National Academy of Sciences of Belarus
B.I. Stepanov Institute of Physics
Belarussian Republican Foundation for Fundamental Research
Belarussian State University
Belarussian Physical Society
Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine

V CONGRESS OF PHYSICISTS OF BELARUS:

Proceedings of the IX International Conference

"METHODS OF NON-EUCLIDEAN GEOMETRY IN PHYSICS AND MATHEMATICS"

Bolyai–Gauss–Lobachevsky-9 (BGL-9)

Minsk, 27 – 30 November 2015

Edited by Yu. Kurochkin and V. Red'kov

Minsk – 2016

International Scientific Committee:

Vladimir Balan
University Politehnica of Bucharest, Romania; vladimir.balan@upb.ro

Vladimir Baryshevsky
Research Institute of Nuclear Problems, BSU, Minsk; bar@inp.mimsk.by

Tamás Sándor Biró
Wigner FK, KFKI, Budapest, Hungary; Biro.Tamas@wigner.mta.hu

Ilya Feranchyk
Belarusian State University; fer@open.by

Artur Ishkhanyan
Insititute of Physical Research, Armenia; aishkhanyan@gmail.co

László Jenkovszky
Bogolyubov Instite for Theoretical Physics, Kiev, Ukraine; jenk@bitp.kiev.ua

Dmitry Kazakov Joint Institute for Nuclear Research, Dubna; kazakovdi@mail.ru

Yu.A. Kurochkin
B.I. Stepanov Institute of Physics NAS of Belarus; y.kurochkin@dragon.bas-net.by

V.I. Kuvshinov
Joint Institute for Power and Nuclear Research "Sosny Belarus; v.kuvshinov@sosny.bas-net.by

Peter-Tibor Nagy
University of Debrecen, Hungary; nagypeti@math.klte.hu

Nikitin Anatoly
Institute of Mathematics, NAS of Ukraine; nikitin@imath.kiev.ua

Tamás Novák
Károly Róbert University, Gyöngyös, Hungary; tnovak@karolyrobert.hu

Dmitry Pavlov
Inst. Hypercomplex Systems in Geometry and Physics, Russia; geom2004@mail.ru

George Poghosyan
Yerevan State University, Armenia; poghosyan@ysu.am

Grigory Polotovskiy
Nizhni Novgorod Lobachevsky State University, Russia; polotovsky@gmail.com

Aristide Papadopoulos
papadop@astro.auth.gr

Emil Petrescu
Faculty of Applied Sciences, University Politehnica of Bucharest. emilpd@yahoo.com

Yury Rybakov
Russian Peoples Friendship University, Moscow; soliton4@mail.ru

Yuri Rudoy
Russian Peoples Friendship University, Moscow; rudikar@mail.ru

G.A. Sardanashvily.
MSU, Russia, sardanashvi@phys.msu.ru

Yuriy Sitenko
Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine; yusitenko@bitp.kiev.ua

Nikolai Shumeiko
National Center of Particles and High Energy Physics, BSU, Minsk; shum@hep.by

Hidezumi Terazawa
Center of Asia and Oceania for Science(CAOS), Japan, terazawa@mrj.biglobe.ne.jp

L.M. Tomil'chik
B.I. Stepanov Institute of Physics, Belarus; lmt@dragon.bas-net.by

Mihai Vişinescu
National Institute for Physics and Nuclear Engineering Magurele,
Bucharest, Romania; mvisin@theory.nipne.ro

Tarics Zoltán
Institute of Electron Physics, Uzhgorod, Ukraine; tarics1@rambler.ru

Local organizing committee:

Chairman: **V. Red'kov**
B.I. Stepanov Institute of Physics, NAS Belarus; v.redkov@dragon.bas-net.by

Vice-chairmanr: **D. Shoukavy**
B.I. Stepanov Institute of Physics, NAS Belarus; shoukavy@ifanbel.bas-net.by

Secretary: **Anastasia Mel'nik**
IP NAS Belarus, asya.s.miller@gmail.com

George Krylov Belarussian State University; krylov@bsu.by

Halina Grushevskaya Belarussian State University; grushevskaja@bsu.by

Alexander Baran IP NAS Belarus, a.baran@dragon.bas-net.by

Igor Dudko
B.I. Stepanov Institute of Physics, NAS Belarus; igordudko89@gmail.com

Andrey Manko
B.I. Stepanov Institute of Physics, NAS Belarus; andrej.j.manko@gmail.com

Фотография участников

FOREWORD

Dear Colleague:

During 27–30 October 2015, in the frame of V-th Congress of Physicists of Belarus, the regular International Conference "Methods of Non-Euclidean Geometry in Physics and Mathematics" (BGL-9) was held in B.I. Stepanov Institute of Physics, National Academy of Sciences of Belarus, Minsk.

It is a series of biennial conferences, initiated by the Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine. Its shorthand subtitle is BGL, from the abbreviation of the names - Bolyai, Gauss, Lobachevsky - of the founders of the new geometry. The first conference of this series was held in Uzhgorod (Ukraine) in 1997, the second one - in Nyiregyháza (Hungary,1999), the third was sheduled such as to match the 200-th anniversary of Jánosh Bolyai, and was held in Targu-Mures (Marosvásárhely, Romania, 2002) - hometown of the Bolayi family. The fourth one, BGL-4, was held in Nizhny Novgorod (Russia, 2004) - hometown of Nikolai Ivanovich Lobachevsky. The fifth conference BGL-5 was held in 2006 in Minsk, Belarus; the sixth one - in Debrecen (Hungary, 2008) the seventh one - in 2010 in Cluj-Napoca (Kolozsvár), Romania. The last one, BGL-8, was held in Uzhgorod (Ukraine) in 2012.

The program of the present conference contains talks on various applications of non-Euclidean geometry in modern physics and mathematics. Among them there will be recent developments in astroparticle physics, namely the evolution of the early Universe, the problem of the dark matter and possible geometrical interpretations. All these topics are closely related to the non-Euclidean geometry. Mathematical part of the discussion will contain recent applications of non-Euclidean geometry to different topics of modern mathematics – geometry, algebra, dynamical systems and other fields.

The electronic version of BGL-9 Proceedings can be found on the official site of the Conference http://dragon.bas-net.by/bgl9/proc.htm.

Chairman of the Belarusian Physical Society, Yu. A. Kurochkin

Chairman of the Organizing Committee of BGL-9, V. M. Red'kov

THE CONTENT

PREFACE

I. HISTORICAL NOTES

1. Губина Е.В. Жизнь и научная деятельность А.А. Андронова 11

2. László Jenkovszky. Bolyai's Language 19

3. Курочкин Ю.А. Специальная теория относительности и геометрия Лобачевского. 21

4. Полотовский Г. Нижегородский математик Артемий Григорьевич Майер 30

II. FIELD THEORY

5. Cherkas S.L., Kalashnikov V.L. Quantum mechanics allows setting initial conditions at a cosmological singularity 46

6. Grushevskaya H.M., Gaisyonok V.A., Krylov G.G. Simulation of graphene with partial degeneration of band structure 56

7. Кокарев С.С. Интегральная гиперболическая динамика частиц в пространстве-времени Минковского 66

8. Rybakov Yu.P., Spinor realization of Skyrme-Faddeev chiral model of particles 79

9. Silenko A. Connection between electromagnetism and gravity 85

10. Sitenko Yu.A. Fiber bundles, non-Euclidean geometry and the Casimir effect 97

11. Ю.П. Выблый, И.Г. Дудко, А.А. Леонович. Тензорная теория гравитации с линейной связью метрики и потенциала 107

III. COSMOLOGY AND DYNAMICS

12. А.О. Шаплов, А.К. Горбацевич. Геодезическая структура сферически симметричного статического решения для центрального тела в рамках 5-мерной проективной теории поля 112

13. Gromov N.A. Elementary particles theory in the early Universe. 121

14. Leonovich A., Vyblyi Yu. Spherical gravitational waves in the weak gravitational field .. 130

15. А. Тарасенко, С. Комаров, А. Горбацевич. Реконструкция движения компактного объекта вблизи черной дыры 136

16. Томильчик Л.М. Движение по геодезической в конформно-плоской метрике как задача одночастичной лагранжевой динамики 144

IV. DYNAMICAL SYSTEMS, MATHEMATICAL ASPECTS

17. Balan V., Neagu M., Oana A., Krylova N.G., Grushevskaya H.V. The geometry of jet Hamiltonians in Langmuir-Blodgett films 150

18. Balan V., Red'kov V.M., Neagu M. The det-induced m-th root Finsler geometry of Mueller-type submanifolds 158

19. Grushevskaya H.V., Krylova N.G. The distortion and S-curvature of Langmuir monolayer space under the action of electro-capillary forces 168

20. Knyazev M.A. Calculation of inflanton field 176

21. Kurochkin Yu.A., Otchik V.S., Petrosian D.R., Pogosyan G.S. Eigenfunction expansions in the imaginary Lobachevsky space 180

22. Otchik V.S. Two Coulomb centers problem in the Lobachevsky space 190

23. Kurochkin Yu.A., Mardoyan L.G., Otchik V.S., Petrosyan D.R., Pogosyan G.S. Kepler motion on single-sheet hyperboloid 197

24. Ушаков Е.А. Применение неевклидовой геометрии в задачах релятивистской физики 207

25. Tarakanov A.N. On the possible trajectories of spinning particles in external electromagnetic fields 214

QUANTUM SYSTEMS

26. Balan V., Veko O.V., Ovsiyuk E.M., Red'kov V.M. The Dirac equation in parabolic cylindrical coordinates, possible effects of the spinor structures in quantum physics 225

27. Капшай В.Н., Фиалка С.И. Ортогональные системы функций в пространстве Лобачевского и их приложение в квазипотенциальном подходе 239

28. Kudryashov V.V. Approximate radial wave functions for a scalar particle with polarizability in the presence of the Coulomb field. 246

29. Ovsiyuk E.M., Kisel V.V., Red'kov V.M. Spin 1/2 particle with anomalous magnetic moment in presence of external magnetic field, exact solutions 250

30. Ovsiyuk E.M., Dashuk K.V., Veko O.V., Ishkhanyan A.M., Red'kov V.M. Hydrogen Atom in de Sitter spaces 260

31. Sanin A.L., Smirnovsky S.A. Quantum Matheu oscillator with quartic potential and ohmic friction 274

32. Veko O.V. Cox's particle in magnetic and electric fields on the background of Lobachevsky geometry 284

V. FUNDAMENTAL INTERACTIONS

33. Hidezumi Terazawa. Varying Fundamental physical constants – dark Energy, dark matter, and strange stars 295

34. Andreev V.V. On solving the Schrodinger equation with hypersingular kernel in momentum space 309

35. Andreev V.V., Deryuzhkova O.M., Maksimenko N.V. Spin dependent dipole polarizabilities and characteristics of the nucleon, related with parity violation 315

36. Andreev V.V., Haurysh V.Yu. Radiative decays of vector mesons in Poincaré-covariant quark model 322

37. Babich K.S., Andreev V.V. A new high precision method for solution of integral equations for two particles bound systems with the Cornell potential in momentum space 331

38. Boyarkin O.M., Boyarkina G.G., Makhnach V.V. The Higgsstrahlung process in the Standard model extensions 338

39. Gilewsky V.V., Satsunkevich I.S. Neutrino mass matrix parameterization 346

40. Maksimenko N.V., Vakulina E.V., Kuchin S.M. Dipole spin polarizabilities and gyrations of spin 1 particles in the Duffin-Kemmer-Petiau formalism 352

41. Панчелюга В.А., Панчелюга М.С. Периоды во временных рядах флуктуаций скорости радиоактивного распада и возможный механизм их образования 361

42. Д.В. Шёлковый. Анализ теста прототипа калориметра эксперимента COMET 367

43. В.А. Шапоров, В.И. Кувшинов. Сжатые и перепутанные состояния глюонов в КХД 373

VI. WAVE EQUATIONS

44. Dvoeglazov V.V. Tensor Fields in Relativistic Quantum Mechanics 383

45. Плетюхов В.А. Об интерпретации электрослабого поля в теории релятивистских волновых уравнений 387

46. Simulik V. Relativistic canonical quantum mechanics of arbitrary spin 396

47. Simulik V. Manifestly covariant field equations for arbitrary spin as a consequences of relativistic canonical quantum mechanics 410

48. Simulik V. Field equations for spins s=3/2 and s=2 420

49. Кисель В.В., Войнова Я.А., Редьков В.М. Частица Дирака с аномальным магнитным моментом во внешнем магнитном поле, построение решений методом квадрирования .. 431

50. Веко О.В., Войнова Я.А., Кисель В.В. О расширенных уравнениях для частиц со спином ноль (единица) в электромагнитных полях 441

51. И.Г. Дудко, Е.А. Толкачев. Три типа дуальности преметрических уравнений электродинамики: геометрический аспект 449

ADVERTISEMENTS .....

Жизнь и научная деятельность А.А. Андронова

Е.В. Губина
gubinael@mail.ru; Нижегородский госуниверситет им. Н.И. Лобачевского, Россия

Аннотация

Приведена краткая научная биография А. А. Андронова и история становления андроновской школы по теории нелинейных колебаний и качественной теории дифференциальных уравнений.

Ключевые слова: А. А. Андронов, биография, научная школа

«Я не знал и не знаю ни одного человека, который бы отличался от моего идеала хорошего человека меньше, чем А. А. Андронов»

Г. С. Горелик

Александр Александрович Андронов родился 11 апреля 1901 года в Москве. Его мать, Лидия Александровна Липская, была домашней хозяйкой. Отца своего А. А. Андронов практически не знал, воспитывал его отчим — Корнелий Адамович Липский, врач одного из Московских роддомов.

В детстве А. А. Андронов решил, что будет врачом. Ему виделась новая медицина — не одно лишь «искусство врачевания», а ещё и наука, использующая достижения математики и физики. Поэтому ещё в школьные годы он занялся изучением высшей математики.

Среднюю школу Андронов закончил в 1918 году, с 1918 по 1920 годы он работал браковщиком на заводе «Пулемёт», монтёром на электростанции, затем вступил добровольцем в Красную Армию, служил в военно-продовольственном отряде на Урале, был лектором политотдела Троицкого укрепленного района. Осенью 1920 года Андронов перенес тяжелую форму плеврита, был признан непригодным к военной службе и поступил на электромеханический факультет Московского высшего технического училища (МВТУ). С 1921 года одновременно с занятиями в МВТУ он стал посещать лекции на физмате МГУ, куда в 1923 году перевёлся из МВТУ и который окончил в 1925 году по специальности «теоретическая физика».

Годы учёбы А. А. Андронова в МГУ совпали с началом расцвета московской математической школы. Курс математики был единым для математиков и для физиков, поэтому Андронов, настойчиво занимавшийся математикой, приобрел математическую культуру значительно более высокую, чем получали обычно физики-теоретики. Андронов проявил большой интерес и к теоретической механике (возможно, под влиянием С. А. Чаплыгина, который произвёл на него сильное впечатление).

Решающую роль в формировании А. А. Андронова как учёного сыграла аспирантура под руководством Л. И. Мандельштама в НИИ физики при МГУ (1925–1929 гг.). Итогом учёбы в аспирантуре стала диссертация «Предельные циклы Пуанкаре и теория автоколебаний», посвящённая основополагающим вопросам теории нелинейных колебаний и ставшая решающим звеном в самом создании этой теории. Краткое изложение диссертации было опубликовано в [1], а затем в докладах Французской академии наук [2]. Сам А. А. Андронов писал, что эта диссертация определила область его дальнейшей научной деятельности — теорию колебаний и смежные вопросы математики и теоретической физики. Андронов ввёл в теорию колебаний настоящую математику, основанную на теории обыкновенных дифференциальных уравнений, дал автоколебаниям их название и чёткое математическое определение, указал путь строгого анализа автоколебательных режимов, определив надолго вперёд подход к исследованию автоколебательных систем.

Г. С. Горелик писал об этом времени[1]: «*Кто знал 25-летнего Шурку Андронова с его могучим телосложением, буйной энергией, голосом, гремевшим на все этажи физического института Московского университета, мог бы подумать, что именно он больше, чем кто-либо, предназначен для дел, не чуждых внешней эффектности, производящих сильное впечатление и на людей, далёких от науки. На деле оказалось совсем иначе. Подвиги А. А. Андронова иного рода. Они совершались в тишине. Для того, чтобы их понять, нужно смотреть в глубь вещей*».

Близкое окружение Андронова замечательно описано Н. М. Леонтович (Левиной): «*. . . в начале двадцатых годов . . . в центре Москвы, на Сивцевом Вражке, организовалась коммуна, вокруг которой собрались эти удивительные люди. Несомненно, это сообщество уникально для советского времени, да, пожалуй, и вообще уникально. В эту компанию входили Николай Николаевич Парийский* (*астроном, впоследствии член-корреспондент АН*[2]), *Александр Александрович Андронов* (*физик, впоследствии академик*), *Михаил Александрович Леонтович* (*физик, впоследствии академик*), *Петр Сергеевич Новиков* (*математик, впоследствии академик*), *Александр Адольфович Витт* (*физик, впоследствии доктор физ.-мат. наук*). *Это основные мужчины, определяющие содержание жизни и дух всего сообщества. Все они учились в университете на физико-математическом факультете. Женщины были объединены учёбой в Лосиноостровской гимназии. Это Лидия Викторовна Птицына, сёстры Свешниковы* (*Татьяна и Наталья*), *Людмила Всеволодовна Келдыш,*

[1] Здесь и ниже все цитаты, не снабжённые ссылками, даются по тексту книги [3].

[2] Это и следующие пояснения, взятые в скобки, в оригинальном тексте Н. М. Леонтович (Левиной) даны ниже, за пределами данной цитаты.

Наталия Лучинская. Входила в эту компанию и Евгения Александровна Леонтович, сестра М. А. Леонтовича, брат и сестра Старокадомские (Михаил — композитор, Екатерина — физик). Несколько на особицу стоял Игорь Евгеньевич Тамм. Он был старше и к нему обращались на Вы и по имени-отчеству <...>

Естественно, образовалось несколько супружеских пар. Н.Н. Парийский и Л.В. Птицына, А.А. Андронов и Е.А. Леонтович, М.А. Леонтович и Т. П. Свешникова, П.С. Новиков и Л.В. Келдыш. Пары оказались очень крепкими — на всю жизнь. Характерно, что в семьях было много детей <...>

Люди эти были очень яркими индивидуальностями и, конечно, очень отличались друг от друга. Однако было многое, что их объединяло. Они составляли содружество, имевшее общее лицо <...> Основой жизни, стержнем, была, конечно, наука. И наукой они занимались ради науки. Своё основное удовольствие они получали от узнавания, открытия научного факта. Всё остальное — вторичное. Они не делали карьеры. Даже представить себе невозможно, чтобы кто-нибудь из них организовывал получение какого-либо звания <...>

Но они не были сухарями и интересовались совсем не одной наукой. Очень большое значение в их жизни играла природа <...> Все они знали и любили литературу, живопись <...>

Они были атеистами. При этом их нравственная планка была очень высока <...>

В двадцатые годы они были «красными». Эти очень умные люди (причём думающие над социальными, общественными вопросами) не поняли преступность Октябрьской революции <...> Видимо, самым революционным из них был А. А. Андронов <...> От чего зависело прозрение? Кроме, конечно, фактов, которые все знали. От психологической необходимости жить «заодно с правопорядком». От того, насколько эти умнейшие люди разрешали себе в этих вопросах думать так же до конца, как они умели делать это в своей науке <...> При всей революционной настроенности этих людей в начале двадцатых годов, ни один из них не был членом партии. Почему? Мне кажется, что для них была невозможной потеря степеней свободы, взятие на себя обязательств что-то делать и говорить не согласно со своими убеждениями, но согласно с чьими-то указаниями <...>

Удивительным был их пуританский образ жизни <...> Они же жили очень аскетично. Очень простая одежда, у женщин абсолютно никаких украшений. Очень простая мебель, никаких занавесок, абажуров <...> Наверняка, неправильно сказать, что все они были очень счастливые люди. Но все они были люди состоявшиеся. И причиной этого были, на мой взгляд, не только заложенные в них способности в сочетании с научным любопытством, но и их нравственные позиции, которые не дали им растратить на мелочи то, из чего получилось нечто по-настоящему стоящее — наука, семья».

После окончания аспирантуры А. А. Андронов сначала работает во Всесоюзном электротехническом институте, а затем в НИИ физики при МГУ. В 1931 году А. А. Андронов и его жена Е. А. Леонтович переехали в Нижний Новгород (Горький). Причин для переезда молодых учёных было много, в том числе — забота о развитии отечественной науки и стремление создать научный центр в провинции. В Нижнем Новгороде Андронов начинает работать в Физико-техническом институте (ГИФТИ) и во вновь открывшемся 1 ноября 1931 года Нижегородском университете. В ГИФТИ он возглавлял отдел теории колебаний и теории автоматического регулирования (1931–1949). В университете по инициативе А. А. Андронова была создана кафедра теории колебаний (1933 г., одна из первых в мире), которой он заведовал до 1945 года.

Первые годы Александр Александрович совмещал работу в Горьком с работой в НИИ физики при МГУ, затем сосредоточил всю работу в Горьком. Свои исследования в 1931–1941 годах сам Андронов делил на три направления:

1. Развитие качественной теории дифференциальных уравнений и её приложений к проблемам теории нелинейных колебаний.

2. Применение теории нелинейных колебаний к задачам радиотехники и механики.

3. Исследования в некоторых разделах теоретической физики, связанных с вопросами физики колебаний.

Главное место в работах А. А. Андронова этого периода занимало первое направление. Особого упоминания заслуживают работы по теории бифуркаций автоколебательных систем и прежде всего — идея о грубых системах, разработанная Андроновым совместно с Л. С. Понтрягиным [4]. Понятие грубости можно трактовать как устойчивость структуры разбиения фазового пространства системы на траектории по отношению к малым изменениям правых частей уравнений системы.

В 1933 году А. А. Андронов получил приглашение сделать доклад на Первой международной нелинейной конференции в Париже, а в 1934 году Ван-дер-Поль представил на конференции в Лондоне обширный доклад о советских работах по нелинейным колебаниям, написанный А. А. Андроновым совместно с другими авторами.

