$, i. e. the function F_η for the dynamical process

$$\eta^\sigma(\tau) := |G^\sigma(\tau)|.$$

This distribution function does not depend on σ for almost all τ_0.

Assume the distribution function $F_\eta(x)$ does not depend on angle variables in a neighborhood of zero. Consider the spherical coordinate system, i. e., $x = rp(\varphi)$ where r is a radial variable and φ are angle variables. Thus, $|x| = r$, $p|(\varphi)| = 1$.

Define the density function for the radial component in a neighborhood of zero by the formula

$$f_\eta(r) = r^{l-1} \int\limits_{\mathbb{S}^{l-1}} f_G(rp(\varphi))J_1(\varphi)d\varphi, \qquad (4.2)$$

$$0 \le r \le r_1 \le 1$$

where $J_1(\varphi) = J(r,\varphi)|_{r=1}$ is the reduced Jacobian determinant of the transformation. Hence, the distribution function of the module is given by the formula

$$F_\eta(r) = \int\limits_0^r f_\eta(r')dr', \quad 0 \le r' \le r \le r_1 \le 1. \quad (4.3)$$

4.3. The Fölner family

Now, consider a family of the nested sets $\Sigma_s \subset \Sigma, t \in \mathbb{R}_+$, i. e., $\Sigma_{s_0} \subset \Sigma_{s_1}$ if $s_0 \le s_1$, (the Fölner family). In addition, it is assumed that the condition

$$\lim_{t\to\infty} \frac{V(\Sigma_{s+\delta}\,\Delta\,\Sigma_s)}{V(\Sigma_s)} = 0$$

is satisfied for any $\delta > 0$ where Δ is the symmetric difference of two sets and V is a measure of volume.

These sets can be chosen as $\Sigma_s = [0,s]^m$.

For a discrete $s \in \mathbb{Z}$, these sets can be selected as the sets of the form $\Sigma_s^0 = \Sigma_s \cap \mathbb{Z}^m$.

4.4. Statistical distribution function of control times

Let η^σ, $\sigma \in \Sigma$ be scalar stationary random process with positive values. For any $\varepsilon > 0$, consider the set

$$\{\sigma|\sigma \in \Sigma_s, \eta^\sigma(\tau_0) < \varepsilon\} =: \Sigma_s \cap B_\eta(\varepsilon,\tau_0)$$

where $\Sigma_s = [0,s]^m$, $B_\eta(\varepsilon,\tau_0) = \{\sigma|\eta^\sigma(\tau_0) < \varepsilon\}$.

Define the value $F_\eta(\varepsilon,\tau_0)$ of statistical distribution function as follows

$$F_\eta(\varepsilon,\tau_0) = \lim_{s\to\infty} \frac{V(\Sigma_s \cap B_\eta(\varepsilon,\tau_0))}{V(\Sigma_s)}. \qquad (4.4)$$

For continuous time, V is a measure of volume. For discrete time, V is a counting measure, i. e. the measure of the discrete set is equal to the number of its constituent elements.

By the ergodic property of the process η^σ, $\sigma \in \Sigma$, the limit value given by (4.4) is the same for almost all τ with respect to P_G. Thus,

$$F_\eta(\varepsilon,\tau_0) = F_\eta(\varepsilon), \quad \mod P_G$$

where the distribution function $F_\eta(\varepsilon)$ is defined by formula (4.3) for $r = \varepsilon$. Thus, eq. (4.4) can be rewritten in the form

$$F_\eta(\varepsilon) = \frac{V(\Sigma_s \cap B_\eta(\varepsilon))}{V(\Sigma_s)} + p_\eta(s, \varepsilon) \qquad (4.5)$$

where $p_\eta(s, \varepsilon) \to 0$ as $s \to \infty$.

4.5. Weyl - Schönberg criterion

The family of transformations G^σ has an invariant measure P_G iff for P_G-almost all τ_0

$$\lim_{s\to+\infty} \frac{1}{V(\Sigma_s)} \int\limits_{\sigma\in\Sigma_s} e^{2\pi i(k, G^\sigma(\tau_0))} d\sigma =$$

$$\int\limits_{\tau\in\mathbb{I}^l} e^{2\pi i(k,\tau)} P_G(d\tau), \quad k \in \mathbb{Z}^l \setminus \{0\} \qquad (4.6)$$

where $\Sigma_s = [0, s]^m$, V is a measure of volume for continuous $s \in \mathbb{R}_+$ or a counting measure for discrete $s \in \mathbb{Z}_+$. Thus, $V(\Sigma_s) = s^m$. The function $(.,.)$ is the scalar product in $\mathbb{R}^l$, (see [9]).

5. Estimates of the rate of convergence of the sequence, which approximates the exact control time

Next, we consider the problem of control by switching among a finite number of controllers in discrete instants of time. In other words, we assume that control times are integers, (see eq. (2.3)).