В 1934 году А. А. Андронову было присуждено звание профессора, а год спустя — степень доктора физико-математических наук. Результаты работ А. А. Андронова и его сотрудников вошли в изданную в 1936 году коллективную монографию [5].

Академик Н. Д. Папалекси писал: «*Крупная заслуга А. А. Андронова и его многочисленных сотрудников и учеников — открытие значения методов качественного анализа дифференциальных уравнений для теории нелинейных колебаний и превращение их в мощное орудие исследования. С* 1927–28 *года ведущая роль исследований нелинейных колебательных систем и развитие их теории постепенно перешла к нам, и в настоящее время ведущая роль наших ученых в области нелинейных колебаний получила всеобщее признание*».

В 1937 г. выходит в свет книга «Теория колебаний», написанная А. А. Андроновым совместно с А. А. Виттом и С. Э. Хайкиным. Однако в этом первом издании книги не было имени А. А. Витта, «*участвовавшего в написании книги наравне с другими авторами, но не указанного в их числе вследствие печальной ошибки*» (С. Э. Хайкин). Эта «печальная ошибка» состояла в том, что А. А. Витт был репрессирован и умер в лагере на Колыме 26 апреля 1938 года[3]. Н. М. Леонтович (Левина) пишет: «*Из всей их компании погиб только один человек — А. А. Витт. Про Витта мы теперь очень мало что знаем. Но одно из его* “*мо*” *вошло в научный фольклор —* “*всё плохое сократится, всё хорошее останется*”».

О книге «Теория колебаний» Н. В. Бутенин, один из учеников А. А. Андронова, пишет: «*Вряд ли можно переоценить значение этой книги в становлении нелинейной теории колебаний как в нашей стране, так и во всём мире. Ведь, в сущности, впервые появилась книга, где с ясной теоретической позиции излагались основы теории нелинейных колебаний как сложившейся науки; эта теория иллюстрировалась многочисленными примерами из различных областей физики и техники. Исследователи получили в руки мощное оружие для решения задач, возникающих при рассмотрении нелинейных динамических систем*».

В 1959 году вышло второе издание «Теории колебаний» (в числе авторов был указан и А. А. Витт) с существенными дополнениями, сделанными Е. А. Леонтович и учеником А. А. Андронова Н. А. Железцовым. Второе издание представляло вершину «колебательных» достижений 50-х годов в области динамики нелинейных автономных систем второго порядка. В 70-е годы сформировалось мнение, что книга А. А. Андронова, А. А. Витта и С. Э. Хайкина — это классика, которую не следует дополнять и перерабатывать, и в 1981 году появилось третье издание знаменитой монографии, тождественное первому.

Г. С. Горелик пишет: «*Особые точки, предельные циклы, сепаратрисы — всё это для*

[3] А. А. Витт был посмертно реабилитирован в 1957 году.

Александра Александровича как бы части живого организма: математической модели той или иной машины, или следящей системы, или ламповой схемы. И недаром он часто говорит: "*Предельные циклы родятся, растут, умирают* (*дохнут*)". *Он* "*просвечивает*" *математическим рентгеном особую точку и выясняет, что она* "*беременна*" *предельным циклом... Некоторые страницы рисунков в книге Андронова и Хайкина прямо напоминают эмбриологический атлас*».

В предвоенные годы А. А. Андронов получил фундаментальные результаты в теории автоматического регулирования. Эти исследования получили большое развитие в последние годы Великой Отечественной войны и в послевоенные годы. Система с автоматическим регулированием (например, самолёт, снабжённый автопилотом) обладает характерной склонностью к автоколебаниям (обычно нежелательным). А. А. Андронов усмотрел глубокую аналогию между автоколебаниями в системах автоматического регулирования и колебаниями в радиофизике. В 1944 году совместно с А. Г. Майером он опубликовал свою первую статью по теории регулирования [6] и статью об автопилоте [7], написанную совместно с Н. Н. Баутиным. В 1945–1947 гг. совместно с Г. С. Гореликом, А. Г. Майером и Н. Н. Баутиным он продолжил этот цикл статей. За работы, выполненные во время Великой Отечественной войны, А. А. Андронов награжден орденом Красной Звезды.

В послевоенные годы А. А. Андронов и его сотрудники много работали над созданием книги, объединяющей результаты по качественной теории дифференциальных уравнений. Эта работа была завершена уже после смерти А. А. Андронова: в 1966 и 1967 гг. вышли в свет коллективные монографии А. А. Андронова, Е. А. Леонтович, И. И. Гордона и А. Г. Майера [8] и [9].

В 1946 г. А. А. Андронов, не являясь членом-корреспондентом АН, единогласно избирается действительным членом АН СССР по отделению технических наук.

Научные достижения А. А. Андронова громадны и удивительны, особенно если учесть, что они достигнуты на фоне очень большой, порой самоотверженной организаторской, административной и педагогической работы.

Александр Александрович начал свою педагогическую деятельность очень рано. Еще до окончания МГУ он стал преподавать во 2-м МГУ (Московский государственный педагогический институт) в качестве ассистента, а затем — в качестве доцента по кафедре теоретической физики и механики. В Горьком А. А. Андронов становится и до конца жизни остаётся профессором университета (ГГУ), в котором с осени 1941 по ноябрь 1942 года он был проректором, а летом 1942 года выполнял обязанности ректора.

А. А. Андронов придавал очень большое значение качеству преподавания. Он разработал множество учебных планов и программ, поставил курс теории колебаний, читал курсы электродинамики и теории относительности, организовал преподавание теоретической физики. По общему признанию слушателей, его лекции были очень яркими, увлекательными, глубоко продуманными. А. А. Андронов чётко следовал выработанным им правилам преподавания: сделать для студентов абсолютно ясными основы науки, после этого студентов можно подводить к вещам, действительно сложным для понимания. Кроме того, воздействовать не только на ум, но и на воображение студентов.

Андронов болезненно переживал отставание большинства периферийных вузов от вузов Москвы и Ленинграда в качестве подготовки специалистов. В 1950 году в статье [10] он указал три основные причины низкого качества работы вузов на периферии: 1) недостаточно высокая квалификация преподавателей; 2) слабая оснащённость лабораторий и библиотек и отсутствие во многих вузах хорошо оборудованных мастерских; 3) поверхностное руководство учебно-педагогическим процессом и формальный подход к оценке качества работы вузов со стороны министерств. А. А. Андронов писал о необходимости реорганизации университетов И. В. Сталину. Вот отрывок из черновика этого письма [11]: «... *В настоя-*

щей докладной записке мы хотим поставить вопрос о состоянии и дальнейшем развитии русских университетов, т. е. университетов, находящихся сейчас в ведении Наркомпроса Р.С.Ф.С.Р., т. к. о состоянии и деятельности других университетов мы имеем лишь косвенные [сведения], однако мы думаем, что мы [неразборчиво]. Мы считаем, что состояние русских университетов в настоящее время, в особенности провинциальных, не соответствует ни достоинству великого русского народа, ни тем задачам, которые сейчас стоят перед нашей родиной. . . »

А. А. Андронов выступал за тесную связь ГГУ с научно-исследовательскими институтами г. Горького и других городов страны. Он приглашал в Горьковский университет крупных учёных из других городов для чтения некоторых курсов и отдельных лекций. Благодаря ему в Горьком работали Г. С. Горелик, С. М. Рытов, С. П. Стрелков, В. Л. Гинзбург, Е. Л. Фейнберг, возглавившие разработку новых научных направлений и воспитавшие большую группу талантливых учеников.

В университете А. А. Андронов создал сеть семинаров, которые стали настоящей школой научно-исследовательской работы. Семинарами по теории нелинейных колебаний, по качественной теории динамических систем, по теории электрических машин он руководил сам, другими руководили его сотрудники (Г.С. Горелик, Е.А. Леонтович, А.Г. Майер). Особое внимание Андронов уделял университетской библиотеке: «. . . *Никто так не заботился о библиотеке ГГУ, как Александр Александрович. Благодаря его хлопотам библиотека ГГУ получала больше, чем другие библиотеки, иностранной* (*валютной*) *литературы. Под его руководством комплектовались старые журналы. При его участии решались все важные для библиотеки дела. . .* » (А. И. Лалетина, работник библиотеки ГГУ). А. А. Андронов вложил много труда и энергии в работу со школьниками и абитуриентами. Он написал справочник для поступающих в университет.

В созданной Андроновым творческой атмосфере развивались настоящая наука и научная школа в том высоком смысле, который вкладывал в это понятие сам А. А. Андронов. В разные годы с ним работали Г. С. Горелик, С. М. Рытов, А. Г. Майер, Н. П. Власов, Я.Н. Николаев, Н. Н. Баутин, Н. В. Бутенин, Г. В. Аронович, Н.А. Железцов, Ю. И. Неймарк, С. А. Жевакин, А. С. Алексеев, Н. А. Фуфаев, И. Л. Берштейн, С. В. Беллюстин, А. В. Гапонов-Грехов. «*Горьковской научной школе А. А. Андронова судьба определила долгую жизнь. Поднятая ею тема оказалась одной из основных, базовых в точном естествознании и технике, требующей длительной разработки и имеющей широчайшие и разнообразные приложения. Теория колебаний — наука об общих закономерностях эволюционных процессов различной природы: физической, химической, биологической, экономической, социальной. Изучаемая ею математическая модель — динамическая система — стала основной математической моделью точных наук*» (Ю. И. Неймарк, [12]).

В 1996 году в Париже состоялась конференция «Андронов и его школа в Горьком».

А. А. Андронов был не только выдающимся физиком и математиком, он внёс большой вклад в историю науки. Начиная заниматься новой задачей, он изучал всю имеющуюся литературу, включая историю исследования этой задачи. При этом Андронов проявлял большой интерес к личностям самих исследователей. Первая работа по истории науки была опубликована им совместно с Е. А. Андроновой-Леонтович в 1930 году — это книга [13], посвящённая жизни и мировоззрению Лапласа. Сейчас она является библиографической редкостью. А. А. Андронов написал замечательную статью [14] о своем учителе Л. И. Мандельштаме, в которой определена роль Мандельштама в истории развития теории нелинейных колебаний. В статье [15] совершенно по-новому изложена история создания теории автоматического регулирования, в качестве главных создателей которой обоснованно названы Д. Максвелл, И. А. Вышнеградский и А. Стодола.

В 1947 году А. А. Андронов начал заниматься биографией Н. И. Лобачевского. По инициативе, под руководством и при личном участии Андронова созданная им исследова-

тельская группа провела большую работу по изучению нижегородского периода жизни Н. И. Лобачевского. В результате этой работы удалось установить точную дату (20 ноября 1792 г. по старому стилю) и место (Нижний Новгород) рождения Лобачевского. По инициативе А. А. Андронова в 1956 году, уже после его смерти, Горьковскому университету было присвоено имя Н. И. Лобачевского.

В 1947 году А. А. Андронов был избран депутатом Верховного Совета РСФСР, а в 1950 году — депутатом Верховного Совета СССР. К своим депутатским обязанностям Андронов относился чрезвычайно добросовестно, не оставляя без внимания ни одно письмо, ни одно обращение. Благодаря его усилиям был электрифицирован большой район Горьковской области, где свыше тридцати лет Советской власти люди жили при свете керосиновых ламп. Он разбирал жалобы и заявления избирателей о предоставлении жилплощади и оказании материальной и медицинской помощи, о помиловании и сокращении срока заключения, о розыске пропавших родственников и т. д. Очень многие люди, которым помог А. А. Андронов, присылали ему письма с благодарностью за тепло и отзывчивость.

А. А. Андронов находил время и для активной просветительской деятельности. Яркий пример — две лекции «Об атомной энергии и атомной бомбе», прочитанные им на Горьковском автозаводе в октябре 1945 года по просьбе руководства завода (общественный интерес к этой тематике возник после атомных бомбардировок Хиросимы и Нагасаки в августе 1945 года). В. И. Широков (ректор ГГУ с 1955 по 1961 гг.) в своих воспоминаниях в [3] пишет: «*Лекция была выслушана с неослабевающим интересом и имела такой успех, который я не могу описать. Весь следующий день мой телефон на рабочем столе не переставал сообщать мне восторженные отзывы о лекции. Все интересовались лектором: откуда он, говорили о его необыкновенной манере разговаривать и улыбаться одновременно, об умении ясно и доходчиво раскрывать перед слушателями сложные физические процессы, сопровождая рассказ о сложных явлениях элементами легко воспринимаемого и понятного юмора*».

И далее о повторной лекции (первую не смогли послушать все желающие): «*Уже в пятницу после очередной селекторной оперативки главный диспетчер автозавода по селектору сообщил <...> что в очередной вторник состоится повторно лекция профессора Горьковского университета Андронова <...> А потом добавил*: "*Это очень интересная лекция, товарищи! И читает её необыкновенный профессор. Советую всем, кто не занят на работе в это время, обязательно посетить эту лекцию*!"»

О жизни и деятельности А. А. Андронова имеется обширная литература. По многочисленным свидетельствам, Андронов обладал исключительными человеческими качествами и был безусловным нравственным образцом и авторитетом для окружающих. В Нижнем Новгороде на здании института, где он работал, установлена памятная доска с его барельефом. Но прочнее всего его имя сохраняет название созданной им всемирно известной андроновской научной школы по теории нелинейных колебаний.

Список литературы

[1] Андронов А. А. Предельные циклы Пуанкаре и теория колебаний // В кн.: VI съезд русских физиков, Москва, Н. Новгород, Казань, Саратов (5–16 августа 1928 года). — М. – Л.: Гос. изд-во, 1928. С. 23–24.

[2] Andronow A. Poincaré limit cycles and the theory of self-sustaining oscillations (Les cycles limites de Poincaré et la théorie des oscillations autoentretenues) // Comptes rendus de l'Academie des Sciences. 1929. V. 189. P. 559–561.

[3] Александр Александрович Андронов (1901–1952) / Серия «Личность в науке». — Нижний Новгород: Изд-во ННГУ им. Лобачевского, 2001. 287 с.

[4] Андронов А. А., Понтрягин Л. С. Грубые системы // ДАН СССР. 1937. Т. 14. № 5. С. 247–250.

[5] Мандельштам Л. И., Витт А. А., Папалекси Н. Д., Андронов А. А., Горелик Г. С., Хайкин С. Э. Новые исследования в области нелинейных колебаний. — М.: Гос. изд-во по вопросам радио, 1936. 96 с.

[6] Андронов А. А., Майер А. Г. Задача Мизеса в теории прямого регулирования и теория точечных преобразований поверхностей // ДАН СССР. 1944. Т. 43. № 2. С. 58–62.

[7] Андронов А. А., Баутин Н. Н. Движение нейтрального самолета, снабженного автопилотом, и теория точечных преобразований поверхностей // ДАН СССР. 1944. Т. 43. № 5. С. 197–201.

[8] Андронов А. А., Леонтович Е. А., Гордон И. И., Майер А. Г. Качественная теория динамических систем второго порядка. — М.: Наука, 1966. 567 с.

[9] Андронов А. А., Леонтович Е. А., Гордон И. И., Майер А. Г. Теория бифуркаций динамических систем на плоскости. — М.: Наука, 1967. 487 с.

[10] Андронов А. А. Нужны решительные меры // Вестник высшей школы. 1950. № 7. С. 33–37.

[11] Архив РАН. Фонд 1938, опись 1, дело 190.

[12] Неймарк Ю. И. Сухой остаток. — Нижний Новгород: Нижегородский гуманитарный центр, 2000. 142 с.

[13] Андроновы А. и Е. Лаплас. — М.: Госиздат РСФСР Московский рабочий, 1930. 192 с.

[14] Андронов А. А. Л. И. Мандельштам и теория нелинейных колебаний. Собр. Трудов. — М.: АН СССР, 1956. С. 441–472.

[15] Андронов А. А., Вознесенский И. Н. О работах Д. К. Максвелла, И. А. Вышнеградского и А. Стодолы в области регулирования машин // В кн.: Д. К. Максвелл, И. А. Вышнеградский и А. Стодола. Теория автоматического регулирования. — М. – Л.: 1949. С. 253–301.

Life and scientific activity of A.A. Andronov

E. V. Gubina

Аннотация

The brief scientific biography of A. A. Andronov and the history of developing of the Andronov's school of the theory of nonlinear oscillations and the qualitative theory of differential.

Keywords: A. A. Andronov, biography, scientific school

László Jenkovszky

BOLYAI's LANGUAGE

jenk@bitp.kiev.ua; Bogolyubov Institute for Theoretical Physics,
National Academy of Sciences of Ukraine, Kiev, 03680 Ukraine

Besides his native Hungarian, János Bolyai was fluent in several European languages. He devoted much effort in searching an "ideal" language, based on Hungarian, applicable to natural sciences.

Keywords: Hungarian scientific language, universal language, shorthand script, root words**,** mathematical linguistics

1. Milestones in János Bolyai's life

Born on 15.12.1802 in Kolozsvár – died on 27.01.1860 in Marosvásárhely; *Parents*: Bolyai [1] Farkas (B**ó**lya, Domáld) and Benkő Zsuzsanna (Kolozsvár). In 1918 Farkas wrote to F. Gauss asking him to invite János for studies (denied by Gauss); studies at the Royal Military College in Vienna (1818-1822); 1931 service in Lemberg (Lwów); wife Rozália (1848-1852); János, unlike his father, was never elected to the Hungarian Academy of Sciences; *NB*: János and Farkas Bolyai were contemporaries of *Count István Széchenyi de Sárvár-Felsővidék* (21.09.1791 – 8.04.1860), Hungarian politician, theorist and writer, founder of the Hungarian Academy of Sciences (1825). *NB:* A precondition to candidate to the Hungarian Academy of Sciences was publication in Hungarian.

Fig. 1. Map of Hungary at Bolyais' times. The sites, Kolozsvár and Marosvásárhely, relevant to the family (see: *Milestones*) are marked by red circles.

Bolyais' native language was Hungarian, which they used in their correspondence, although their scientific works were written in Latin or German. János was fluent also in Latin, German, Italian, French and Romanian.

2. Bolyais' ideas about language reforms

Among many passions, János was obsessed also by the idea of reforming the Hungarian language, to make it applicable to scientific texts. *A digression*: *the* Hungarian belongs to the Ugro-Finnic group of languages. Before Latin symbols were adopted, ancient Runic script was used.

The modern Hungarian literary language was developed and established in the 18-th century, however, similar to the rest of the Western world, Latin remained the language of natural sciences. Reforms aimed at adapting native languages to scientific purposes started in Hungary, like in most West-European countries, at the beginning of the 18-th century, and János took active part in this process. He believed that Hungarian, due to its particular grammar, offers a perfect basis for a future, universal scientific language. Its basis should be semantic (symbolic). János was obsessed (wrongly?) by the importance of the so-called "root words**"** *(gyökszavak)* or *honominas***,** unique for the Hungarian, as he believed (see Ref. [1]). He was arguing e.g. by comparing Hungarian with Latin and German in the phrase:

Péter ember (Hungarian),
Petrus est homo (Latin),
Peter is ein Mensch (German).

The first, Hungarian version seems the simplest (two words only), and thus, according to János Bolyai is superior(?) to others. Let me add, "in parentheses", that in Russian (Bolyai was not familiar with) this phrase is: *Петр человек*, still two words, as in Hungarian, although a symbol*, a hyphen* may be appropriate between the two. Bolyai's idea in building the future language for natural sciences was to use the simplicity and uniqueness of the Hungarian language/grammar and

extend it with modern European vocabulary as well as by the logical/rational mathematical language.

Similarly to most of such reformers, he failed in his attempts. One reason is that any living language, including that of natural science, cannot and should not be rational. The language of natural sciences is a *transmitter* between the fully rational language of mathematics and logic (e.g. the statement $a=b$ is unambiguous, while the description of what is a and b, and what does the equality = (or identity) mean allow for various interpretations. Many attempts to build a universal and ideal language exist. The best known is *Esperanto,* another failure. The reason of these and similar failures is that any living language, including that of natural sciences cannot be constrained artificially. (NB: a "partially successful" counterexample is *Ivrit*. By "partial" I mean that: a) it was not completely artificial since its rudiments survived from ancient times, and b) it is still not clear whether *Ivrit* will be able to resist the dominance of great language like German, Russian or English, similarly to the Ukrainian, that despite the official pressure, cannot resist the Russian linguistic dominance.

Bolyai János' heritage contains over 15000 pages of manuscripts, written in special codes, and stored at the Teleky library in Marosvásárhely. Most of them are still awaiting to be decoded. Kiss Elemér [2] contributed largely to their decoding. It is not always clear whether the texts of János imply shorthand-writing (*stenography*) or his *honomina* (the thought new language).

A specific feature of the Hungarian language is the great variety of vowels, namely: *ú (u), ű (ü), ó (o), ő (ö), í(i), é (e),* too many?! János, being a professional and able musician was meditating on how to harmonize the music of the Hungarian language by making comparison with the Italian. He was reasoning about the correspondence between the pronunciation (phonetics) and musical tones (height) of the above vowels, bringing parallels between the two. As noted by János, the Hungarian does not contain diphthongs (like the German (*Sie, Liebe,…*), for more details see Ref. [3].

Another linguistic aspect he was interested in is related to optimizing the symbols, e.g. by replacing the multitude of the tonic (upper) accents, as above, by various underscores. These proposals was never realized.

3. Modern developments in shorthand writing

János Bolyai's ideas about the construction of a laconic language for natural sciences are being developed in many directions. Let us mention the "telegraphic language", in which "auxiliary" words are dropped, although they are required by regular grammar. A somewhat similar development occurs in electronic communication, e.g. in e-mails. Moreover, the e-mail style and the Internet produced a large number of abbreviations (see below) and, in many cases, neglect of tonic accents (implied!). The reason for the latter is that the computers are adapted to a limited number of languages (in most cases two: the local language and English). A further logical step in this direction would be the neglect of vowel (as e.g. in Ivrit). LA(TEX) is another example in this process. For example, in LA(TEX), the (operation of) square root is symbolized by "sqrt" , and an exponential is "exp", *etc*. Even more amusing (and efficient?) are the following widely used abbreviations: *at*=@, *cross section*=xsection, *I do not know* = idk, *as soon as possible* – ASAP, *together*=2gether, *forever*=4ever, *to do*=2do, *to be*=2be, *you*=u, *for you*=4you, *your*=ur, are=r, *please*=plz, *thank you*=thy. Emotions can be expressed by simple icons ("hieroglyphs") like: :)=smile, ;), :\, etc. Would János have liked it? Maybe, we do not know. In any case, these simplifications may contribute to the degradation of any language and culture on the whole.

Summary

The human language, even scientific, cannot be reduced to rational mathematical relations**;** any living language is *irrational* and (necessarily!) contains polysemantism. Reducing an ambiguous literature phrase to a rational formula may kill the language. *NB:* In the past, many papers in natural sciences (*e.g.,* by Galileo and Lomonosov) were written in verses, today difficult to conceive! The relation (and proportions) between the two is a rich and interesting subject for further studies and discussions.

Acknowledgement

I thank Benkő Samu and Lovas István for useful and pleasant conversations on the subject of this contribution as well as Oláh-Gál Robert for a helpful correspondence.

References

1. The Hungarians write first their family name (*e.g.,* Bolyai), followed by the given name (*e.g.,* Farkas).

2. Kiss Elemér: M*atematikai kincsek BolyaI János kéziratai hagyatékából.*Akadémiai kiadó, Budapest, 1999.

3. Marácz Sándor, *Bolyai János és a magyar mint tökéletes nyelv,* http://www.kincseslada.hu/ magyarsag/content.php?article.328.

Специальная теория относительности и геометрия Лобачевского

Ю.А. Курочкин
y.kurochkin@dragon.bas-net.by
Институт физики им. Б.И. Степанова, НАН Беларуси

Аннотация

Дан историко-методологический анализ проблемы вывода релятивистского закона сложения произвольно направленных скоростей как векторов трехмерного пространства Лобачевского - упорядоченных пар точек (направленных отрезков геодезических).

1 Н. Лобачевский и Ф. Миндинг

В жизни великого геометра Н.И. Лобачевского (в личной и творческой) есть несколько тайн. Одна из них связана с вопросом признания его геометрии. Вот, что пишет на сайте Казанского университета профессор В.В. Вишневский: "Недавно стало известно, что у Лобачевского был все-таки реальный шанс убедиться в конкретной применимости его геометрии, которая, таким образом, перестала бы быть воображаемой. В 1839-1840 гг. в журнале Крелля, где печатались наиболее известные работы европейских математиков, дерптский профессор Миндинг опубликовал пару статей по внутренней геометрии поверхностей постоянной кривизны. Профессор КГУ Б.Лаптев, изучая журнал Научной библиотеки КГУ, в который заносились книги, выданные профессорам для домашнего чтения, обнаружил, что Лобачевский систематически брал журнал Крелля, но в 1838-1840 гг. несколько номеров этого журнала им взяты не были, поэтому он так и не узнал, что внутренняя геометрия поверхностей постоянной отрицательной кривизны совпадает с его планиметрией. Это обнаружил лишь в 1868 г. итальянский геометр Бельтрами. Публикация в те годы дневников и писем Гаусса, в которых работам Лобачевского давалась весьма высокая оценка, послужила толчком к признанию и распространению в Европе идей неевклидовой геометрии".

Выше приведен абзац из статьи бывшего декана математического факультета Казанского университета профессора В.В. Вишневского. Указанную информацию автору этих строк пришлось слышать из уст В.В. Вишневского. на одной из конференций, но в трудах был опубликован другой материал. Тем не менее, когда возникла необходимость, сообщение без труда было найдено на сайте Казанского университета. Теперь дадим пояснения. Фердинанд Готлибович Миндинг (1806–1885) профессор Дерптского (Тартуского) университета, занимаясь теорией поверхностей установил, что тригонометрические формулы поверхности отрицательной постоянной кривизны можно получить из формул сферической тригонометрии путем замены соответствующих тригонометрических функций на гиперболические. Иными словами, если взять известную, и уже несколько столетий до Миндинга, используемую в астрономии формулу теоремы косинусов сферического треугольника

$$\cos\frac{r_1}{R}=\cos\frac{r_2}{R}\cos\frac{r_3}{R}+\sin\frac{r_2}{R}\sin\frac{r_3}{R}\cos\vartheta_{23}\quad , \tag{1}$$

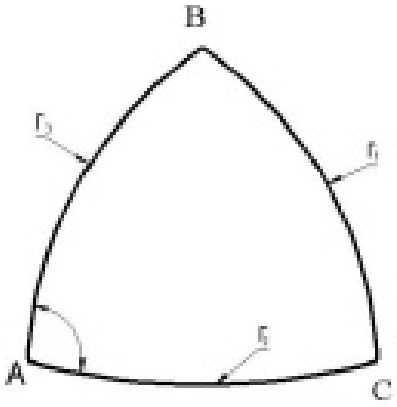


Рис. 1: Сферический треугольник

и, осуществить замену $R \to i\rho$ в результате получим формулу

$$\cosh\frac{r_1}{\rho} = \cosh\frac{r_2}{\rho}\cosh\frac{r_3}{\rho} - \sinh\frac{r_2}{\rho}\sinh\frac{r_3}{\rho}\cos\vartheta_{23} \quad , \tag{2}$$

которую получил Ф. Миндинг, но что еще важнее ранее неявно использовал Н.И. Лоба-

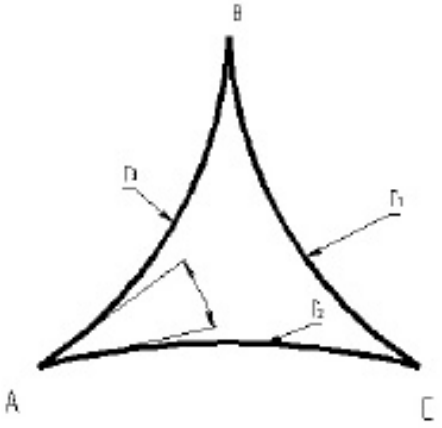


Рис. 2: Гиперболический треугольник

чевский в своей геометрии, для вывода ряда ее основных положений. В формуле (2), при осуществленной замене тригонометрические функции преобразовались в гиперболические. Поясним, также что под кривизной поверхности в математике понимают величинуK обратную квадрату радиуса кривизны. Таким образом, для сферы$K = 1/R^2$, а для поверхности с мнимым радиусом кривизны, кривизна отрицательна, например, $K = -1/\rho^2$. Поэтому сделав в формуле (1) замену $R \to i\rho$ мы получили формулу, которую получил Миндинг, т. е. теорему косинусов пространства Лобачевского. Отметим, что здесь всюду будет идти речь о поверхностях постоянной кривизны.

2 А. Эйнштейн и А. Зоммерфельд

В своей классической работе [1] А. Эйнштейн вывел формулу сложения произвольно направленных скоростей в новой теории, которую с точностью до минимального изменения

обозначений с целью их приближения к современным, воспроизводим ниже

$$U = \frac{\sqrt{(v^2 + w^2 + 2vw\cos\alpha) - (vw\sin\alpha/c)^2}}{1 + vw\cos\alpha/c^2} \tag{3}$$

Здесь v – модуль скорости частицы относительно некоторой произвольной системы отсчета, а w – модуль скорости этой системы отсчета относительно неподвижного наблюдателя, α – угол между направлениями данных скоростей, U – модуль скорости частицы относительно неподвижного наблюдателя. Обратим внимание читателя, что формула (3) получена А. Эйнштейном в скалярной форме. Векторная форма выражения (3) отсутствует. Знаменитый немецкий физик-теоретик А. Зоммерфельд вскоре после выхода работы А. Эйнштейна, в 1909 году дал интерпретацию формулы сложения релятивистских скоростей в терминах сферы мнимого радиуса. Если не вникать в некоторые детали работ А. Зоммерфельда, то смысл им сделанного можно резюмировать так, введем следующие замены в формуле (3):

$$U = c\tanh\frac{r_1}{c},\ v = c\tanh\frac{r_2}{c},\ w = c\tanh\frac{r_2}{c},\ \vartheta_{12} = \alpha; \tag{4}$$

в результате получим формулу (2), в которой роль радиуса кривизны ρ пространства играет скорость света c. Таким образом, выясняется, что в пространстве скоростей специальной теории относительности (в пространстве релятивистских скоростей) реализуется геометрия Лобачевского, а формулы (2), (3) определяют метрику в данном пространстве. Можно сказать, используя понятие мнимой сферы, в идеологическом смысле А. Зоммерфельд следовал линии Ф. Миндинга, а не Лобачевского. Однако, вскоре после выхода работы А. Зоммерфельда, югославский ученый В. Варичак отметил, именно в связи с работой А. Зоммерфельда, что сфера мнимого радиуса несет в себе геометрию Лобачевского и тем самым обратил внимание на связь теории относительности А. Эйнштейна и геометрии Лобачевского. В дальнейшем А. Зоммерфельд еще несколько раз обращался к формуле (3), используя ее геометрическую интерпретацию. Так в 1931 году на конференции по ядерной физике в Риме на основе своего геометрического подхода к формуле сложения релятивистских скоростей (3) он дал вывод явления томасовской прецессии, релятивистского эффекта, состоящего в том, что для частицы движущейся, например по окружности с релятивистской скоростью (со скоростью, когда необходимо учитывать новый закон сложения скоростей (3)), вектор ее скорости получает дополнительный доворот и соответственно необходимо это учитывать в частоте обращения точки по орбите. Эффект использовался для объяснения несоответствия экспериментальных данных по аномальному эффекту Зеемана с теоретическими расчетами. А. Зоммерфельд получил недостающий множитель 1/2 в формуле спектра энергий атома в магнитном поле, используя свой геометрический подход. Вывод данной поправки на основе геометрического подхода дан в приложении к первому тому знаменитой книги А. Зоммерфельда [2]. Мы не будем здесь останавливаться более подробно на описании вывода эффекта по методу А. Зоммерфельда. Анализу работ А. Зоммерфельда по применению геометрии пространств постоянной кривизны к проблемам специальной теории относительности посвящена работа [3], в которой справедливо указывается, что открытие П.А.М. Дираком своего знаменитого уравнения решило проблему с объяснением особенности аномального эффекта Зеемана на фундаментальном уровне и более не требует привлечения томасовой прецессии. Тем не менее, как правильно отметили авторы: "Вывод поправки Томаса и сейчас вызывает трудности у педагогов, особенно если речь идет о вводном курсе квантовой механики что подтверждается в обстоятельном обзоре [4] проблемы. Возникновение эффекта прецессии Томаса выглядит достаточно естественным образом, с точки зрения дифференциальной геометрии, когда изменение вектора релятивистской скорости рассматривается как его перенос в неевклидовом пространстве релятивистских

скоростей [5]. Геометрическую интерпретацию произвольно направленных релятивистских скоростей А. Зоммерфельд использует также в "Электродинамике" своего знаменитого курса [6]. При этом вероятно, будучи неуверенным в познаниях своих читателей в гиперболической (Лобачевского) геометрии, он использует формулу (1), а не (2) указывая правда, что аргументы в тригонометрических функциях от длин сторон треугольника нужно брать мнимыми. Отметим, геометрический подход А. Зоммерфельда к проблеме сложения скоростей в специальной теории относительности в известной мере компенсирует отсутствие векторной формы эйнштейновского выражения (3) в том смысле, что геометрический подход обеспечивает инвариантность описания. Однако проблема векторной формы формулировки закона сложения релятивистских скоростей остается.

3 В.К. Клиффорд и А.П. Котельников

Чтобы понять смысл и значение вышесказанного, приведем кратко некоторые сведения из истории формирования векторного исчисления.

Можно смело сказать, что формирование этого раздела математики в основном проходило в физике, в связи с проблемами механики и электродинамики. В частности, голландец С. Стевин в 16 веке занимаясь проблемой равновесия тела на наклонной плоскости, сформулировал теорему о треугольнике сил и предложил способ изображения сил с помощью линий [7, 8]. И. Ньютон в своих "Математических началах натуральной философии" рассматривает правило параллелограмма сил и скоростей как следствие сформулированных им законов механики [9]. Однако, возможность последовательной алгебраической формулировки теории векторов возникла только с созданием алгебры кватернионов. С интересующим нас направлением развития векторного исчисления, историю в определенном смысле можно считать завершенной только сравнительно недавно, в первой половине 20-го века, а именно в 1918 г. с созданием Г. Вейлем аксиоматики аффинного и евклидового точечного пространства. В частности, авторское изложение аксиоматики приведено в монографии Вейля [10].

Как уже имел возможность почувствовать читатель, речь в двух предыдущих параграфах не идет о координатном пространстве материальных частиц – множестве радиус-векторов, частиц для которых координаты частиц являются компонентами. Речь идет о пространстве скоростей частиц. В случае классической механики Ньютона пространство скоростей, как и пространство координатное является пространством Евклида, это означает, что имеет место закон сложения векторов

$$\vec{U} = \vec{v} + \vec{w} \tag{5}$$

смысл букв в котором тот же, что и в формуле (3). Возводя в квадрат выражение (5), т.е. умножив скалярно на себя и извлекая корень квадратный из его левой и правой частей для определения модуля U, мы получим формулу для теоремы косинусов евклидового пространства скоростей дорелятивистской классической физики

$$U = \sqrt{v^2 + w^2 + 2vw\cos\alpha} \quad , \tag{6}$$

где α по-прежнему угол между векторами $\vec{v}$ и $\vec{w}$. Легко видеть, что формула (6) получается из формулы (3), когда v иw по модулю много меньше скорости света c , т.е когда последними слагаемыми в числителе и знаменателе формулы (3) можно пренебречь. Такое приближение с точки зрения физики называется нерелятивистским (приближением механики Ньютона), а с точки зрения геометрии пространства скоростей - плоским (евклидовым) пределом.

Последнее означает, что в малой окрестности точки на сфере или в пространстве Лобачевского реализуется евклидова геометрия -геометрия плоского пространства. Формулы (5), (6) называют правилом параллелограмма или правилом треугольника.

После создания своей геометрии Н.И. Лобачевский в заключении к работе “О началах геометрии” [11] писал: – "Оставалось бы исследовать, какого рода перемена произойдет от введения воображаемой Геометрии в Механику и не встретится ли здесь принятых уже и несостоятельных понятий о природе вещей, но которые принудят нас ограничить и или совсем не допускать зависимость линий от углов. Однако можно предвидеть, что перемены в механике при новых началах геометрии будут такого ж рода, какие показал Г. Лаплас (Mécanique céleste. I Liv.I Ch.II), предполагая всякую зависимость скорости от силы, или – выразимся вернее – предполагая силы, измеряемые всегда скоростью, подчиненными другому закону, нежели принятому сложению их”. Гениальное предвидение теории типа специальной теории относительности с законом сложения векторов скоростей отличным от принятого в механике Ньютона.

Приведенные слова великого геометра служили побудительным мотивом для многих ученых к их попыткам построения механики в пространстве Лобачевского. Примеры таких попыток можно найти в книге [12], но нас будут интересовать работы русского ученого А.П. Котельникова. Тематика векторного исчисления в работах А.П. Котельникова возникла со времени подготовки и защиты им магистерской диссертации на тему "Винтовое счисление и некоторые его применения в геометрии и механике". Алгебраической основой исследований послужило кватернионное исчисление. В.Р. Гамильтон был одним из открывателей кватернионов. Он в отличие от других авторов открытия (К.Ф. Гаусса, Б.О. Родрига) активно развивал кватернионное исчисление и искал области его приложений. Гамильтон ввел в математику также кватернионы над комплексными числами, названные им бикватернионами. В.К. Клиффорд, вслед за Гамильтоном, обобщил понятие чисел и ввел такие алгебраические объекты как бикватернионы, определенные над двойными и дуальными числами (эллиптические и параболические комплексные числа соответственно по терминологии Клиффорда) и показал связь эллиптических бикватернионов с римановой геометрией, а параболических с геометрией Евклида. Считается, что знаменитой работой Клиффорда 1873 года "Предварительный очерк бикватернионов" положил начало изучению механики в пространстве Римана. Спустя 10 лет подход Клиффорда был перенесен Г. Коксом на пространство Лобачевского с помощью бикватернионов, определенных над комплексными числами, которые по терминологии Клиффорда назывались гиперболическими. Алгебраическая сущность векторов содержится в кватернионом исчислении. Недаром В.Р. Гамильтон представлял кватернион как линейную комбинацию трехмерного вектора и скаляра. При этом вектор у Гамильтона представлял то, что сегодня мы и понимаем под векторами трехмерного евклидового пространства. Поскольку для кватернионов определена операция деления и в результате ее получался кватернион общего вида (вектор+скаляр), даже при делении трехмерных векторов как части кватернионов, такие конструкции были названы винтами, т.к. через них выражались повороты от одного из векторов к другому. В построении алгебраических объектов такого типа состояла основная идея построения исчисления, которое для трехмерного пространства обеспечивало бы то, что обеспечивают комплексные числа на плоскости (двумерном евклидовом пространстве) при их геометрической интерпретации. Т.е. введенные алгебраические объекты кватернионы могли выступать как преобразуемые величины, так и величины задающие преобразования. Однако теперь возникала проблема построения геометрической теории векторов пространств постоянной кривизны. Т.е. необходимо было определить геометрические образы векторов в такого рода пространствах и операции именно над данными геометрическими объектами подобно тому как это следовало из механики Ньютона для трехмерного евклидового пространства.

Этим и занялся А.П. Котельников, который построил свою теорию проективных векторов, опубликованную в Казани 1899 г. и защищенную им в качестве докторской диссертации [13,14].

Не вникая в детали работы, отметим, что введенный им закон сложения векторов, определенных как упорядоченные пары точек (направленные отрезки геодезических) давал для теоремы синусов выражение в случае двумерной сферы (простейшее пространство Римана)

$$\frac{\tan\frac{r_1}{R}}{\sin\vartheta_{23}} = \frac{\tan\frac{r_2}{R}}{\sin\vartheta_{13}} = \frac{\tan\frac{r_3}{R}}{\sin\vartheta_{12}} \tag{7}$$

в то время как теорема синусов согласно той же теореме косинусов (1) для сферы должна иметь вид

$$\frac{\sin\frac{r_1}{R}}{\sin\vartheta_{23}} = \frac{\sin\frac{r_2}{R}}{\sin\vartheta_{13}} = \frac{\sin\frac{r_3}{R}}{\sin\vartheta_{12}}. \tag{8}$$

Формула (7) справедлива для треугольника (параллелограмма) построенного в плоскости касательной к сфере в точке, являющейся общим началам векторов, отнесенных к ней вдоль прямых на которых они лежат. С такой процедурой А.П. Котельников связал определение своего правила сложения векторов – правила параллелограмма. Однако почему он не воспользовался известными из сферической астрономии формулами, остается загадкой.

4 Ф.И. Федоров и другие

Возможна векторная форма сложения векторов, соответствующих упорядоченным парам точек (направленным отрезка геодезических) пространств постоянной кривизны. Реализована она была в последовательной теории векторов в пространствах постоянной кривизны, построенной на основе установленной связи между векторной параметризацией преобразований групп движений данных пространств с бикватернионами в трудах белорусских ученых. Итогом усилий, осуществляемых на протяжении десятилетий явилась аксиоматическая схема построения проективных трехмерных пространств как частный случай решающая проблему сложения векторов в пространствах постоянной кривизны [15, 16].

Основными математическими объектами аксиоматики явились трехмерные векторы для которых определены известные со школы векторное и скалярное произведения. Правда векторы, описывающие геометрию эллиптического пространства (или сферы) определены над двойными числами, а векторы гиперболического пространства, расширенного пространства Лобачевского определены над комплексными числами как у Клиффорда и Кокса соответственно.

Аксиомы:

1. Трехмерные векторы над двойными числами

$$\vec{q} = \vec{a} + i\vec{b}, \quad i^2 = 1, \ (\vec{q}^{\,2} = \vec{q}^{\,2*}), \quad (\vec{q}^{\,*} = \vec{a} - i\vec{b}), \tag{9}$$

ставятся в соответствие упорядоченным точкам (или прямым линям) в трехмерном эллиптическом пространстве.

1. Трехмерные векторы над комплексными числами

$$\vec{q} = \vec{a} + i\vec{b}, \quad i^2 = -1, \ \ (\vec{q}^{\,2} = \vec{q}^{\,2*}), \ (\vec{q}^{\,*} = \vec{a} - i\vec{b}), \tag{10}$$

ставятся в соответствие упорядоченным точкам (или прямым линям) в трехмерном гиперболическом (или расширенном Лобачевского) пространстве. Символ $_*$ обозначает здесь

сопряжение в системе двойных чисел в первом случае и комплексных во втором случае. Допустимые области изменения компонентов векторов $\vec{q}$ определяются выражениями:

a) для случая эллиптического пространства

$$\vec{q}^{\,2} \geq 0, \tag{9}$$

b) для случая расширенного пространства Лобачевского :

$$1. \quad \vec{q}^{\,2} \leq 0, \quad 2.\vec{q}^{\,2} \geq 0, \quad \vec{q}^{\,2} = 0, \ (\vec{a} = \vec{b} \neq 0). \tag{12}$$

Расстояние r между двумя точками на прямой, которым ставится в соответствие вектор$\vec{q}$ в каждом из пространств определяется выражениями:

a) для эллиптического пространства

$$\sqrt{\vec{q}^2} = \sqrt{q_1^2 + q_2^2 + q_3^2} = \tan \frac{r}{R} \quad , \tag{10}$$

b) для расширенного пространства Лобачевского

$$\sqrt{\vec{q}^2 =} i \tanh \frac{r}{R} (-1 \leq q^2 \leq 0), \quad \sqrt{\vec{q}^2 =} i \coth \frac{r}{R}, (q^2 \leq -1). \tag{14}$$

$$2. \quad \sqrt{\vec{q}^2} = \tan \frac{r}{R}, (\vec{q}^2 \geq 0) \quad 3. \quad r = 0, \quad (q^2 = 0, \vec{a} = \vec{b} \neq 0),$$

где R – радиус кривизны пространства.