Thus, we shall examine the rate of convergence of the estimates $\hat{\tau} \in \mathbb{Z}^l_+$, which approximate the values of control times $\tau \in \mathbb{R}^l_+$ satisfying the equation $x_* - F(\tau, x_0) = 0$, i. e. condition (3.1).

To solve this problem, we will use the properties of the associated polysystem $G^\sigma(\tau_0)$, $\sigma \in \Sigma$ (see section 3). The result will be given in terms of the invariant measure of the associated polysystem (see section 4).

Let F_η be the distribution function of values

$$\eta^\sigma(\tau_0) = |G^\sigma(\tau_0)|,\ \tau_0 \in \mathbb{T}^l \qquad (5.1)$$

where $G^\sigma(\tau_0)$, $\sigma \in \Sigma$ is basic process in the form (3.9). Since the process $G^\sigma(\tau_0)$, $\sigma \in \Sigma$ is stationary, the function F_η does not depend on σ for almost all τ_0.

Suppose that the values $\eta^\sigma(\tau_0)$ are dense in the unit interval $[0, r_1]$ for P_G -almost all τ_0 where $\sigma \in \Sigma$. Hence, for any $r \in [0, r_1]$, there exists a sequence of the values σ such that $\eta^\sigma(\tau_0) \to r$. In particular, there exists a subsequence σ_s depending on τ_0, i. e.

$$\sigma_s \in \Sigma_s, \quad s = 1, 2, \dots, \qquad (5.2)$$

such that

$$\eta^{\sigma_s}(\tau_0) \to 0, \quad s \to +\infty. \qquad (5.3)$$

Condition (5.3) means that the distance

$$\mathrm{dist}(\hat{\tau}(\sigma_s, \tau_0), \mathbb{Z}^l_+) \to 0, \quad s \to +\infty \qquad (5.4)$$

where the value $\hat{\tau}(\sigma_s, \tau_0)$ defined by eq. (3.4).

Define the rate ε_s of convergence for sequence $\eta^{\sigma_s}(\tau_0)$ as follows. Let

$$B_{\eta,s}(\varepsilon) := \Sigma_s \cap B_\eta(\varepsilon, \tau_0).$$

For any s, let us define the value ε_s of parameter ε by the equation

$$V(\Sigma_s \cap B_\eta(\varepsilon, \tau_0)) = v \qquad (5.5)$$

where v is a finite positive integer. It means that the set Σ_s contains only a finite number of points σ such that $\eta^\sigma(\tau_0) < \varepsilon$. For simplicity, let $v = 1$.

Let us denote $V(\Sigma_s)$ by v_s. Express the values of ε_s by the values of v_s. From eq. (4.5), it follows that the value $\varepsilon = \varepsilon_s$ satisfies the following equality

$$F_\eta(\varepsilon_s) = \frac{v}{v_s} + p_\eta(s, \varepsilon_s), \quad \mod P_G, \qquad (5.6)$$

i.e for almost all τ_0. Therefore, in eq. (5.6) and further, τ_0 is omitted.

Let us estimate the value $p_\eta(s, \varepsilon)$ with respect to the value $\frac{v}{v_s}$. Rewrite the value $F_\eta(\varepsilon)$

in the form

$$F_\eta(\varepsilon) = F_\eta(\varepsilon)\gamma_\eta(s,\varepsilon) + p_\eta(s,\varepsilon) \qquad (5.7)$$

where a suitable value

$$\gamma_s := \gamma_\eta(s,\varepsilon) \to 1, \quad s \to \infty.$$

From (4.5) and (5.7), it follows that

$$F_\eta(\varepsilon_s)\gamma_s v_s = V(\Sigma_s \cap B_\eta(\varepsilon)).$$

Hence,

$$F_\eta(\varepsilon_s)\gamma_s v_s = v \iff F_\eta(\varepsilon_s) = \frac{v}{\gamma_s v_s}. \quad (5.8)$$

Assume that F_η is strictly monotonic. Then from (5.6), (5.8), it follows that

$$\begin{aligned} \frac{v}{\gamma_s v_s} = \frac{v}{v_s} + p_\eta(s,\varepsilon_s) \quad &\iff \\ \frac{v}{v_s}\left(\frac{1-\gamma_s}{\gamma_s}\right) = p_\eta(s,\varepsilon_s) \quad &\iff \\ p_\eta(s,\varepsilon_s) = o\left(\frac{v}{v_s}\right). \quad &(5.9) \end{aligned}$$

From (5.6), (5.9), it follows that

$$\varepsilon_s = F_\eta^{-1}\left(\frac{v}{v_s} + o\left(\frac{v}{v_s}\right)\right). \qquad (5.10)$$

Thus, we have proved the following theorem.