III Векторы (прямые соответствующие им) назовем пересекающимися, когда для них имеет место условие

$$(\vec{q}_1 \vec{q}_2)^* = (\vec{q}_1 \vec{q}_2). \tag{11}$$

В качестве единой формулы, определяющей выражение для закона сложения пересекающихся векторов трехмерного эллиптического и расширенного пространства Лобачевского формула, совпадающая по форме с законом композиции векторов параметров Ф.И. Федорова,

$$\vec{q} = \langle \vec{q}_1, \vec{q}_2 \rangle \, \frac{\vec{q}_1 + \vec{q}_2 + [\vec{q}_1 \vec{q}_2]}{1 - (\vec{q}_1 \vec{q}_2)}. \tag{12}$$

однако существенно отличающуюся по содержанию от последней. У Ф.И. Федорова формула, совпадающая по форме с формулой (12) имеет групповой смысл. Некоторые естественные ограничения на комплексные вектор-параметры преобразований группы Лоренца и на вектор-параметры преобразований других групп у Ф.И. Федорова не противоречат групповым свойствам формулы (12) в то время как условия (9), (10) и (11) нарушают групповые свойства формулы (12). Однако, формула (12) с упомянутыми условиями отражает тот факт, что вообще говоря, пространства постоянной кривизны не являются группами, а являются аналитическими лупами Муфанг. Возводя в квадрат обе стороны формулы (12) и учитывая определения векторов в аксиомах и условие (11) можно получить теорему косинусов (1). Для трехмерного пространства постоянной положительной кривизны в случае (10), и теорему косинусов расширенного пространства Лобачевского в случае (14) и в частном случае формулу (2). Следует отметить, что теорема косинусов расширенного пространства Лобачевского может содержать косинусы мнимых углов и мнимые расстояния. В этом нет ничего удивительного. Скажем только, что развитый поход с физической точки зрения учитывает существование гипотетических сверхсветовых объектов (тахионов), с геометрической то, что существующие в реальном пространстве расходящиеся прямые с проективной точки зрения можно рассматривать как прямые пересекающиеся в идеальной точке (за абсолютом, за световым конусом).

5 Релятивистская кинематика

Как уже отмечалось, переход от нерелятивистской кинематики к релятивистской можно рассматривать как переход от евклидовой геометрии пространства нерелятивистских скоростей и геометрии трехмерного пространства Лобачевского релятивистских скоростей.

Формула (12) в этом случае есть формула сложения относительных скоростей. Непосредственно через стандартно определяемые скорости комплексные векторы типа (12) определяются согласно той же формуле сложения (12) например, так

$$\vec{q}_{12} = \frac{i\vec{\mathrm{v}}_{01} - i\vec{\mathrm{v}}_{02} + [\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02}]}{1 - (\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02})}. \tag{13}$$

где $\vec{\mathrm{v}} = \vec{u}/_{u_0}$, $\quad u = iu_0 + \vec{u}$- четырехмерная скорость. Аналогичная связь трехмерной относительной скорости с четырехмерным импульсом. Четырехмерные скорости и четырехмерные импульсы представляются бикватернионами, что и обеспечивает связь трехмерного и четырехмерного способов описаний независящих от выбора конкретных систем отсчета [17, 18]. В данной формуле принято, что скорость света $= 1$.

В то время как $\vec{q}_1$,$\vec{q}_2$,$\vec{q}_3$ соответствуют парам произвольных точек пространств постоянной кривизны, чисто мнимые векторы следует рассматривать как векторы соответствующие парам точек одна из которых соответствует покоящейся точке (покоящейся системе отсчета). Комплексность означает, что скорость определяется относительно системы отсчета, связанной не с покоящейся, а с движущейся точкой (частицей). Квадрат $\vec{q}$ (13) величина вещественная и равная квадрату соответствующей относительной скорости с противоположным знаком.

Комплексность введенных относительных скоростей объясняет эффект прецесии Томаса.

$$\vec{q}_{12} = \vec{a} + i\vec{b} = \langle i\vec{\mathrm{v}}_{01}, -i\vec{\mathrm{v}}_{02}\rangle = \frac{i\vec{\mathrm{v}}_{02} - i\vec{\mathrm{v}}_{02} + \vec{\mathrm{v}}_{01} \times \vec{\mathrm{v}}_{02}}{1 - (\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02})}, \tag{14}$$

где очевидно,

$$\vec{a} = \frac{\vec{\mathrm{v}}_{01} \times \vec{\mathrm{v}}_{02}}{1 - (\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02})}, \quad \vec{b} = \frac{\vec{\mathrm{v}}_{01} - \vec{\mathrm{v}}_{02}}{1 - (\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02})}, \tag{15}$$

причем $(\vec{a}\vec{b}) = 0$. Данный вектор-параметр комплексный. Его можно представить в виде композиции,

$$\vec{q} = \langle \vec{c}, i\vec{u}\rangle = \vec{c} + i\vec{u} + i(\vec{c} \times \vec{u}). \tag{16}$$

Здесь учтено, что, как это следует из (18), $(\vec{q}^2)^* = (\vec{q}^2)$, поэтому $(\vec{c}\vec{u}) = 0$, где $\vec{c}$-вектор определяющий поворот вектора $\vec{u}$ по направлению совпадающего со скоростью, которая получается в результате нерелятивистского правила сложения. Действительно, сравнивая (18) с (16) получаем

$$\vec{c} = \vec{a} = \frac{\vec{\mathrm{v}}_{01} \times \vec{\mathrm{v}}_{02}}{1 - (\vec{\mathrm{v}}_{01}\vec{\mathrm{v}}_{02})}, \quad \vec{u} = \frac{1 - \vec{a}^{\times}}{1 + \vec{a}^2}\vec{b}, \tag{17}$$

где $\vec{a}^{\times}$, оператор векторного произведения.

Таким образом, как видно из (18)-(17) вектор $\vec{a}$ характеризует дополнительный поворот вектора $\vec{b}$, совпадающего по направлению с вектором полученным сложением по нерелятивистским правилам с вектором полученным в результате сложения по правилам релятивистской кинематики. Именно наличие данного поворота и приводит к томасовской прецессии.

В заключение отметим, что мы сознательно не касались здесь темы применения геометрии Лобачевского для решения задач кинематики процессов столкновений и распадов частиц в целом. В разработку данного подхода большой вклад внесли Н.А. Черников и

Я.А. Смородинский (см. напр. [19, 20], а также [18]). Мы ограничились проблемой закона сложения произвольно направленных скоростей, в решение которой белорусские физики внесли, как нам кажется определяющий вклад.

Литература

1. А. Эйнштейн К электродинамике движущихся тел // Собрание сочинений – М.: Наука, 1965, Т. 1, С. 7–35.

2. А. Зоммерфельд. Строения атома и спектры. Т. 1, М.; ГИТТЛ 1956, 579 с.

3. Л. Беллони, Ч. Рейна Процессия Томаса: подход Зоммерфельда. Эйнштейновский сборник 1984–1985. М.: Наука 1988. С. 201–214.

4. Г.Б. Малыкин. Прецессия Томаса: корректные и некорректные решения. УФН (2006) Т. 176, № 8, С. 865–882.

5. B.M. Barbasov, A.B. Pestov Fermi–Walker transport and the Weyl connection Teoret. Mat. Fiz. 1999, V. 119, N 1, P. 136–141.

6. Зоммерфельд А. Электродинамика М.: ИЛ (1958) 501 с.

7. Розенбергер Ф. История физики. Т.1 М.-Л.: ОНТИ ГТТИ (1934), 148 с.

8. Розенфельд Б.А. История неевклидовой геометрии. М.: Наука, 1976. – 412 с.

9. Ньютон И. Математические начала натуральной философии. М.: Наука, 1989.

10. Вейль Г.Пространство, время, материя. М.: УРСС. 2004. 456 с.

11. Лобачевский, Н.И. О началах геометрии / Н.И. Лобачевский // Об основаниях геометрии. Сборник классических работ по геометрии Лобачевского и развитию ее идей. – М.: ГИТТЛ, 1956. – С. 27–49.

12. Борисов А.В., Мамаев И. С. Классическая динамика в неевклидовых пространствах. Москва-Ижевск, 2004. 348 с.

13. ПутятаТ.В., Б.Н., Лаптев Б.Л., Розенфельд Б.А., Фрадлин Б.Н./Александр Петрович Котельников. М.: Наука 1968. 120 с.

14. Теория векторов и комплексные числа (Начала механики в неевклидовом пространстве). В кН. А.П. Котельников, В.А. Фок Некоторое применение идей Лобачевского в механике и физике. М.-Л. 1950. С. 7–47.

15. Ю.А. Курочкин, А.А.Богуш, Вектор-параметры Ф.И.Федорова и аксиоматическое о писание геометрии пространств постоянной кривизны $S_3, {}^1S_3$. Весці АНБ, Сер. фіз.-мат. навук. № 4, 1995. С. 70–75.

16. Богуш, А.А., Ю.А. Курочкин, Кинематические модели трехмерных пространств постоянной кривизны // Гравитация и электромагнетизм. Мн. Университетское. 1998, С. 20–26.

17. Богуш, А.А. Векторы пространства Лобачевского и релятивистская кинематика / А.А. Богуш, Ю.А. Курочкин, Ф.И. Федоров // ДАН СССР. – Т. 236, № 1. – 1977. – С. 58–60.

18. Березин, А.В. Кватернионы в релятивистской физике / А.В. Березин, Ю.А. Курочкин, Е.А. Толкачев – Минск: Наука и техника, 1989. – 197 с.

19. Черников Н.А. Геометрии Лобачевского и релятивистская кинематика / ЭЧАЯ 1973. Т.4, вып.3, С. 773–810.

20. Смородинский Я.А. Геометрия Лобачевского и кинематика Эйнштейна // Эйнштейновский сборник 1971. М.: Наука 1972. С. 272–301.

Нижегородский математик Артемий Григорьевич Майер

Г.М. Полотовский
polotovsky@gmail.com; Нижегородский госуниверситет им. Н.И. Лобачевского, Россия

Аннотация

Очерк о жизни и научно-педагогической деятельности нижегородского математика профессора Артемия Григорьевича Майера.

Ключевые слова: качественная теория дифференциальных уравнений, профессор Артемий Григорьевич Майер

Артемий Григорьевич Майер был математиком высокой культуры, который наряду с чисто-математическими работами не только мог, но и любил заниматься прикладными задачами. Математические работы А.Г. Майера в настоящее время являются классическими.

Е.А.Леонтович-Андронова

Имя и научные результаты замечательного нижегородского математика Артемия Григорьевича Майера хорошо известны специалистам по качественной теории дифференциальных уравнений. Однако, насколько мне известно, до самого последнего времени не было ни одной сколько-нибудь обстоятельной публикации о его жизни и научно-педагогической деятельности.

Артемий Григорьевич Майер
(6 сентября 1905 – 20 сентября 1951)

1 Научные результаты и публикации

Список опубликованных работ А.Г. Майера по нынешним меркам весьма короткий – он насчитывает 25 позиций,[1] причём в известных библиографических справочниках [1] (стр. 435–436) и [2] (стр. 826) опущены 8 следующих публикаций А.Г. Майера:

М1.*К теории связанных колебаний двух самовозбуждённых генераторов* // Горький, Учён. зап. ГГУ, вып. 2 (1935). С. 3–12.

М2.*Исследование уравнений Рэлея и Ван-дер-Поля* // Изв. ГГУ, вып. 2 (1936).

М3. *К теории вынужденных колебаний в сложном генераторе* // Горький, Учён. зап. ГГУ, вып. 6 (1937). С. 25–36.

М4. *De trajectoires sur les surfaces orientees* // Comptes Rendus (Doklady) de L'Academie Des Sciences de L'URSS. Vol. 24, №1 (1939). P. 673–675.

М5. *О задаче Вышнеградского в теории прямого регулирования* // ДАН СССР, 43 (1945). С. 345–348 (совм. с А.А. Андроновым).

М6. *Задача Вышнеградского в теории прямого регулирования. I* // Автоматика и телемеханика, т. 8, вып. 5 (1947). С. 314–334 (совм. с А.А. Андроновым).

М7. *Задача Вышнеградского в теории прямого регулирования. II. Теория регулятора прямого действия при наличии кулоновского и вязкого трения* (*продолжение*) // Автоматика и телемеханика, т. 14, вып. 5 (1953). С. 505–530 (совм. с А.А. Андроновым).

М8. *Теория «ударной машины»*// Горький, Учён. зап. ГГУ, вып. 27 (1954). С. 3–22 (совм. с А.А. Андроновым).

Попытаюсь кратко охарактеризовать математическое наследие А.Г. Майера, опираясь на имеющуюся у меня небольшую (менее двух машинописных страниц) неопубликованную заметку Е.А. Леонтович-Андроновой[2] «О работах Артемия Григорьевича Майера», ссылки на которую помечены ниже инициалами «Е.Л.». Из этой же заметки взята и фраза, приведённая выше в эпиграфе.

Представляется естественным разделить все результаты А.Г. Майера на следующие пять групп.

I. *«Это, во-первых – его работа, касающаяся динамических систем на двумерных поверхностях»* [Е.Л.]. Этот круг результатов представляют три статьи (ДАН СССР, 14 (1937), с. 251–254 (совм. с Е.А. Леонтович); М4 из списка выше; Матем. сб. 12(54) (1943), с. 71–84) и кандидатская диссертация Майера «О траекториях на ориентируемых поверхностях», защищённая им в Институте Математики при МГУ в 1939 г., содержанием которой *«является установление возможного характера траекторий на поверхностях. Для систем на двумерных поверхностях им создана теория, аналогичная классической теории Пуанкаре-Бендиксона в случае плоскости* (*или сферы*) *<:> Эта работа Артемия Григорьевича является основой почти любого изучения динамических систем на двумерных поверхностях»* [Е.Л.].

II. Статья «Грубое преобразование окружности в окружность» (Учён. зап. ГГУ, вып. 12, с. 215–229 – Горький, 1939), в которой *«установлены некоторые основные факты – например, необходимые и достаточные условия грубости диффеоморфизмов окружности, непрерывная зависимость числа вращения от параметра и указано, что функция, дающая зависимость числа вращения от параметра, есть, вообще говоря, функция Кантора»* [Е.Л.].

[1]Трудно удержаться от замечания, что с такой «публикационной активностью» – за без малого 20 лет работы «всего» 25 публикаций, да ещё не входящих в Web of Science или Scopus, – А.Г. Майер не мог бы в наши дни претендовать на руководство проектом по гранту РНФ.

[2]Евгения Александровна Леонтович-Андронова (1905–1997) – профессор Нижегородского университета, жена академика А.А. Андронова, сотрудничала с А.Г. Майером на протяжении всего периода его работы в Нижнем Новгороде. О жизни и деятельности Е.А. Леонтович-Андроновой см. [3] – [5].

III. *«Наиболее сильной математической работой Артемия Григорьевича является его работа "О центральных траекториях в проблеме Биркгофа"»* [Е.Л.]. Здесь речь идёт о пяти статьях (ДАН СССР, 55 (1947), с. 477–480; ДАН СССР, 55 (1947), с. 583–586; УМН, том 2, вып.2(18) (1947), с. 193–194; ДАН СССР, 59 (1948), с. 1393–1396; Матем. сб., 26 (1950), с. 265–290), на которых основана докторская диссертация А.Г. Майера «О центральных траекториях и проблеме Биркгофа[3]», защищённая им в 1947 г.

IV. Работы об инварианте, названном «схема динамической системы». После двух совместных с Е.А. Леонтович статей (ДАН СССР, 14 (1937), с. 251–254, и ДАН СССР, 103 (1955), с. 557–560) построение схемы динамической системы, а также многие другие вопросы качественной теории динамических систем на плоскости, были подробно изложены в двухтомной монографии [6], [7], которая до настоящего времени остаётся настольной книгой специалистов по качественной теории и студентов, изучающих этот предмет. В предисловии к [6] написано: «Настоящая книга была начата в 1949 году А.А. Андроновым совместно с Е.А. Леонтович и А.Г. Майером и после смерти А.А. Андронова (в 1952 г.) и А.Г. Майера (в 1951 г.) дописана Е.А. Леонтович и И.И. Гордоном[4]. Окончательный вариант принадлежит Е.А. Леонтович».

V. Работы, посвящённые прикладным задачам, о которых Е.А. Леонтович написала: *«Трудно указать математика, который при решении конкретных задач проявлял такую быстроту ориентировки, способность использовать всевозможные математические средства и мастерство»* [Е.Л.]. Среди этих работ – самая первая статья А.Г. Майера и Е.А. Леонтович «Об одном неравенстве, связанном с интегралом Фурье» (ДАН СССР, 4 (1934), с. 353–360), о которой Л.П. Шильников[5] писал в [3]: *«Она относится к теории линейных систем и содержит ответ на вопрос Л.И. Мандельштама о соотношении между продолжительностью радиоимпульса и размытостью. По существу, Е.А. Леонтович и А.Г. Майер в этой заметке дали точную формулировку классического аналога квантового соотношения неопределённости, имеющего фундаментальное значение для теории связи»*.

О другой работе из этой группы – «Задача Мизеса в теории прямого регулирования и теория точечных преобразований поверхностей» (ДАН СССР, 43 (1944), с. 58–60 (совм. с А.А. Андроновым)) во введении к книге [12] написано: «Трёхмерная нелинейная задача, получающаяся при учёте сухого трения в муфте, оставалась нерешённой из-за своей большой математической трудности, несмотря на то, что ею занимались Н.Е. Жуковский, Стодола, Грдина, Мизес и многие другие исследователи. Эта задача была решена А.А. Андроновым и А.Г. Майером».

В правоте Е.А. Леонтович, определившей работы А.Г. Майера как классические (см. эпиграф), сомневаться не приходится: основные идеи и результаты Майера получили развитие в работах математиков последующих поколений. Так, динамические системы на поверхностях интенсивно изучались и изучаются до сих пор, в том числе в Нижнем Новгоро-

[3]Дж. Биркгоф ввёл инвариант динамической системы, называемый «порядковым числом центральных траекторий», и, в частности, поставил вопрос о существовании трёхмерной динамической системы, у которой этот инвариант больше трёх. А.Г. Майер показал, что для любого заданного трансфинита 2-го класса можно построить трёхмерную систему с порядковым числом множества центральных траекторий, превышающим этот трансфинит.

[4]Израиль Исаакович Гордон (1910–1985) – выпускник Ленинградского университета, первый аспирант Л.С. Понтрягина. В 1935 г. в своей диссертации (см. также [8]) ввёл кольцо когомологий. Таким образом, когомологии были независимо и одновременно введены А.Н. Колмогоровым, Дж. Александером и И.И. Гордоном, причём все они сделали на эту тему доклады на международной топологической конференции 1936 года в Москве. С 1944 г. И.И. Гордон – доцент Горьковского университета. Подробнее об И.И. Гордоне см. [9].

[5]Леонид Павлович Шильников (1938–2011) – профессор Нижегородского университета, один из крупнейших специалистов по теории бифуркаций многомерных динамических систем. О жизни и научных результатах Л.П. Шильникова см. [10], [11].

де. В частности, результаты Майера, касающиеся асимптотического поведения траекторий динамических систем на поверхностях, и идеи А. Вейля и Д.В. Аносова позволили С.Х. Арансону и В.З. Гринесу получить топологическую классификацию транзитивных потоков на ориентируемых поверхностях, подробности см. в [13].

О роли преобразований окружности в изучении динамических систем говорить излишне. Отмечу только, что некоторые результаты из упомянутой в п.II статьи 1939 года были переоткрыты В.И. Арнольдом [14] и В.А. Плиссом [15]. Исследования Майера о проблеме Биркгофа, показавшие, что трёхмерные системы могут быть чрезвычайно сложными, тоже имели продолжение: один из примеров Майера был усилен Л.П. Шильниковым в 1969 г. [16], а в 2000 г. результаты Майера были передоказаны другим методом в диссертации С.А. Шаповалова (МГУ). Наконец, продолжалась и тематика, связанная со схемой динамической системы. Так, в работах С.Ю. Пилюгина [17] и Я.Л. Уманского [18] подобные конструкции введены для определённых классов динамических систем размерности > 3, а одним из истоков работ Л.М. Лермана и Я.Л. Уманского по четырёхмерным гамильтоновым интегрируемым системам (см., например, [19]) также является геометрический подход Майера.

2 Родословная и биография А.Г. Майера

Фрагмент генеалогического дерева «наших»[6] Майеров приведён на рис.1. Из него видно, что Артемий Григорьевич является прямым потомком Николая Васильевича Майера (1806–1846). Как следует из многочисленных воспоминаний[7], Н.В. Майер был человеком очень ярким[8]. В 1827 г. он окончил в Санкт-Петербурге Медико-хирургическую академию и стал врачом. С 1830 г. он служил в Ставрополе, где в 1837 г. познакомился с М.Ю. Лермонтовым, которому послужил прототипом для образа доктора Вернера из «Княжны Мери»[9].

Вот словесный портрет доктора Вернера из «Княжны Мери»: *«Вернер человек замечательный по многим причинам. Он скептик и материалист, как почти все медики <:> Его наружность была из тех, которые с первого взгляда поражают неприятно, но которые нравятся впоследствии, когда глаз выучится читать в неправильных чертах отпечаток души испытанной и высокой <:> Вернер был мал ростом, и худ, и слаб, как ребёнок; одна нога была у него короче другой, как у Байрона; в сравнении с туловищем голова его казалась огромна: он стриг волосы под гребёнку, и неровности его черепа, обнаруженные таким образом, поразили бы френолога странным сплетением противоположных наклонностей. Его маленькие чёрные глаза, всегда беспокойные, старались проникнуть в ваши*

[6] Фамилия Майер довольно распространена на территории Германии и России и происходит, скорее всего, от немецкого слова «meier», которое означает «фермер», либо «владелец молочной фермы».

[7] Г.И. Филипсона (1809–1883) (русский генерал, сенатор, участник Кавказской войны), Н.П. Огарёва (1813–1877) (поэт, ближайший друг А.И. Герцена), Н.М. Сатина (1814–1873) (поэт-переводчик) и других – см. очень интересную статью [20].

[8] Среди его потомков помимо А.Г. Майера много других неординарных личностей; за недостатком места – кратко только о Евгении Мальм-Майере, племяннике А.Г. Майера. Отец Евгения был репрессирован. С началом войны семья из Севастополя эвакуировалась в Элисту, но всё же оказалась под немецкой оккупацией. Затем через лагерь для перемещённых лиц в Австрии семья попала в Аргентину, где Евгений окончил консерваторию, был организатором и руководителем русских хоров «Ивушка» и «Знаменный». В 1976 г. он участвовал в постановке «Бориса Годунова» в оперном театре Буэнос-Айреса, позже был стипендиатом Института им. Гнесиных в Москве. Вернувшись на короткий срок в Аргентину, затем он переехал с женой в Барселону, где живёт уже более 30 лет. Е. Мальм-Майер работал с хором Каталонской православной церкви, в настоящее время он один из руководителей камерного хора «Арс-Аниме» (сайт этого хора в Интернете http://www.arsanimae.org).

[9] По утверждениям литературоведов, доктор Вернер – единственное действующее лицо романа «Герой нашего времени», чей прототип не вызывает сомнений.

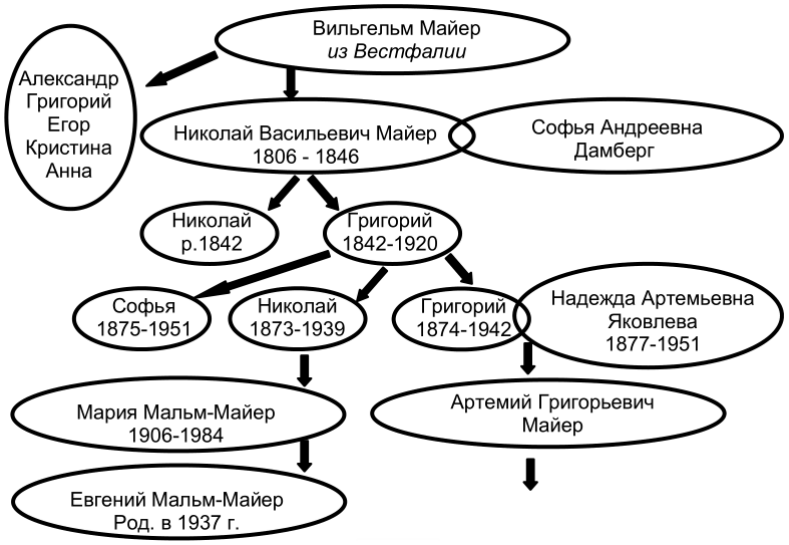


Рис. 1: Фрагмент генеалогического дерева Майеров.

мысли. В его одежде заметны были вкус и опрятность; его худощавые, жилистые и маленькие руки красовались в светло-жёлтых перчатках. Его сюртук, галстук и жилет были постоянно чёрного цвета. Молодёжь прозвала его Мефистофелем: Мы друг друга скоро поняли и сделались приятелями, потому что я к дружбе неспособен".

Николай Васильевич Майер
(1806 – 1846)

У Артемия Григорьевича хранился приведённый здесь автопортрет Н.В. Майера, который он предоставил известному лермонтоведу И.Л. Андроникову (1908–1990) по просьбе последнего. Андроников считал, что «кроме глаз Вернера, которым Лермонтов сообщил чёрный цвет, в остальных деталях его наружность совершенно совпадает с автопортретом Майера и впечатлениями очевидцев».

Перейдём к биографии А.Г. Майера. В архиве сохранился рукописный черновик автобиографии, расшифровка[10] которого приведена ниже.

[10]Здесь «расшифровка» – всего лишь общепринятый термин: почерк А.Г. Майера вполне разборчивый.

Автобиография Майера А.Г.

Родился в 1905 *г.* (В г. Дмитриев Курской губернии – Г.П.) *Отец – инженер* (*умер в* 1942 *г.*), *мать – дом. хозяйка* (*ныне живёт в Москве*).

Окончил в 1926 *г. физ.-матем. факультет Моск. Гос. Университета по специальности «теория чисел». В* 1930 *г. закончил аспирантуру в Ин-те Матем. и Мех. при МГУ.* (Руководителем А.Г. Майера в аспирантуре был известный математик А.Я. Хинчин (1894–1959) – Г.П.).

Во время аспирантуры работал:

В 26/27 *г. вёл курс высшей математики на геолого-географическом факультете МГУ.*

(?)–1928 *гг. работал ст. статистиком в Центр. Статист.* [управлении] *СССР.*

В 8/9 *г. преподавал на подготов. отделении Мех.-Маш. Ин-та им. Каган-Шабшая.*

В 29/30 *г. был ст. ассистентом в Моск. Мех.-Маш. Ин-те им. Баумана.*

По окончании аспирантуры был направлен НКПрос'ом в Горьк. Педаг. Ин-т, доцентом по кафедре математики, где и работал в 1930/31 *и* 1931/32 *гг.*

С 1931 *г. работаю ст. научн. сотр. Горьк. Иссл. Физико-Техн. Ин-та, с осени* 1931 *г. – доцентом кафедры математич. анализа Горьк. Гос. Университета; с сент.* 1946 *г. – заведую ею*[11].

Научную работу вёл в основном в области качественной теории дифференц. уравнений и её приложений к теории колебаний и к теории автоматич. регулир. Всего напечатано 10 *работ,* [готовятся] 2.

Диссертацию на степень кандид. ф.-м. наук защитил в 1939 *г., в Ин-те Матем. при МГУ.*

Имею семью: жена, трое детей.

1947 *А. Майер*

О том, как А.Г. Майер попал в группу Андронова в ГИФТИ[12], рассказывала[13] Е.А. Леонтович-Андронова: *«Он* [А.Г. Майер] *был по распределению распределён в Нижний Новгород. Так что когда мы сюда приехали с Александром Александровичем* [Андроновым – Г.П.], *то он уже там был, он работал в пединституте, и Александр Александрович его к себе перетянул и пригрел. В пединституте он чем-то занимался, чем-то очень трудным, по-моему, он занимался теорией чисел. Он говорил, что вот когда он работал по той тематике, которая у него была в Москве, что он как-то доходил до пределов мысли, что это очень тяжело. Так что он с большим удовольствием, по-моему, связался с Александром Александровичем. Сменил тематику, которая для него была очень тяжела».*

Ещё один пункт в автобиографии Майера требует разъяснения: что такое «Мех.-Маш. Ин-т им. Каган-Шабшая», на подготовительном отделении которого А.Г. Майер преподавал в 1928 – 1929 гг.?

В 1920 году на свои личные средства Я.Ф. Каган-Шабшай[14] организовал в Москве ВТУЗ

[11] А.Г. Майер не упомянул, что с апреля 1933 г. по февраль 1934 г. он был деканом физико-математического факультета (Центральный архив Нижегородской области (ЦАНО), ф.377, оп.6, д.9, л.53а) – Г.П.

[12] Горьковский исследовательский физико-технический институт, образован в 1930 г. как научное учреждение республиканского значения, в 1932 г. включен в систему Горьковского госуниверситета.

[13] В 1996 г. Е.И. Гордон (профессор Восточного Иллинойского университета, сын И.И. Гордона; в то время был профессором Нижегородского университета) и я попросили Е.А. Леонтович-Андронову рассказать о Майере; приводится фрагмент из аудиозаписи её рассказа.

[14] Яков Фабианович Каган-Шабшай (1877–1939) был совершенно неординарной личностью. Он получил разностороннее образование и благодаря чрезвычайной энергичности и работоспособности воплотил в жизнь многие свои проекты. Коллекционировал предметы искусства, поддерживал молодых художников, в том числе Марка Шагала. Более подробно о Каган-Шабшае можно прочитать, например, в [21], [22].

для подготовки полноценных инженеров за 2–3 года против обычных в то время 6–7 лет. Позже этот ВТУЗ получил название «Государственный Электромашиностроительный Институт имени Каган-Шабшая» (ГЭМИКШ). Вступительных экзаменов было пять (алгебра, геометрия, тригонометрия – устно, геометрия и алгебра с тригонометрией – письменно). Студент 4 дня в неделю работал на заводе, а 2 дня по 10–12 часов (!) проходил теоретическое обучение. Институт имел шесть курсов и только один месяц каникул, так что в год проходилось по три курса, и через два года студент получал звание инженера, если не проваливался на каком-нибудь экзамене. В случае хотя бы одного провала студент оставался на курсе второй раз. Третий раз оставаться на одном курсе было нельзя. Максимальный срок пребывания студента в институте – три года.

В 1933 году ГЭМИКШ был расформирован. Позднее в его здании (Страстной бульвар, дом 27/16) располагался Московский авиационный технологический институт (МАТИ), а станкоинструментальный факультет ГЭМИКШа был преобразован Каган-Шабшаем в институт, который сейчас называется МГТУ «Станкин».

Вернёмся к автобиографии Майера – к пункту «Имею семью: жена, трое детей». А.Г. Майер женился в 1926 году на Нине Фёдоровне Морошкиной (1901–1971), внучке известного русского юриста, профессора Московского университета Ф.Л. Морошкина (1804–1857). Их сыновья стали известными химиками: старший, Александр (1927–1997), был заслуженным деятелем науки и техники, профессором Российского химико-технологического университета им. Д.И. Менделеева, заведовал кафедрой химической технологии керамики и огнеупоров, затем кафедрой химии и технологии кристаллов; второй сын, Николай (1932–2012), был членом-корреспондентом НАН Беларуси[15]. Младшая дочь, Наталия, в замужестве Казакова – кандидат физико-математических наук, преподавала в Нижегородском архитектурно-строительном университете. В 1950 г. А.Г. Майер оставил семью, вне семьи у него в 1950 г. родился сын Алексей.

3 А.Г. Майер в Нижегородском (Горьковском) университете

Артемий Григорьевич Майер начал работать в Нижегородском университете осенью 1931 г. доцентом кафедры математического анализа, а с 1946 г. до конца своей жизни заведовал этой кафедрой; в 1948 г. он был утверждён ВАК в звании профессора. Этот короткий послужной список не раскрывает участие А.Г. Майера в жизни университета и его роль в развитии математики в Нижнем Новгороде. Об этом и будет рассказано ниже.

Начну с заметки из архива профессора Д.А. Гудкова[16], которая, насколько мне известно, не публиковалась и, вероятно, предназначалась для местной или стенной печати. Краткость заметки позволяет привести её текст полностью.

Артемий Григорьевич Майер

Профессор А.Г. Майер играл заметную роль в повышении математической культуры в Горьковском госуниверситете. А.Г. Майер участвовал в разработке курса математиче-

[15]В органической химии есть реакция Разуваева-Ольдекопа-Майера; Н.А. Майер и член-корреспондент НАН Беларуси Ю.А. Ольдекоп (1918–1992) были учениками академика АН СССР Г.А. Разуваева (1895–1989), основателя Института металлоорганической химии (в настоящее время – Институт металлоорганической химии им. Г.А. Разуваева РАН).

[16]Дмитрий Андреевич Гудков (1918–1992) – замечательный математик, решивший задачу о кривых степени 6 из 16-й проблемы Гильберта, а также автор книги [23] о нижегородском периоде биографии Н.И. Лобачевского. О жизни и деятельности Д.А. Гудкова см. [24], [25].

ского анализа (наряду с проф. И.Р. Брайцевым[17] *и проф. А.Г. Сигаловым*[18] *).*

Традиции, созданные им, сильны и сейчас. Читал А.Г. Майер и многие другие курсы: дифференциальные уравнения, теорию чисел, качественную теорию дифф. уравнений и т. д. Отмечу содержательный курс истории математики. Этот курс в Горьковском университете теперь забыт. Я особенно сожалею об этом, т. к. ценю исторический подход к развитию математических теорий в современной математике.

А.Г. Майер принадлежал к коллективу учёных, организованному академиком А.А. Андроновым. А.Г. Майер вёл большую научную работу и был симпатичным и привлекательным научным руководителем для своих учеников.

Вспоминается, что создание кафедры математики на радиофаке и «уход» с физмата был идеей Артемия Григорьевича. Очень жаль, что он не дожил до осуществления этой идеи, т. к. живо интересовался приложением математики к радиофизическим проблемам.

Артемий Григорьевич любил стихи, математику и людей.

23.9.76 *г.* *Ученик проф. А.Г.Майера проф. Д. Гудков.*

Отмечу, что именно Д.А. Гудков реализовал идею Майера «ухода с физмата»: Гудков организовал в 1961 г. кафедру математики радиофизического факультета, которой руководил до 1978 г. Перейдя на мехмат в качестве заведующего кафедрой геометрии и высшей алгебры (1978–1988 гг.), Гудков восстановил чтение лекций по истории математики – он разработал новый курс, который, постоянно перерабатывая, читал до 1992 года.

Приведу несколько воспоминаний о лекциях А.Г. Майера. Так, Б.Н. Верещагин[19], вспоминал:

«Одним из близких и талантливых сотрудников Андронова, игравшим также весьма серьёзную роль в преподавании таких важных дисциплин, как теория дифференциальных уравнений и уравнения в частных производных математической физики, был доцент, впоследствии профессор математик Артемий Григорьевич Майер. Его прекрасные лекции давали нам, студентам, очень много».

Об отношении студентов к А.Г. Майеру (или А.Г. Майера к студентам?) много говорит фрагмент из воспоминаний неизвестного мне автора А. Борисова (газета «Горьковский Университет» за 28 сентября 1965 года):

«Студент получил двойку и уходит с экзамена без тени недовольства. Ему ясно, где его ошибки, что он недоработал. Вам случалось видеть такое? Мне вспоминается в этих случаях экзамен по математическому анализу у профессора А. Майера».

Вот пример буквально восторженных воспоминаний о лекциях А.Г. Майера из книги М.А. Миллера[20][29]:

«Напомню, что главным Математиком у нас можно было считать Артемия Григорьевича Майера. Это был пример антипеданта. Никакого занудства. Никакого "профессорства". Блестящая сообразительность и лёгкость. Последняя служила предметом

[17]Иван Романович Брайцев (1870–1947) – инициатор создания (1931 г.) и первый декан (до 1939 г.) физико-математического факультета Нижегородского университета, с 1942 г. заведовал кафедрой теории функций, созданной по его инициативе. И.Р. Брайцеву посвящена книга [26].

[18]Александр Григорьевич Сигалов (1913–1969) – выдающийся математик, решил 20-ю проблему Гильберта. Об А.Г. Сигалове см. статью [27].

[19]Борис Николаевич Верещагин (1918–2008), однокурсник Д.А. Гудкова, студентом участвовал в работе семинара Андронова, защитил дипломную работу «Некоторые случаи рождения циклов для квадратичного уравнения». Однако обстоятельства сложились так, что он оставил математику и стал крупным дипломатом-китаистом. Приводимый фрагмент взят из http://www.mccme.ru/ smirnoff/family/

[20]Михаил Адольфович Миллер (1924–2004) – физик-теоретик, профессор и заведующий кафедрой электродинамики радиофизического факультета Нижегородского университета. См. о нём книгу [28].

осуждения со стороны некоторых его коллег. Я думаю, он был Математиком Полёта. Поэтому иногда его заносило.<...>

В пламенных лекциях, экспромтных, свежеизготовляемых прямо на виду у публики, он достигал максимального успеха, какой только можно представить: в конце каждого присутствия мне хотелось стать математиком! Возможно, не только мне! Пожалуй, не такой плохой критерий оценки "захватываемости профессией"!»

Из воспоминаний члена-корреспондента РАН физика В.А. Зверева (личное сообщение):

«Будучи студентом, я с невероятным наслаждением, прямо с упоением слушал лекции Артемия Григорьевича. В его изложении математика выглядела необычайно увлекательной наукой, с необыкновенной захватывающей интригой. <:> Лекции он читал без бумажки, беседуя с аудиторией запросто, все формулы выписывал по памяти. Он пояснял нам при этом, что каждая лекция от него требует длительной и тщательной подготовки, что у него есть и припасена шпаргалка, которая лежит в кармане и в любой момент может быть задействована, но таких моментов не было. Но всего того, что сказано выше о педагогической деятельности Артемия Григорьевича, ещё крайне мало. Он умел много больше. Он передавал слушателям, а точнее мне, как его слушателю, своё наслаждение научной работой. Это неизвестно, как и чем передаётся, но как-то это передаётся. Для этого необходимо самому испытывать это наслаждение в такой сильной степени, чтобы оно, существенно ослабляясь с дистанцией, достигало до слушателя ещё в такой сильной дозе, которая способна его зажечь».

Ещё об одной стороне педагогической деятельности Майера рассказал мне П.Э. Сыркин[21]: в 1939 году, учась в 10-м классе, он был слушателем цикла лекций для школьников, читавшихся известными нижегородскими учёными, в том числе А.А. Андроновым, Г.С. Гореликом[22], С.С. Четвериковым[23] и другими. Был в числе лекторов и А.Г. Майер, который рассказывал школьникам, что такое топология. По-видимому, это была первая лекция по топологии, прочитанная в Нижнем Новгороде.

Следует сказать, что А.Г. Майер вообще уделял школьному математическому образованию большое внимание – он был хорошо известен школьным учителям как лектор Горьковского областного института усовершенствования учителей, организовал в университете математическую олимпиаду для школьников.

Помимо интенсивной научной и преподавательской работы, А.Г. Майер активно участвовал в других сторонах университетской жизни. В характеристике[24], подписанной ректором университета, говорится: «За время работы в университете тов. Майер принимал и принимает активное участие в общественной работе – работал членом бюро секции научных работников, хорошо работал на трассе оборонного строительства, за что получил грамоту Горьковского Комитета Обороны. Сейчас работает председателем ревизионной комиссии Месткома и является председателем студенческого научного общества при университете». В военные годы А.Г. Майер был назначен членом «столовой комиссии» университета, ездил со студентами на лесозаготовки, сдавал кровь для раненых.

Приведу фрагменты из воспоминаний А.Я. Левина[25] (личное сообщение), который со-

[21]Павел Эммануилович Сыркин (род. в 1922 г.) – профессор Нижегородского государственного технического университета, создатель семейства V-образных автомобильных двигателей.

[22]Габриэль Симонович (Семёнович) Горелик (1906–1956) – физик, автор знаменитого учебника «Колебания и волны», за «идеологические ошибки» в котором в 1952 г. подвергся травле. О жизни и деятельности Г.С. Горелика см. книгу [30].

[23]Сергей Сергеевич Четвериков (1880–1959) – выдающийся советский генетик; о его жизни и деятельности см. книгу [31].

[24]Хранится в ЦАНО, ф.377, оп.8а, д.44, л.46.

[25]Авраам Яковлевич Левин (1922 г.р.) в 1951 г. окончил с отличием исторический факультет Горьковского университета. Заведовал кафедрой психологии Нижегородского университета. С 1999 г. живёт в США.

стоял в руководстве студенческого научного общества в то время, когда председателем общества был А.Г. Майер.

«Мы постоянно встречались с А.Г. Майером на заседаниях президиума общества. Он обычно приносил с собой виноград или другие фрукты для угощения студентов. <:> Он уговаривал меня заняться историей русской денежной системы времён Ивана Грозного. По его мнению, это очень интересная тема, так как сложность этой системы свидетельствует, что математические знания того времени были более продвинутыми, чем принято считать[26]*. Я тогда сказал, что уже влез в историю Флоренции* 14 – 16 *веков. Он со знанием дела стал обсуждать со мной историю дома Медичи. Разносторонность его знаний была потрясающей. На юбилее профессора Некрасова*[27] *он заявил, что может прочитать получасовую лекцию по любой заданной теме. Не помню, какие две темы ему предложили, но, как рассказывали очевидцы, он справился с ними блестяще. <:>*

А.Г. был остроумен и быстро реагировал на ситуацию. Был у нас в университете профессор Воронцов[28]*. Он заведовал кафедрой зоологии на биофаке и занимал должность проректора по науке при ректоре Мельниченко*[29]*.*

На Учёном Совете университета : шла защита физической диссертации. Я не помню, был Горелик руководителем соискателя или оппонентом. В ходе обсуждения Воронцов взял слово и, признавшись, что он ничего не понимает в содержании диссертации, упрекнул диссертанта в злоупотреблении иностранными словами. Горелик взорвался: "Так что, вместо скин-эффекта говорить шкур-эффект? Есть международные термины".

После защиты в повестке дня Совета был отчёт Воронцова о работе кафедры зоологии. Докладывая о различных направлениях, он сказал: "В области орнитологии ... "

Тут Майер прервал его и бросил реплику: "Уважаемый коллега, Вы, вероятно, имеете в виду птицеведение?" Воронцов застыл, раскрыв рот. Члены совета рассмеялись. Потом хохотал весь университет.

Он любил всякие розыгрыши. Например, когда кто-нибудь в отделе ГИФТИ чихал, он не реагировал, Потом, после какого-нибудь чихания, неожиданно говорил: "Будьте здоровы". Оказывается, он считал эти чихания и желал здоровья только на пятый чих. Объяснял эту странность поговоркой: "На каждый чих не наздравствуешься"».

На ту же тему из книги М.А. Миллера [29]:

«Даже в житейском юморе он умел придавать своим высказываниям теоремно-подобную форму. <:> А вот ещё одна забавная и вполне педагогичная "теоремка Майера" (увы! Но в моей обработке!): "Всякую мысль, сколь угодно сложно выраженную, можно представить в ещё более сложном виде, причём любым сколь угодно сложным образом усложняя её далее и далее"».

4 «Об идеологических ошибках профессора Майера в курсе истории математики»

Наряду с лекциями по различным чисто математическим дисциплинам А.Г. Майер много лет читал курс истории математики. Долгое время мне была известна только одна лекция

[26]Эта мысль А.Г. Майера находит подтверждение в недавних исследованиях – см., например, [32], [33] (Г.П.).

[27]Алексей Дмитриевич Некрасов (1874–1960) – советский зоолог, эмбриолог и историк биологии. С 1928 г. заведовал кафедрой зоологии Нижегородского университета (Г.П.).

[28]Евгений Михайлович Воронцов (1899–1971), с 1 января по 22 апреля 1946 г. – и.о. ректора ГГУ, с 1947 г. заведовал кафедрой зоологии (Г.П.).

[29]Андрей Николаевич Мельниченко (1904–1998), биолог, ректор ГГУ с апреля 1946 г. по август 1952 г. (Г.П.).