Theorem 1 *For P_G-almost all τ_0, the rate of convergence of sequence (5.3)of estimates $\eta^{\sigma_s}(\tau_0)$ is given by eq. (5.10) where $\eta^\sigma(\tau_0)$ is defined by eq. (5.1).*

Remark 1 *Formula (5.10) defines the Smirnov inverse transformation. There is a generalization of the Smirnov inverse transformation if F_η is weakly monotonic.*

Further, for the dynamical process $\eta^\sigma = |G^\sigma|$, $\sigma \in \Sigma$, we consider the cases when this dynamical process depends on some parameters.

6. The scale of the distribution functions of control times for dynamical polysystem

Consider a family $\{G^\sigma_\alpha | 0 \le \alpha < \infty\}$ of control processes of the form (3.9). Let the radial distribution function of values $\eta^\sigma_\alpha = |G^\sigma_\alpha|$, $\sigma \in \Sigma$ be defined by the formula

$$F_\alpha(r) = \begin{cases} Cr^\alpha, & 0 \le r \le r_1, \\ 1, & r_1 < r \le 1 \end{cases}$$

where $C = \frac{1}{r_1^\alpha}$, $0 < r_1 \le 1$.

For $v_s = s^m$ and $v = 1$, let us use eq. (5.10) which takes the form

$$\varepsilon_s = F_\alpha^{-1}\left(\frac{1}{s^m} + o\left(\frac{1}{s^m}\right)\right) =$$
$$\left(\frac{1}{s^m} + o\left(\frac{1}{s^m}\right)\right)^{\frac{1}{\alpha}} = \frac{1}{s^{\frac{m}{\alpha}}} + o\left(\frac{1}{s^{\frac{m}{\alpha}}}\right). \quad (6.1)$$

By analyzing the scale of distribution functions F_α, we can conclude the following.

1. For $\alpha = 0$, $F_0(0) = 0$ and $F_0(r) = 1$ for $r > 0$. Hence, the density function $f_0(r) = \delta(r)$, i. e., it is the δ-function. Thus, the case $\alpha = 0$ means that it is extremely fast convergence of sequence (5.3) (or (5.4)) in a finite number of steps.

2. The case $0 < \alpha < 1$ means that there is fast convergence of of sequence (5.3) (or (5.4)). The density function is a function unbounded in a neighborhood of zero.

3. For $\alpha = 1$, $F_1(r) = \frac{1}{r_1}r$ where $0 \le r \le r_1$. Hence, F_1 is the function of a uniform distribution. Thus, the case $\alpha = 1$ means that there is normal convergence of of sequence (5.3) (or (5.4)). The density function is a separated from zero and bounded function.

4. The case $1 < \alpha < \infty$ means that there is slow convergence of of sequence (5.3) (or (5.4)). The density function vanishes at $r = 0$. With increasing values of the parameter α, the rate of convergence decreases.

5. The case $\alpha = \infty$ formally means that the density function is identically zero in a neighborhood of $r = 0$. The case $\alpha = \infty$ can be

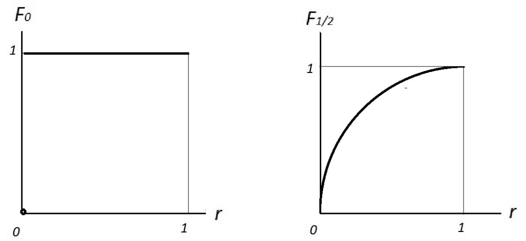

FIG. 1. Graphs of the distribution functions $F_0, F_{1/2}$.

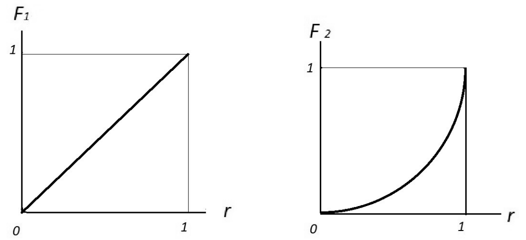

FIG. 2: Graphs of the distribution functions F_1, F_2.

interpreted as a lack of convergence of of sequence (5.3) (or (5.4)), i. e. the points of the sequence η^{σ}_{∞}, $\sigma \in \Sigma$ are separated from zero.

Thus, we have proved the following theorem.

Theorem 2 *The scale of the rates of convergence for estimates is described by the cases 1 – 5.*

Graphs of the distribution functions $F_\alpha(r), 0 \leq r \leq 1$, corresponding to the cases 1–4, are shown in FIG. 1, 2.

7. Conclusion

In order to investigate the controllability polysystem in discrete time, probabilistic methods are used. To define the control times, it is required to find the numerical solutions of inequality (2.9) with a given accuracy. The rate of convergence of estimates of control times is established (see Theorem 1). For estimates of control times, the scale of the rates of convergence is described (see Theorem 2).

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