из этого курса[30] – у меня была «Стенограмма лекции по Истории математики, прочитанной т. Майер для студентов 6-го курса Университета, группа математиков» (17 страниц машинописного текста формата А4) с указанием даты «7 февраля 1950 г.» и с несколько неожиданной пометкой «отп. 3 экз. л.к.».

Три года назад в фонде академика А.А. Андронова в Архиве РАН обнаружились четыре документа, касающиеся курса Майера по истории математики. Один из них (ф.1938, оп.1, д.461, лл.33–42) – другой экземпляр только что упомянутой стенограммы, второй (ф.1938, оп.1, д.461, л.1) – написанный рукой Майера черновик «Программа по курсу истории математики» – вполне стандартное оглавление начала хронологически построенного курса. Третий документ (ф.1938, оп.1, д.461, лл.2, 2об., 3) – рукописные наброски Майера, озаглавленные «О природе (слово «природе» зачёркнуто) математике». Это девять тезисов о происхождении математики, её сопоставление с другими науками, о появлении и роли математических доказательств. Приведу только замечания по поводу «Начал» Евклида – *«Никчемность ряда постулатов. Отсутствие у Эвклида ряда математических фактов. Особое внимание к математическим проблемам, имеющим мистический смысл»* – и геометрии Лобачевского: *«Крах абсолютной истинности – роль Лобачевского. Его неосознанность до настоящего времени. Концепция аксиом в современной математике, воззрения Энгельса. Смысл доказательства в настоящее время в его фактическом употреблении».*

Четвёртый, наиболее объёмный документ (ф.1938, оп.1, д.461, лл.4–31) – это разлинованный блокнот формата А4 «История математики. 1950 – 1951 год». Однако это не запись лекций Майера, а весьма краткий (некоторые лекции занимают всего один лист) их конспект, написанный неизвестным мне почерком. Как по упомянутой выше стенограмме, так и по этому конспекту составить чёткое мнение о лекциях трудно, но можно отметить два обстоятельства: отсутствие каких-либо точных математических формулировок и стремление А.Г. Майера излагать историю математики прежде всего как историю идей, что, конечно, непросто.

Конспект вообще производит довольно странное впечатление: в нём совершенно нет геометрических чертежей, почти нет указания конкретных дат, конкретных задач и т. п., зато на каждом листе красным карандашом отведены довольно широкие поля, на которых отмечаются ссылки, данные лектором на труды классиков марксизма-ленинизма, а также моменты, когда на лекции возникала дискуссия. Запись весьма аккуратная, причём первые после титульного три листа документа первоначально вполне разборчиво написаны на отдельных листах, а затем дословно переписаны более аккуратно тем же почерком в блокнот. По-видимому, этот конспект составлялся по какому-то особому заданию.

Всего законспектированы 19 лекций: один раз в неделю с 2.09.1950 до 30.12.1950, плюс последняя лекция 12.02.1951, посвящённая предыстории анализа (Кеплер, Роберваль, Ферма, Паскаль, Валлис, Кавальери). Не имея возможности из-за большого объёма привести здесь конспект лекций полностью, ограничусь только тем, что необходимо для дальнейшего изложения.

Из конспекта лекции 7: «Эвклид и его “Начала”, обычная оценка Эвклида как высокого образца логической строгости, оказавшего тем самым существенное и положительное воздействие на последующее развитие математики. Моя личная точка зрения: сознательный отрыв от практики, закреплённый в логической строгости изложения, гибелен для науки. Наличие в Эвклиде обобщений ради обобщений – первый признак ложного пути».

Из конспекта лекции 9: «Общая оценка Эвклида, идеалистический пифагорейский дух его и сознательный подчёркнутый отрыв его от практических задач. Применение ленинской формулы о гносеологических корнях идеализма. Последующее влияние Эвклида. <:>

[30]Кроме этого, в архиве Д.А. Гудкова я обнаружил написанный А.Г. Майером в 1948 г. неоконченный текст «Александр Михайлович Ляпунов» (6 машинописных страниц).

Резюме: в последующем Эвклид икона, на которую охотно молятся, но которой фактически не пользуются (масса почтения, мало применения)».

В архивных материалах есть много других фрагментов, ясно показывающих, что А.Г. Майер был «евклидоборцем»[31] и этого не скрывал[32]. Это послужило основным поводом для развёрнутой советом факультета травли Майера. Однако истинные мотивы предпринятого «обсуждения» – это присущее homo sapiens «видовое свойство»: людские сообщества, в частности, профессиональные, не любят выдающихся индивидуумов, особенно если те ведут себя независимо. При этом в стремлении повысить или сохранить своё положение, не имея для этого достаточных оснований, всегда удобно использовать в качестве универсального инструмента официальную идеологию. «Вообще, Д.А.[Гудков] считал, что организаторы этой кампании травли А.Г. Майера сознательно хотели довести его до смерти, прекрасно зная о его тяжелой гипертонии. Естественно, эти люди преследовали чисто корыстные цели. После изгнания настоящих математиков и педагогов с механико-математического факультета они сами возглавили кафедры, бездельничали и бдительно стремились избавляться от постоянно появлявшихся способных молодых людей, чувствуя в них потенциальную угрозу» (из статьи Е.И. Гордона [24] в его переводе).

Кульминация наступила на трёх заседаниях совета физико-математического факультета: 20 и 23 декабря 1950 года и 5 января 1951 года. В Центральном архиве Нижегородской области имеется дело 130 (фонд 377, опись 7), которое содержит тезисы и текст доклада А.Г. Майера «Предыстория создания математического анализа» (20.12.1950), ответы докладчика на вопросы (43 машинописных листа формата А4), стенограмму обсуждения этого доклада на заседании 23.12.1950 (50 листов) и стенограмму заседания 05.01.1951 (продолжение обсуждения, заключительное слово докладчика, обсуждение проекта решения Совета и принятие решения (всего 66 листов). Не только привести, но даже обсудить здесь такой объёмный материал не представляется возможным. Подробное изложение см. в [35][33], а само «Дело 130» доступно в Интернете: см. http://www.unn.runnet.ru/nnmo/abstracts/2014-04-08/

Но есть короткий документ, характеризующий существо и стиль обсуждения, позиции участников, принятые решения (и, конечно, дух времени) – это статья, название которой вынесено в заголовок этого раздела, в многотиражной газете «За Сталинскую науку»[34] от 19.02.1951. Приведу выдержки из этой статьи (текстовые выделения мои).

Выступавшие отметили, что проф. Майер допустил в докладе, а ранее и в вводной лекции по истории математики, **серьёзные ошибки антимарксистского и космополитического характера.**

Несмотря на стремление некоторых членов Совета **(доцент Сигалов, доц. Гордон, доц. Неймарк)** *смазать остроту критики при обсуждении решения, Совет принял развёрнутое и острое решение, осуждающее ошибки проф. Майера. <:>*

Проф. Майер ошибочно указывает на отсутствие сознательного использования метода диалектического материализма советскими математиками.

Безусловно, ошибочна оценка эвклидовых «Начал», данная проф. Майером. **Он пытался охарактеризовать «Начала» и лежащее в их основе стремление к формальной логической строгости как явление реакционное, сыгравшее «очень тяжёлую роль» в развитии математики.**

[31]Такая позиция не является исключением – резким критиком Евклида был французский математик Пётр Рамус (1515–1572), погибший во время событий «Варфоломеевской ночи» (см., например, [34]).

[32]Е.А. Леонтович-Андронова рассказывала: «С ужасной яростью Майер говорил: “Я с этого Евклида штаны сниму!”»

[33]Эти материалы обсуждаются также в [36].

[34]Сейчас газета называется «Нижегородский университет».

Позиция проф. Майера в этом вопросе находится в резком противоречии с общепринятой в советской математике точкой зрения. <:>

Следует отметить, что в заключительном слове проф. Майер в категорической форме отверг критику членов Совета и продолжал упорно отстаивать свои взгляды.

Странным было и поведение доц. Сигалова, который не присутствовал на первых двух заседаниях Совета (следовательно, не слушал доклад и прения) и в то же время нашёл возможным выступить против всех пунктов проекта решения. В грубой и демагогической форме он обвинил всех членов Совета в незнании основных фактов истории математики.

Напомню, что этот текст, подписанный *«Доц. Беневоленский – декан физико-математического факультета, проф. Я. Шапиро, проф. В. Котов, доц. И. Лохин, доц. Н. Отроков»*, опубликован в начале 1951 года: он вполне мог инициировать уголовное преследование.

Из стенограммы заседаний очевидно, что в роли главного обвинителя – преследователя Майера выступал профессор В.Ф. Котов, а «дирижировали» процессом доценты В.И. Беневоленский[35] и Н.Ф. Отроков.

Про В.Ф. Котова следует сказать несколько слов. По специальности он был механиком, с 1936 г. заведовал кафедрой теоретической механики, организовал аэродинамическую лабораторию. Я не знаю, имел ли он какие-либо результаты в области механики: основные его публикации – в журналах «Под знаменем Марксизма», в Трудах института истории естествознания и техники, в Историко-математических исследованиях. Все эти работы сильно идеологизированы – точнее, написаны под знаком критики с идеологических позиций. Нет сомнений, что в этой «идеологической борьбе под знаменем марксизма» и состояло истинное призвание профессора Котова: кроме лидерства в травле А.Г. Майера, 1952 г. он сыграл аналогичную роль в травле замечательного физика Г.С. Горелика[36]. Приведу в подтверждение сказанного две цитаты.

«Главным же травителем ГС [Габриэля Семёновича Горелика – Г.П.] – вне всяких групп – был «теормеханик» В.Ф. Котов – он то работал воистину с упоением хищника! <:> и довёл ГС до состояния агрессивной истерики, а по протоколу всего лишь до обзыва Котова «бесплодной смоковницей» ([29], с. 147).

«Как-то ещё в мои студенческие годы Майер завёл меня в общежитие, где его поджидали В.Ф. Котов и И.Ф. Лохин (кстати, по моим представлениям, довольно талантливый математик, он преподавал у нас матфизику, а потом куда-то уехал). Ждал нас полунакрытый стол, так что Майер стал «третьим», а я «четвёртым». После нескольких «приложений» Котов вдруг встал и произнёс фразу, которую я запомнил на всю доставшуюся мне жизнь. Он сказал: “Должен вас покинуть. Меня ждёт работёнка. Мне надо мозги вправлять интеллигентам на семинаре”. (Философском, между прочим, и даже, кажется, общегородском)» ([29], с. 148).

Из стенограммы (да и из газетной заметки) видно, что организовать «единодушное осуждение» не удалось: помешало «стремление некоторых членов Совета (доцент Сигалов, доц. Гордон, доц. Неймарк[37]) смазать остроту критики», т. е. стремление названных учёных, а также Д.А. Гудкова[38], защитить Майера и перевести осуждение в деловое об-

[35]Показательно прозвище, данное ему студентами, о котором мне сообщил А.Я. Левин: «нибениминоленский».

[36]О «Деле Горелика» см., например [29], [30], [37].

[37]Юрий Исаакович Неймарк (1920–2011) – доктор технических наук, профессор, один из организаторов в ГГУ первого в СССР факультета Вычислительной математики и кибернетики.

[38]Д.А. Гудков, получив диплом с отличием об окончании университета 21 июня 1941 г., сразу попал на ускоренные артиллерийские курсы и затем прошёл всю войну, участвовал во взятии Берлина. В 1951 г. в свои 32 года он был аспирантом и не являлся членом совета факультета.

суждение. А.Г. Сигалов, в частности, сказал: *«Рассмотрение этих вопросов требует от тех, кто берётся эти вопросы решать, каких-то конкретных определённых знаний по истории математики. Я этих знаний не имею и голосовать за резолюцию не буду, и я сомневаюсь,* [что] *хотя бы один член совета обладает такими знаниями, включая и всю комиссию.*

Я думаю, что составители этой резолюции поступили, как начётчики, не зная существа дела, взяв отдельные положения марксизма, цитируя их вкривь и вкось».

Поразительное по прямоте и смелости выступление, особенно если учесть, что в это время в совете МГУ находилась докторская диссертация[39] А.Г. Сигалова!

Что касается поведения на совете А.Г. Майера, то никакого покаяния не было. В своём «заключительном слове» он, в частности, сказал:

«Курс истории математики я веду не по доброй воле – каждый год я прошу меня от него освободить. Почему он мне труден? Он требует охвата огромного материала – большего, чем тот, которым я располагаю, и у меня нет даже надежды овладеть всем этим материалом. Таково, например, положение с историей Индии, историей арабов – иногда я просто заявляю студентам, что не знаю истории этих народов, по крайней мере, не знаю настолько, что мог бы её им вкратце рассказать».

Уже практически завершая своё выступление, А.Г. Майер заявил: *«Должен сказать ещё раз: читать не то, что я думаю, я не буду».*

***

Разумеется, в лекциях А.Г. Майера по истории математики были недостатки (да и никто не обязан разделять точку зрения Майера – скажем, на роль Евклида в развитии математики; кстати, в лекциях Майер всегда указывал, где он излагает свою личную точку зрения). В любом случае это не может служить оправданием развязанной травли замечательного математика.

Официальному преследованию после описанных заседаний совета Майер не подвергался, но курс истории математики не читался после этого 30 лет. Да и кто может поручиться, что ранняя смерть А.Г. Майера (ему было только 46 лет!), последовавшая 20 сентября 1951 года от инсульта, спровоцированного гипертонией, не зависима от описанных событий? М.А. Миллер пишет ([29], с. 112) что на похоронах А.Г. Майера звучало: «Затравили!». А гонители Майера ещё долгое время «правили бал» на факультете.

Я хочу закончить этот текст, посвящённый памяти Артемия Григорьевича Майера, словами А.Я. Левина (личное сообщение, 2014 г.):

«Мне очень интересно всё, что относится к жизни и творчеству Артёма Григорьевича. Удивительное обаяние исходило от него. Каждая минута общения с ним была подарком».

Список литературы

[1] Математика в СССР за сорок лет 1917–1957. Том второй. – М.: Наука, 1959. – 821 с.

[2] Математика в СССР 1958 – 1967. Том второй, выпуск второй. – М.: Наука, 1970. – 762 с. (нумерация тома начинается со стр. 821)

[3] Шильников Л.П. Леонтович-Андронова Евгения Александровна. – Сб. «Личность в науке. Женщины-ученые Нижнего Новгорода», вып.2. Нижний Новгород: Изд-во ННГУ, 1999. С. 83–102.

[39] Напомню, что эта диссертация, успешно защищённая позже в том же 1951 году, содержит решение 20-й проблемы Гильберта.

[4] Shilnikov L.P. Evgeniya Aleksandrovna Leontovich-Andronova // Methods of Qualitative Theory of Differential Equations and Related Topics, AMS Translations, Ser.2. Vol.200, 2000. P. 1–14.

[5] Шильников Л.П. К 100-летию со дня рождения Евгении Александровны Леонтович-Андроновой (1905 – 1997) // Вестник ННГУ, серия Математика. Вып 1(3). 2005. С. 191–204.

[6] Андронов А.А., Леонтович Е.А., Гордон И.И., Майер А.Г. Качественная теория динамических систем второго порядка. – М.: Наука, 1966. – 568 с.

[7] Андронов А.А., Леонтович Е.А., Гордон И.И., Майер А.Г. Теория бифуркаций динамических систем на плоскости. – М.: Наука, 1967. – 487 с.

[8] Gordon I.I. On intersection invariants of a complex and its complementary space // Ann. of Math. Vol.37, No 3. 1936. P. 519–525.

[9] Гордон Е.И. Адресат Л.С. Понтрягина – И.И. Гордон (Вступительные заметки) // Историко-математические исследования, Вторая серия, вып. 9(44). – М.: Янус-К. 2005. С. 14–26. (См.также http://7iskusstv.com/2011/Nomer11/EGordon1.php)

[10] Аносов Д.В., Афраймович В.С., Бунимович Л.А., Гонченко С.В., Гринес В.З., Ильяшенко Ю.С., Каток А.Б., Кащенко С.А., Козлов В.В., Лерман Л.М., Морозов А.Д., Нейштадт А.И., Песин Я.Б., Самойленко А.М., Синай Я.Г., Трещёв Д.В., Тураев Д.В., Шарковский А.Н., Шильников А.Л. Леонид Павлович Шильников (некролог) // УМН. Том 67, вып.3(405), 2012. С. 175–178.

[11] Afraimovich V.S., Gonchenko S.V., Lerman L.M., Shilnikov A.L., Turaev D.V. Scientific heritage of L.P. Shilnikov // Regular and Chaotic Dynamics. Vol. 19, Issue 14, 2014. P. 435–460. (См. также ещё: Editorial Leonid Pavlovich Shilnikov // International Journal of Bifurcation and Chaos, vol.24, №8. 2014. P. 1–7.)

[12] Памяти Александра Александровича Андронова – М.: Изд. АН СССР, 1955. – 718 с.

[13] Арансон С.Х., Гринес В.З. Топологическая классификация потоков на замкнутых двумерных многообразиях // УМН. Том 41, вып.1(247), 1986. С. 149–169.

[14] Арнольд В.И. Малые знаменатели. I. Об отображениях окружности на себя // Изв. АН СССР. Сер. матем. Том 25, вып.1, 1961. С. 21–86.

[15] Плисс В.А. Нелокальные проблемы теории колебаний. – М.–Л.: Наука, 1964. – 369 с.

[16] Шильников Л.П. К работам А. Г. Майера о центральных движениях // Мат. заметки, т.5, №3, 1969. С. 335–339.

[17] Пилюгин С.Ю. Фазовые диаграммы, определяющие системы Морса-Смейла без периодических траекторий на сферах // Дифф. уравн., Т. 14, №2, 1978. С. 245–254.

[18] Уманский Я.Л. Необходимые и достаточные условия топологической эквивалентности трёхмерных динамических систем Морса-Смейла с конечным числом особых траекторий // Матем. сб., Т.181, вып.2, 1990. С. 212–239.

[19] Lerman L.M., Umanskii Ya.L. Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects) // Translations of Mathem. Monographs, AMS, v.176, 1998.

[20] Бронштейн Н. Доктор Майер //Литературное наследство. Т. 45/46. 1948, с. 473–496.

[21] Топоровский Я. Человек Запада // Еврейское слово, №№ 31, 32. Москва, 2006. (В настоящее время доступны на сайте http://chitayu-i-zapisyvayu.blogspot.ru/)

[22] Брук Я. Яков Каган-Шабшай и Марк Шагал // Бюллетень Музея Марка Шагала. Вып. 16–17. Витебск, 2009. С. 85–101. (См. также http://chagal-vitebsk.com/node/229)

[23] Гудков Д.А. Н.И. Лобачевский. Загадки биографии. – Нижний Новгород: изд-во ННГУ, 1992. – 241 с.

[24] Gordon E.I. Recollection of D.A. Gudkov // In «Topology of Real Algebraic Varieties and Related Topics», AMS Translations, Ser. 2, Vol.173, 1996. P. 11–16.

[25] Polotovskiy G.M. Dmitrii Andreevich Gudkov // In «Topology of real algebraic Varieties and Related Topics», AMS Translations, Ser. 2, Vol.173, 1996. P. 1–9.

[26] Иван Романович Брайцев (1870–1947) (Серия «Личность в науке») / сост. Кузнецова Н.Б. – Нижний Новгород: изд-во Нижегородского ун-та, 2004. – 192 с.

[27] Жислин Г.М. О работах А.Г. Сигалова по математической физике (к 100-летию со дня рождения) // Математика в высшем образовании, №11. 2013. С. 105–114.

[28] Михаил Адольфович Миллер. «Из спектров чувств и сгустков дум. . . » Жизнь. Воспоминания. Творчество / Рос. акад. наук, Ин-т приклад. физики. – Нижний Новгород: ИПФ РАН, 2015. – 448 с.

[29] Миллер М.А. Избранные очерки о зарождении и взрослении радиофизики в горьковско-нижегородских местах. – Н. Новгород: изд-во ИПФ РАН, 1997. – 224 с.

[30] Личность в науке. Г.С. Горелик. Документы жизни / Сост. Н.В. Горская, М.Б. Локтева – Нижний Новгород.: ННГУ, 2006. – 298 с.

[31] Артёмов Н.М., Калинина Т.Е. Сергей Сергеевич Четвериков. 1880 – 1959. – М.: Наука, 1994. – 160 с.

[32] Цайгер М.А. Арифметика в Московском государстве XVI века. – Беэр-Шева: Берилл, 2010. – 72 с.

[33] Симонов Р.А. К истории счёта в допетровской Руси (рецензия на книгу: Цайгер М.А. Арифметика в Московском государстве XVI века. Беэр-Шева: Берилл, 2010) // Математика в высшем образовании, №8. 2010. С. 133–140.

[34] Матвиевская Г.П. Рамус. 1515–1572. – М.: Наука, 1981. – 152 с.

[35] Полотовский Г.М. Нижегородский математик Артемий Григорьевич Майер и его курс истории математики / В кн. Полотовский Г.М. Очерки истории российской математики – Нижний Новгород: изд-во Нижегородского госуниверситета. 2015. – 320 с. – С. 210–294 (см. также http://7iskusstv.com/2015/Nomer2/Polotovsky1.php)

[36] Идеология и наука. Дискуссии советских ученых середины XX века / Отв. ред. Касьян А.А. – М.: Прогресс-Традиция, 2008. – 288 с.

[37] Касьян А.А. История с физикой (Горьковский университет, середина XX века) – Нижний Новгород, 2004. –187 с.

Artemiy Grigorievich Maier

G. M. Polotovskiy

Sketch about life and scientific and pedagogical activity of the Nizhny Novgorod mathematician professor Artemy Grigoryevich Mayer.

Keywords: qualitative theory of the differential equations, professor Artemiy Grigorievich Maier.

Quantum mechanics allows setting initial conditions at a cosmological singularity: Gowdy model example

S.L. Cherkas
cherkas@inp.bsu.by; Institute for Nuclear Problems,
Belarusian State University, Minsk, Belarus
V.L. Kalashnikov
v.kalashnikov@aston.ac.uk; Aston Institute of Photonic Technologies,
Aston University, UK

Abstract

It is shown, that initial conditions in the quasi-Heisenberg quantization scheme can be set at an initial cosmological singularity per se. This possibility is provided by finiteness of some quantities, namely momentums of the dynamical variables, at a singularity, in spite of infinity of the dynamical variables themselves. The uncertainty principle allows avoiding a necessity to set values of the dynamical variables at singularity, as a wave packet can be expressed through the finite momentums. Influence of the initial condition set in the singularity to the issue of a vacuum state, arising during evolution when the gravitational waves appear, is addressed as well. It is shown that, even if one try to choose some state in the singularity minimizing late time expansion rate, some amount of matter appear in the late time evolution. Thus singularity could be considered not only as some information seed, but also as source of part of the matter in the universe.

1 Introduction

Singularity is a critical problem of modern cosmology [1]. In classical physics, it is considered as an impediment since the initial conditions for the universe evolution can not be formulated in terms of finite quantities at a cosmological singularity. One may expect that a quantization of gravity would avoid a singularity. Nevertheless, simple quantum cosmological models including anisotropic ones remain singular [2]. Some class of models predict a "singular-escaping" bounce-like behavior [3]. Extensions of the General Relativity (GR) provides with some additional scenarios of "escape from a singularity" (e.g., see Ref. [4]). However, there exists a possibility when a singularity does not prevent from a quantum evolution from "Beginning".

To consider such a possibility, one has to emphasize two strategics in a gravity quantization, which are based on two alternative pictures: Schrödinger's and Heisenberg's ones, which are equivalent in the ordinary quantum mechanics but not in quantum gravity. In quantum gravity, the Schrödinger picture is represented by the Wheeler-DeWitt equation [5, 6] which does not provide with an explicit time-evolution of universe. As an alternative, an analog of the Heisenberg's picture has been developed. It consists in the quantization of the equations of motion. Such a (quasi-)Heisenberg picture has been described in details for mini- and midi- superspace models [7–9]. It could serve a prototype of quantization for a complete set of the equations of motion for the (3+1)-dimensional GR. The aim of the present work is to consider more closely the singularity problem and partly the cosmological constant problem in the frameworks of this approach.

The quantization of the equations of motion requires setting the initial conditions for the quasi-Heisenberg operators and determining the corresponding Hilbert space in which they act. Though the initial singularity remains, the situation differs substantially from the classic one. It appears, that one may set initial conditions at singularity directly. It will be demonstrated on an example of the Gowdy model. The last admits an analytical solution within a whole time domain. Choosing an out-vacuum state is also possible, as the gravitational waves evolve against a classical background. It should be noted that the quasi-Heisenberg picture admits a more general case, when the background is quantum. However in this case, one needs to solve the operator equations of motion, that is tricky mathematical and computational problem.

2 Gowdy Model

Gowdy model [10] (see also [2,11]) corresponds to an anisotropic universe, in which gravitational waves travel unidirectionally. We use slightly different gauge than the original Gowdy's one. In this gauge, the equations of motion contain a difference of the potential and kinetic energies of the field oscillators. In absence of evolution, this quantity is zero by virtue of the virial theorem. When a system evolves, the virial theorem is violated [12]. As was shown earlier, the difference of the potential and kinetic energies provides a value of the universe acceleration parameter for the Friedman universe which is comparable with the observed one [13].

Let us come to the Gowdy model and take a metric in the form of

$$ds^2 = e^{\tau-\lambda}(d\eta^2 - dx^2) - e^{2\tau+2\sqrt{3}V}dy^2 - e^{2\tau-2\sqrt{3}V}dz^2, \tag{1}$$

where the quantities τ, λ, V depend on the variables η, x.

The Einstein equations lead to three equations of motion

$$V'' - \partial_{xx}V + 2\tau'V' - \partial_x V\partial_x\tau = 0, \tag{2}$$

$$\tau'' - \partial_{xx}\tau = 2(\partial_x\tau)^2 - 2(\tau')^2, \tag{3}$$

$$\lambda'' - \partial_{xx}\lambda = 4(\partial_x\tau)^2 - 4(\tau')^2 - 6(\partial_x V)^2 + 6(V')^2, \tag{4}$$

and to two constraints

$$\mathcal{H}(\eta,x) = e^{2\tau}\left(\frac{1}{3}(\partial_x\tau)^2 + \frac{1}{2}(\partial_x V)^2 + \frac{1}{6}\partial_x\tau\partial_x\lambda + \frac{1}{3}\partial_{xx}\tau - \frac{1}{3}\tau'^2 + \frac{1}{2}V'^2 + \frac{1}{6}\tau'\lambda'\right) = 0,$$

$$\mathcal{P}(\eta,x) = e^{2\tau}\left(\frac{1}{6}\partial_x\lambda\,\tau' + \partial_x V V' + \frac{1}{6}\partial_x\tau\lambda' + \frac{1}{3}\partial_x\tau'\right) = 0.$$

Let us discuss structure of the equations of motion (2)-(4). Eqs. (2)-(4) contain a part corresponding to the wave equation. Remaining parts belong to two different types. The first one is of $(\tau'V' - \partial_x V\partial_x\tau)$-type. In this case we refer to V as a "field" variable, whereas τ plays role of a "background" against which the field V oscillates. Equations for the "background" variable contain a difference of the kinetic and potential energies, e.g. $(\tau')^2 - (\partial_x\tau)^2$ or $(V')^2 - (\partial_x V)^2$. The situation is analogous to the simple model representing a string against a curved background [8]. However, the equations for the background variable τ differ from those considered in [8] because Eq. (3) for τ is isolated, whereas the field variables contribute to the corresponding equation for the "background" in the toy of Ref. [8]. On the other hand, there is another "background" variable λ here, because the Gowdy model is anisotropic and one needs two variables τ and λ to describe a background. It should be noted, that the "background" variable λ does not influence oscillations of the "field" V.

The most interesting case is when the background variable τ is quantum and spatially nonuniform. However, a goal of the present paper is to consider the singularity problem so we take a simple gauge, where τ is nonquantum and spatially uniform. This allows obtaining an analytical solution considered in the original Gowdy model [10] and detailed in Ref. [2].

The way is to expand the variables into the Fourier series

$$V(\eta,x)=\sum_{k=1}\mathcal{V}_k(\eta)e^{ikx},$$
$$\lambda(\eta,x)=\sum_{k=1}\Lambda_k(\eta)e^{ikx},$$
$$\tau(\eta,x)=\sum_{k=1}T_k(\eta)e^{ikx}. \tag{5}$$

The equation of motion (3) for τ is isolated from the other ones. Thus, the spatially uniform initial conditions for τ make it to be spatially independent in the course of evolution. So one can take the initial conditions

$$T_k(0)=\delta_{0,k}\mathcal{T}_0,$$
$$T_k'(0)=\delta_{0,k}\,e^{-2\mathcal{T}_0}\Pi, \tag{6}$$

where Π and $\mathcal{T}_0$ are some constants. We will refer to $T_0(\eta)$ simply as $\tau(\eta)$ below.

Advancing in such a way and using the aforementioned gauge, one comes to the following equations of motion and constraints:

$$\tau''+2(\tau')^2=0, \tag{7}$$
$$\mathcal{V}_k''+k^2\mathcal{V}_k+2\tau'\mathcal{V}_k'=0, \tag{8}$$
$$\Lambda_0''=-4(\tau')^2+6\sum_q\mathcal{V}_q'\mathcal{V}_{-q}'-q^2\mathcal{V}_q\mathcal{V}_{-q}, \tag{9}$$

$$\mathcal{H}_k=e^{2\tau}\left(-\delta_{k,0}\frac{1}{3}\tau'^2+\frac{1}{6}\tau'\Lambda_k'+\frac{1}{2}\sum_q\mathcal{V}_q'\mathcal{V}_{k-q}'-q(k-q)\mathcal{V}_q\mathcal{V}_{k-q}\right)=0, \tag{10}$$
$$\mathcal{P}_k=e^{2\tau}\left(\frac{1}{6}ik\Lambda_k\,\tau'+\sum_q(iq)\mathcal{V}_q\mathcal{V}_{k-q}'\right)=0. \tag{11}$$

The equations of motion (7)-(9) can be obtained from the Hamiltonian $H=\mathcal{H}_0$. It should be noted that Λ_k at $k\neq 0$ is completely defined by the momentum constraint equation (11), namely

$$\Lambda_k=-\frac{6}{k\tau'}\sum_q q\mathcal{V}_q\mathcal{V}_{k-q}'. \tag{12}$$

One can introduce the momentums

$$\pi_k=\frac{\partial H}{\partial\mathcal{V}_k'}=e^{2\tau}\mathcal{V}_{-k}',$$
$$P_\Lambda=\frac{\partial H}{\partial\Lambda_0'}=e^{2\tau}\tau'/6,$$
$$P_\tau=-\frac{\partial H}{\partial\tau'}=e^{2\tau}\left(\frac{2}{3}\tau'-\Lambda_0'/6\right), \tag{13}$$

and rewrite the Hamiltonian in the terms of these momentums

$$H = e^{-2\tau}\left(-6P_\Lambda P_\tau + 12P_\Lambda^2 + \frac{1}{2}\pi_0^2 + \sum_{k\geq 1}\pi_k\pi_k^*\right) + e^{2\tau}\sum_{k\geq 1}k^2\mathcal{V}_k\mathcal{V}_k^*, \tag{14}$$

where it is taken into account that $\pi_{-k} = \pi_k^*$ and $\mathcal{V}_{-k} = \mathcal{V}_k^*$.

The (quasi-)Heisenberg quantization consists in quantization of the equations of motion that was described thoroughly [8, 9]. Briefly, this procedure can be described in the following way. The operator initial conditions for the equations of motion include the conditions (6) rewritten in terms of τ and the remaining conditions

$$\begin{aligned}
\hat{\mathcal{V}}_k(0) &= \hat{v}_k,\\
\hat{\mathcal{V}}_k'(0) &= e^{-2\mathcal{T}_0}\hat{p}_{-k},\\
\hat{\Lambda}_0(0) &= L_0,\\
\hat{\Lambda}_0'(0) &= e^{-2\mathcal{T}_0}(24P_\Lambda(0) - 6\hat{P}_\tau(0)),\\
\tau(0) &= \mathcal{T}_0,\\
\tau'(0) &= 6e^{-2\mathcal{T}_0}P_\Lambda(0),
\end{aligned} \tag{15}$$

where $\mathcal{T}_0$, L_0 are some c-numbers,

$$\begin{aligned}
&P_\Lambda(0) = \Pi,\\
&\hat{P}_\tau(0) = \frac{1}{6\Pi}\left(12\Pi^2 + \frac{1}{2}\sum_k \hat{p}_k\hat{p}_{-k} + e^{4\mathcal{T}_0}k^2\hat{v}_k\hat{v}_{-k}\right),
\end{aligned} \tag{16}$$

and Π is c-number, as well. The operators $\hat{p}_k$ and $\hat{v}_k$ do not depend on time and satisfy the standard commutation relations $[\hat{p}_k, \hat{v}_{k'}] = -i\delta_{k,k'}$, where $\delta_{k,k'}$ is the Kronneker symbol. One may realize the above operator commutation relations by the representation $\hat{v}_k = v_k$, $\hat{p}_k = -i\frac{\partial}{\partial v_k}$, or by the momentum representation $\hat{p}_k = p_k$, $\hat{v}_k = i\frac{\partial}{\partial p_k}$.

The next step is to define a Hilbert space, where the quasi-Heisenberg operators act. For this aim one should return to the Hamiltonian (14) and consider it as the Wheeler-DeWitt equation in the vicinity of $\mathcal{T}_0 \to -\infty$. Besides, the momentum P_Λ should be excluded with the help of the gauge condition $P_\Lambda = \Pi$.

The corresponding Wheeler-DeWitt equation in the vicinity $\tau = \mathcal{T}_0 \to -\infty$ is given as

$$\left(-i6\Pi\frac{\partial}{\partial\tau} + 12\Pi^2 - \frac{1}{2}\frac{\partial^2}{\partial v_0^2} - \sum_{k\geq 1}\frac{\partial}{\partial v_k}\frac{\partial}{\partial v_k^*}\right)\Psi(\tau, ..v_1^*, v_0, v_1...) = 0. \tag{17}$$

The solution of Eq. (17) has a form of the wave packet

$$\begin{aligned}
\Psi(\tau, ..v_1^*, v_0, v_1...) = &\int C(..p_1^*, p_0, p_1...)\\
&\exp\left(-\frac{i}{6\Pi}\left(12\Pi^2 + \frac{1}{2}p_0^2 + \sum_{k\geq 1}p_kp_k^*\right)\tau + i\sum_{k\geq 0}v_kp_k^*\right)dp_0dp_1dp_1^*...,
\end{aligned} \tag{18}$$

where the integral over $dzdz^* \equiv \frac{\rho d\rho d\phi}{2\pi i}$, $z = \rho e^{i\phi}$ is understood in the holomorphic representation as it has been implied [14]. In the momentum representation, the wave function (18) takes the form

$$\psi(\tau, ..p_1^*, p_0, p_1...) = C(..p_1^*, p_0, p_1...) \exp\left(-\frac{i}{6\Pi}\left(12\Pi^2 + \frac{1}{2}p_0^2 + \sum_{k\geq 1} p_k p_k^*\right)\tau\right). \tag{19}$$

Mean values of the quasi-Heisenberg operators in the momentum representation are given by the formula

$$< \psi|\hat{A}(\eta, \mathcal{T}_0, p_i, \frac{\partial}{\partial p_i})|\psi >= \int \psi^*(\mathcal{T}_0, p_i)\hat{A}(\eta, \mathcal{T}_0, p_i, \frac{\partial}{\partial p_i})\psi(\mathcal{T}_0, p_i)dp_0 dp_1 dp_1^* \ldots\Big|_{\mathcal{T}_0\to-\infty}. \tag{20}$$

Eq. (20) for a mean value is analogous to that used earlier (Cherkas and Kalashnikov 2012, 2015). One has to note, that $\mathcal{T}_0$ in Eq. (20) is chosen to be initially finite that allows avoiding a singularity, but finally the limiting process of $\mathcal{T}_0 \to -\infty$ is carried out.

For this simple model, the analytical solution exists that allows demonstrating the mean values in detail. The solution of Eq. (7) is

$$\tau(\eta) = \mathcal{T}_0 + \frac{1}{2}\ln\left(1 + 12\Pi e^{-2\mathcal{T}_0}\eta\right). \tag{21}$$

First, let us consider a solution of Eq. (8) in the vicinity of $\tau \sim \mathcal{T}_0 \to \infty$. It takes the form

$$\hat{\mathcal{V}_k}(\eta) \approx \hat{v}_k + \frac{1}{12\Pi}p_k^* \ln\left(1 + 12\Pi e^{-2\mathcal{T}_0}\eta\right). \tag{22}$$

If $\mathcal{T}_0$ tends to infinity then the expression for $\tau(\eta)$ becomes $\tau(\eta) = \frac{1}{2}\ln(12\Pi\eta)$. However, the expression for the operator $\hat{\mathcal{V}}_k(\eta)$ diverges formally under $\mathcal{T}_0 \to -\infty$. This reflects a fact that it is impossible to set the field values at a singularity in the classic picture. Below, we demonstrate that the quantum picture validates the limit of $\mathcal{T}_0 \to -\infty$ for the mean observable values.

Let us consider the mean value of (22) over the wave packet (19)

$$\begin{aligned}
< \psi|\hat{\mathcal{V}}_k|\psi >= \int (C(..p_1^*, p_0, p_1...))^* \exp\left(\frac{i}{6\Pi}(12\Pi^2 + \sum_{q\geq 0} p_q p_q^*)\mathcal{T}_0\right)\Big(i\frac{\partial}{\partial p_k} + \frac{1}{12\Pi}p_k^*\ln(1 \\
+12\Pi e^{-2\mathcal{T}_0}\eta)\Big)\exp\left(-\frac{i}{6\Pi}(12\Pi^2 + \sum_{q\geq 0} p_q p_q^*)\mathcal{T}_0\right)C(..p_1^*, p_0, p_1...)dp_0 dp_1 dp_1^*\ldots\Big|_{\mathcal{T}_0\to-\infty} \\
= \int (C(..p_1^*, p_0, p_1...))^*\Big(\frac{1}{12\Pi}p_k^*\ln(1 + 12\Pi e^{-2\mathcal{T}_0}\eta) + \frac{1}{6\Pi}p_k^*\mathcal{T}_0 + i\frac{\partial}{\partial p_k}\Big) \qquad (23) \\
C(..p_1^*, p_0, p_1...)dp_0 dp_1 dp_1^*\ldots\Big|_{\mathcal{T}_0\to-\infty} \\
= \int (C(..p_1^*, p_0, p_1...))^*\Big(\frac{1}{12\Pi}p_k^*\ln(12\Pi\eta) + i\frac{\partial}{\partial p_k}\Big)C(..p_1^*, p_0, p_1...)dp_0 dp_1 dp_1^*\ldots \qquad (24)
\end{aligned}$$

One can see from Eq. (24) that the divergent terms with $\mathcal{T}_0 \to -\infty$ cancel each other and the mean value of $\hat{\mathcal{V}}_k$ is finite. Hence, the wave packet defined at a singularity determines whole evolution of a system.

The approximate expression for $\hat{\mathcal{V}}_k$ has been used above. It is valid for $\eta \sim -\infty$. However, it is intensional to consider an exact expression and a contribution of the $V-$ quantum fluctuations

to the $\lambda-$ evolution. The exact solution of the equation of motion (8) with $\tau(\eta)$ given by (21) takes the form of

$$\hat{\mathcal{V}}_k(\eta) = \frac{\pi}{24\Pi}\left(p_k^* J_0\left(\frac{e^{2\mathcal{T}_0}k}{12\Pi}\right)Y_0\left(k\left(\eta+\frac{e^{2\mathcal{T}_0}}{12\Pi}\right)\right) - J_0\left(k\left(\eta+\frac{e^{2\mathcal{T}_0}}{12\Pi}\right)\right)\right.$$
$$\left.\left(p_k^* Y_0\left(\frac{e^{2\mathcal{T}_0}k}{12\Pi}\right) + ke^{2\mathcal{T}_0}\hat{v}_k Y_1\left(\frac{e^{2\mathcal{T}_0}k}{12\Pi}\right)\right) + ke^{2\mathcal{T}_0}\hat{v}_k J_1\left(\frac{e^{2\mathcal{T}_0}k}{12\Pi}\right)Y_0\left(k\left(\eta+\frac{e^{2\mathcal{T}_0}}{12\Pi}\right)\right)\right). \quad (25)$$

Here $J_0(z), Y_0(z), Y_1(z), J_1(z)$ are the Bessel functions. Second derivative of the Λ_0 can be determined from the equation of motion (9), whereas the first derivative of this quantity can be determined from the Hamiltonian constraint (10):

$$\hat{\Lambda}_0' = \frac{1}{\tau'}\left(2\tau'^2 - 3\hat{\mathcal{V}}_0'^2 - 6\sum_{k\geq 1}\hat{\mathcal{V}}_k'\hat{\mathcal{V}}_k'^{+} + k^2\hat{\mathcal{V}}_k\hat{\mathcal{V}}_k^{+}\right). \quad (26)$$

Here $\hat{\mathcal{V}}_k^{+}$ should be obtained from Eq. (25) by the changing $\hat{v}_k \to \hat{v_k^{+}} = i\frac{\partial}{\partial p_k^*}$, $p_k^* \to p_k$. Thus the most intrigue problem is to calculate a mean value of the quantities $\hat{\mathcal{V}}_k'\hat{\mathcal{V}}_k'^{+}$ and $k^2\hat{\mathcal{V}}_k\hat{\mathcal{V}}_k^{+}$, which are constituents of Eqs. (9) and (26) for $\hat{\Lambda}_0'$, $\hat{\Lambda}_0''$. Tracing this quantities allows calculating the $\hat{\Lambda}_0-$evolution.

Let us take the Gaussian form of a wave packet to determine the evolution of a system

$$C(..p_1^*, p_0, p_1...) = \prod_{k=0}^{\infty} N_k \exp\left(-a_k p_k p_k^*\right), \quad (27)$$

where the constant a_k determines the width of a packet for every mode and N_k is the normalization factor. The calculation according to (20) leads to the following expressions for $k \neq 0$ defining a mean value of the potential energy Ξ_k and a value of the kinetic energy K_k of every mode:

$$\Xi_k \equiv < \psi|k^2\hat{\mathcal{V}}_k\hat{\mathcal{V}}_k^{+}|\psi > = \frac{k^2}{1152 a_k\Pi^2}\left(\left(4J_0^2(k\eta)\left(144a_k^2\Pi^2 + \log^2\left(\frac{k}{24\Pi}\right)\right.\right.\right.$$
$$\left.\left.\left. + 2\gamma\log\left(\frac{k}{24\Pi}\right) + \gamma^2\right) - 4\pi\left(\log\left(\frac{k}{24\Pi}\right) + \gamma\right)J_0(k\eta)Y_0(k\eta) + \pi^2 Y_0^2(k\eta)\right)\right),$$
$$K_k \equiv < \psi|\hat{\mathcal{V}}_k'\hat{\mathcal{V}}_k'^{+}|\psi > = \frac{k^2}{1152 a_k\Pi^2}\left(\left(4J_1^2(k\eta)\left(144a_k^2\Pi^2 + \log^2\left(\frac{k}{24\Pi}\right)\right.\right.\right.$$
$$\left.\left.\left. + 2\gamma\log\left(\frac{k}{24\Pi}\right) + \gamma^2\right) - 4\pi\left(\log\left(\frac{k}{24\Pi}\right) + \gamma\right)J_1(k\eta)Y_1(k\eta) + \pi^2 Y_1^2(k\eta)\right)\right), \quad (28)$$

where $J_0(z), Y_0(z), J_1(z), Y_1(z)$ are the Bessel functions is Euler's constant.

In a spatially uniform mode, sole kinetic energy term is

$$K_0 \equiv \frac{1}{2} < \psi|\mathcal{V}_0'^2|\psi > = \frac{1}{576\, a_0^2\Pi^2\,\eta^2}.$$

For a further analysis, it is convenient to consider a quasi-classical sector corresponding to late times. This insight can be provided by an expansion of the Bessel function into series over

a large argument with keeping the leading terms:

$$Y_0(z) \approx \frac{1}{\sqrt{\pi z}}\left(\left(-\frac{9}{128z^2}-\frac{1}{8z}+1\right)\sin(z)+\left(\frac{9}{128z^2}-\frac{1}{8z}-1\right)\cos(z)\right),$$
$$J_0(z) \approx \frac{1}{\sqrt{\pi z}}\left(\left(-\frac{9}{128z^2}+\frac{1}{8z}+1\right)\sin(z)+\left(-\frac{9}{128z^2}-\frac{1}{8z}+1\right)\cos(z)\right),$$
$$J_1(z) \approx \frac{1}{\sqrt{\pi z}}\left(\left(\frac{15}{128z^2}+\frac{3}{8z}+1\right)\sin(z)+\left(-\frac{15}{128z^2}+\frac{3}{8z}-1\right)\cos(z)\right),$$
$$Y_1(z) \approx \frac{1}{\sqrt{\pi z}}\left(\left(-\frac{15}{128z^2}+\frac{3}{8z}-1\right)\sin(z)+\left(-\frac{15}{128z^2}-\frac{3}{8z}-1\right)\cos(z)\right).$$

Then, a simple estimation results from change of the oscillating multipliers by their time-averaged values. Thus, one has $\cos^2(k\eta) \to \frac{k}{2\pi}\int_0^{2\pi/k}\cos^2(k\eta)d\eta = \frac{1}{2}$, $\sin^2(k\eta) \to \frac{1}{2}$, $\sin(k\eta)\cos(k\eta) \to 0$.

Using Eqs. (9), (26) results in

$$< \psi|\hat{\Lambda}_0'|\psi > \approx \frac{1}{\eta}\left(1-\frac{1}{96a_0\Pi^2}\right)-\sum_{k\geq 1}\frac{kF_k}{6\Pi^2\pi a_k}+\frac{12a_k k}{\pi}-\frac{1}{\eta^2}\left(\frac{F_k}{48\pi\Pi^2 k a_k}+\frac{3a_k}{2\pi k}\right), \qquad (29)$$

$$< \psi|\hat{\Lambda}_0''|\psi > \approx -\frac{1}{\eta^2}\left(1-\frac{1}{96a_0\Pi^2}\right)+\frac{1}{\eta^3}\sum_{k\geq 1}\frac{F_k}{24\pi\Pi^2 k a_k}+\frac{3a_k}{\pi k}, \qquad (30)$$

where $F_k = \left(\frac{\pi^2}{8}+\frac{\gamma^2}{2}+\frac{1}{2}\ln^2\left(\frac{k}{24\Pi}\right)+\gamma\ln\left(\frac{k}{24\Pi}\right)\right)$.

It should be noted, that Eq. (30) describing an averaged second derivative of Λ_0 in a sense of the time-averaged evolution can be obtained from Eq. (29) by differentiation over η. Turning to a continuous limit of $\sum_k \to \frac{1}{2\pi}\int dk$, one can see that the second term in Eq. (29), corresponding to a vacuum energy diverges for any asymptotic of a_k at large k.

The most divergent term $\frac{kF_k}{6\Pi^2\pi a_k}+\frac{12a_k k}{\pi}$ vanishes under differentiation of Eq. (29). Concerning the remaining term which is mean value of the difference of the potential and kinetic energies of field oscillators, this issue has been discussed by for Friedman universe [13]. It has been found that this term defines a value of the acceleration parameter of universe, which is compatible with the observed one. One has note, that the UV cut-off of momentums was used in the estimates for Friedman universe [13]. Certainly, no anisotropic expansion of the universe exists in reality so one cannot compare the results of the above calculations with some observational values.

3 Evolution of a vacuum state

In the gauge considered, the background variable τ is not quantum. For this particular case, one can use an ordinary quantization using the creation and annihilation operators. Thus, we consider quantization of the field V against the time dependent background $\tau(\eta) = \frac{1}{2}\ln(12\Pi\eta)$. In this case, the field V is represented as [15]

$$\mathcal{V}_k(\eta) = \sum_k \hat{a}_k u_k(\eta) + \hat{a}_k^+ u_k^*(\eta), \qquad (31)$$

where

$$[\hat{a}_k, \hat{a}_k^+] = 1. \qquad (32)$$

The function $u_k(\eta)$ has to satisfy the condition

$$e^{2\tau(\eta)}\left(u_k^*(\eta)u_k'(\eta) - u_k(\eta)u_k^{*\prime}(\eta)\right) = i. \tag{33}$$

Mean values of the kinetic and potential energies of the mode k in a vacuum state equal to

$$\begin{aligned} \Xi_k &=< 0|k^2\mathcal{V}_k\mathcal{V}_k^+|0 >= k^2u_k^*u_k, \\ K_k &=< 0|\mathcal{V}_k'\mathcal{V}_k^{+\prime}|0 >= u_k^{*\prime}u_k'. \end{aligned} \tag{34}$$

Thus, one has to determine the functions u_k. A vacuum state is defined as a state vanishing under action of the annihilation operator: $\hat{a}_k|0 >= 0$. However, the definition of u_k is ambiguous. It should be noted that there exists a family of the functions u_k, which satisfy Eq. (33) and are interrelated by the Bogolubov's transformation. It has suggested to define a vacuum state through minimization of some functional containing a difference of the potential and kinematic energies of the field oscillators [16]. In such a way, one comes to the function

$$u_k(\eta) = \frac{1}{4}\sqrt{\frac{\pi}{3\Pi}}\,H_0^{(2)}(|k|\eta), \tag{35}$$

where $H_0^{(2)}(z)$ is the Hankel function of second kind. There is no particle (i.e., graviton) creation here, because the difference of the kinetic and potential energies is not oscillating quantity [16].

Using the asymptotic of the Hankel function for large arguments

$$H_0^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}}\,e^{-i(z-\pi/4)}\left(1 + \frac{i}{8z} - \frac{9}{128z^2}\right)$$

one can obtain for a vacuum state

$$\begin{aligned} < 0|\hat{\Lambda}_0'|0 >&\approx \frac{1}{\eta} - \sum_{k\geq 1}\frac{k}{\Pi} + \frac{1}{8k\eta^2\Pi}, \\ < 0|\hat{\Lambda}_0''|0 >&\approx -\frac{1}{\eta^2} + \sum_{k\geq 1}\frac{1}{4k\eta^3\Pi}. \end{aligned} \tag{36}$$

It is interesting to compare the above results with those from a quasi-Heisenberg quantization. For this aim one has to find the value a_k in the Eqs. (29),(30) which minimizes the constant part contribution $\frac{kF_k}{6\Pi^2\pi a_k} + \frac{12a_kk}{\pi}$ of every mode to Λ_0' given by Eq. (29). That gives $a_k = \frac{1}{6\Pi}\sqrt{\frac{F_k}{2}}$. Substitution of this value to Eqs. (29), (30) leads to

$$< \psi|\hat{\Lambda}_0'|\psi >\approx \frac{1}{\eta}\left(1 - \frac{1}{96a_0\Pi^2}\right) - \sum_{k\geq 1}\sqrt{1 + \frac{4}{\pi^2}\left(\gamma + \ln\left(\frac{k}{24\Pi}\right)\right)^2}\left(\frac{k}{\Pi} + \frac{1}{8k\eta^2\Pi}\right), \tag{37}$$

$$< \psi|\hat{\Lambda}_0''|\psi >\approx -\frac{1}{\eta^2}\left(1 - \frac{1}{96a_0\Pi^2}\right) + \sum_{k\geq 1}\sqrt{1 + \frac{4}{\pi^2}\left(\gamma + \ln\left(\frac{k}{24\Pi}\right)\right)^2}\,\frac{1}{4k\eta^3\Pi}. \tag{38}$$

Comparison with Eq. (36) demonstrates that the non-vanishing term supplements the vacuum state term in the (quasi-)Heisenberg quantization scheme. The sense is that any momentum wave packet at singularity has an inevitable counterpart corresponding to a matter. Thus, there is no need in a "matter creation from nothing" in the quasi-Hiesenberg picture because it exists primordially.

4 Discission and Conclusion

It is well known, that the GR is invariant relatively the general transformations of coordinates. It is difficult to expect that this property could remain in quantum theory. The point is that a detector moving with acceleration detects quanta of fields even in the Minkowsky space-time. Certainly, there exists absolutely untenable gauges for quantization in which a detector position oscillates, that causes the detection of the created particles by a detector. One may expect, that not all gauges (i.e., the systems of reference) are available for the quantization of gravitation and we adhere such a point of view. One may assume that it should exist some preferable reference frame suitable for quantization. As was pointed, the system of reference considered here is one in which the equations of motion for the background variable contain the difference of kinetic and potential energies of the field oscillators. Here massless fields have been under consideration. For massive fields, one has to take into account a concrete mechanism of mass generation and possible condensates. Thus, the problem becomes more complicated for the fields with a nonlinear interaction.

When an appropriate gauge is chosen, the quasi-Heisenberg quantization scheme solves the problems of time and singularity that is describes an explicit quantum evolution. The problem of singularity is solved in this scheme in a sense that a singularity does not prevent a description of evolution within a whole time diapason.

An additional question arises in connection with a possible generalization of the GR which would provide with a consistent quantum theory. We consider this issue from the vacuum energy issue viewpoint. On the one hand, the fluctuations of the scale factor can compensate zero point fluctuations of the matter fields in framework of the toy model of quantum gravity [8], which describes a string against a curved background. This conjecture is not enough proved in the GR. However, there is no such a compensation for the Gowdy model considered in the most natural gauge where the background variable is initially spatially uniform. Hence, one needs to remove a constant contribution to the Λ_0'-term by hand.

The toy model describing a string against a curved background [8] is conformal where an initially uniform scale factor becomes spatially nonuniform. Thus, some extensions of GR towards a conformal theory of gravity are promising. Recently, interesting steps in this direction were made [17, 18].

Authors are grateful for Belinsky V. A. for the discussion concerning the initial conditions in classical and quantum cosmology.

References

[1] S. Hawking, R. Penrouse. The nature of space and time. – Princeton U. Press: Princeton, 1996.

[2] B.K. Berger. Quantum graviton creation in a model universe. Ann. Phys. 1974. Vol. 83. P. 458–490.

[3] M. Bojowald, G.M.A. Paily. A no-singularity scenario in loop quantum gravity. Class. Quant. Grav. 2012. Vol. 29. 242002–8.

[4] A. V. Minkevich. Gauge approach to Gravity and regular Big Bang theory. Grav. Cosmol. 2006. Vol. 12. 11–20.

[5] B.S. DeWitt. Quantum Theory of Gravity. I. The Canonical Theory. Phys. Rev. 1967. Vol. 160. 1113–1148.

[6] J. A. Wheeler. Superspace and Nature of Quantum Geometrodynamics. In: DeWitt, C., Wheeler,J.A. (eds.) Battelle Rencontres – Benjamin: New York, 1968.

[7] S. L. Cherkas, V. L. Kalashnikov. Quantum evolution of the Universe in the constrained quasi-Heisenberg picture: from quanta to classics? Grav.Cosmol. 2006. Vol. 12. 126. (*Preprint* gr-qc/0512107).

[8] S. L. Cherkas, V. L. Kalashnikov. An inhomogeneous toy-model of the quantum gravity with explicitly evolvable observables. Gen. Rel. Grav. 2012. Vol. 44. 3081.

[9] S.L. Cherkas, V.L. Kalashnikov. Quantization of the inhomogeneous Bianchi I model: quasi-Heisenberg picture. Nonlin. Phen. Compl. Syst. 2015. Vol. 18. 1. (*Preprint* 1302.2229[gr-qc])

[10] R.H. Gowdy. Vacuum Spacetimes with Two-parameter Spacelike Isometry Groups and Compact Invariant Hypersurfaces: Topologies and Boundary Conditions. Ann. Phys. (N.Y.) 1974. Vol. 83. 203.

[11] C.W. Mizner. A minisuperspace Example: The Gowdy T3 Cosmology. Phys. Rev. 1973. Vol. 8. 3271.

[12] S.V. Anischenko. Violation of the virial theorem for the ground state of the time-dependent oscillator. Vestnik Belarus State U., ser. Fiz.-Mat.2008. Vol. 2. 43. [in Russian]

[13] S.L. Cherkas, V.L. Kalashnikov. Determination of the UV cut-off from the observed value of the Universe acceleration. J. Cosmol. Astropart. Phys. 2007. Vol. 01 028. (*Preprint* gr-qc/0610148).

[14] L. D. Faddeev, A.A. Slavnov. Gauge Fields: An Introduction To Quantum Theory. – Addison-Wesley: London & New York, 1991.

[15] N.D. Birrell, P.C.W. Davis. Quantum fields in curved space. – Cambridge U. Press: Cambridge, 1982.

[16] S. V. Anischenko , S.L.Cherkas, V.L. Kalashnikov. Functional minimization method addressed to the vacuum finding for an arbitrary driven quantum oscillator. Nonlin. Phen. Compl. Syst. 2009. Vol. 12. 16.

[17] H. Gomes, S. Gryb, T. Koslowski. Einstein gravity as a 3D conformally invariant theory Class.Quant.Grav. 2011. Vol. 28. 045005.

[18] L. Smolin. Linking shape dynamics and loop quantum gravity. Phys. Rev. D 2014. Vol. 90. 044070.

Simulation of graphene with partial degeneration of band structure

H.V Grushevskaya
grushevskaja@bsu.by; Belarusian State University, Belarus
V.A. Gaisyonok
National Institute of Higher Education, Belarus
G.G. Krylov
krylov@bsu.by; Belarusian State University, Belarus

December 1, 2015

Abstract

A secondary quantized field gauge theory with a number of flavors $N = 3$ has been proposed to describe optical and electrophysical properties of monolayer graphene. Charge carriers in this graphene model are modelled as Majorana pseudo-fermions. Partial unfolding of the Dirac bands results in coupling between anti-ordered pseudo-spins and valley currents. It has been shown that splitting of Dirac cone replicas on Weyl-like node and anti-node leads to polarization effects similar as graphene doping.

1 Introduction

Structurally stable genuine two dimensional material graphene [1] with linear energy dispersion law in the so called Dirac points of the Brillouin zone and chiral electric charge carriers is considered as a promising material for further miniaturization in electronics. There are performed numerous attempts to construct quantum devices based on magnetoelectric effects in 2D Dirac materials [2]. All such attempts in constructing of new electronic devices encounter an intensively investigated but still unsolved theoretical problem on clear understanding of both electrical properties and features of Dirac 2D-material band structures. As an example of one of such feature is an asymmetry of interaction between pseudo-fermions and right or left polarized photons incident obliquely on a single atomic layer [3]. By virtue of the pseudo-helicity (chirality) conservation law for massless pseudo-fermions, an electron (hole) belonging to triangular sublattice {A}({B}) of the hexagonal lattice is forced to jump through the nearest site belonging to triangular sublattice {B}({A}), to the next nearest site that leads to emergence of a valley current of excitonic quasiparticle-excitations.

At the same time, the modern quantum field theory (QFT) [4] applied for pseudo-Dirac quasiparticles [5, 6, 7, 8] predicts strong screening which destroys the excitonic pairing instability for a dynamic fermion mass m_q small in comparison with the chemical potential μ: $m_q \leq \mu$. Dynamic screening for such a system has been calculated by Eliashberg based on self-consistent technique with the number $N = 2$ of physical flavors (pseudo-spin and pseudo-charge). The existence of dynamic screening gives the value of m_q non-zero for momentums $|q| < 10^{-6}$ [4]. Unfortunately, such a calculated range of values of q leads also to practically vanishing mass $m_{k_F} \sim q/v_F < 10^{-12}$ for the Fermi momentum k_F and Fermi velocity $v_F \sim 10^6$ m/c. Additional possible drawback of the last approach is that pseudo-Dirac fermions act in $2 + 1$ dimensional

space-time whereas an electromagnetic field has to be defined as in ordinary QFT in $3+1$ dimensions [4, 14, 15].

The asymmetry of pseudo-fermion interaction with photons is possible to explain for valley currents being conserved currents in the theory admitting a symmetry transformation group that determines rotations in a time-space of charge carriers. Such sort of a gauge model $N = 3$ with physical flavors: pseudo-spin, pseudo-charge, and orbital pseudo-momentum could be background to describe electrophysical properties of the 2D semimetals.

In the paper we propose the secondary quantized gauge field theory with a number of flavors $N = 3$ to describe a monolayer graphene. Charge carriers in this model turns out to be Majorana pseudo-fermions. In the model, the exchange operators acting on sublattices provide narrowing of states. A goal of the paper is to establish effects of coupling of pseudo-spin with valley currents on low frequency conductivity and optical dielectric permeability for monolayer graphene accounting for spatial dispersion as well.

2 Monolayer Graphene Model with Partial Unfolding of Dirac Bands

Graphene is known to be a bipolar material with half-filled valence and conduction bands. A distinctive feature of graphene band structure is the existence of Dirac cones in the edges K, K' of the Brillouin zone. In the paper these points are designated as Dirac points K_A, K_B. Let us assume that pseudo-spins of hexagonally packed carbon atoms in the monolayer graphene are anti-ordered as shown schematically in Fig. 1a. Due to the fact that the pseudo-helicity (chirality) conservation law forbids massless charge carriers to be in lattice sites with opposite sign of pseudo-spin, there exist valley currents due to jumps through the forbidden sites as shown schematically in Fig. 1a. Coupling between the pseudo-spin and the valley current in Majorana representation of bispinors can be determined in the following way.

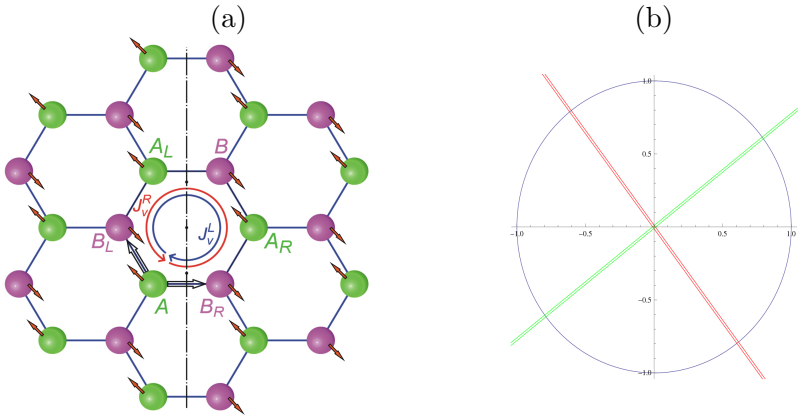


Figure 1: (a) Graphene lattice, comprised of two sublattice $\{A\}$ with spin "up" and $\{B\}$ with spin "down" . Right left valley currents J_v^R and J_v^L are shown as rounded curves with arrows. Double arrows from site A to site B_L and from A to B_R indicate clockwise and anti-clockwise directions. The axis of mirror reflection from A_R to B_L is marked by bar dotted line. (b) Transformation of k-circumference into ellipses under exchange operator action of $(\Sigma^x_{rel})_{AB}$ and $(\Sigma^x_{rel})_{BA}$ for sublattices $\{A\}$ and $\{B\}$, respectively (in color)

According to Fig. 1a, a particle can travel from a lattice site A to, for example, a lattice site A_R through right or left sites B_R or B_L, respectively. Since the particle is symmetrical,

its description in the right and left reference frames has to be equivalent. Therefore a bispinor wave function Ψ' of graphene has to be chosen in a Majorana form, and its upper and lower spin components ψ', $\dot{\psi}'$ are transformed by left and right representations of the Lorenz group:

$$\Psi' = \begin{pmatrix} \psi'_\sigma \\ \dot{\psi}'_{-\sigma} \end{pmatrix} = \begin{pmatrix} e^{\frac{\kappa}{2}\vec{\sigma}\cdot\vec{n}}\psi_\sigma \\ e^{\frac{\kappa}{2}(-\vec{\sigma})\cdot\vec{n}}\dot{\psi}_{-\sigma} \end{pmatrix}. \tag{1}$$

The wave-function $\widehat{\chi^\dagger_\sigma}(\vec{r}_A)\,|0,+\sigma\rangle$ of a particle, (in our case of electron-hole pair) located on the site A, behaves as a component ψ_σ, and the wave-function $\widehat{\chi^\dagger_{-\sigma}}(\vec{r}_B)\,|0,-\sigma\rangle$ of a particle located on the site B, – as a component $\dot{\psi}_{-\sigma}$ of the bispinor (1).

Relativistic particles with non-zero spin posses the helicity h which is the projection of particle's spin to the direction of motion [16]:

$$h \equiv \vec{p}\cdot\vec{S} = \frac{1}{2}p_i \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{2}$$

where $\vec{p}$ is the particle momentum, $\vec{S}$ is the spin operator for a particle, $\vec{\sigma}$ is the vector of Pauli matrices σ_i, $i = x,\ y$. In quantum relativistic field theory, the value of the helicity of a massless particle is preserved at the transition from one reference frame moving with velocity v_1, to another one moving with velocity v_2 [16, 17].

Let us denote $\vec{S}_{AB} = \hbar\vec{\sigma}_{AB}/2$ and $\vec{S}_{BA} = \hbar\vec{\sigma}_{BA}/2$ two-dimensional spin of quasi-particle in valleys K_A and K_B, respectively. A valley current J^R_v or J^L_v, on right or left closed contour $\{A \to B_R \to A_R \to B \to A_L \to B_L \to A\}$ or $\{A \to B_L \to A_L \to B \to A_R \to B_R \to A\}$, respectively, in Fig. 1, is created by an electron (hole) with pseudo-angular momentum $\vec{l}_{AB_R}$ and momentum $\vec{p}_{AB_R}$ or by an electron (hole) with $\vec{l}_{AB_L}$ and $\vec{p}_{AB_L}$. Pseudo-helicity of bispinors (1), describing the particles right or left from lattice site A, is defined by expressions which are analogous to (2):

$$h_{B_RA} \equiv \vec{p}_{AB_R}\cdot\vec{S}_{B_RA}, \tag{3}$$

$$h_{B_LA} \equiv \vec{p}_{AB_L}\cdot\vec{S}_{B_LA}. \tag{4}$$

Let us act by the parity operator P, which mirrors the bispinor (1) in respect to the line passing through points A and B. Pseudo-helicity of the mirrored bispinor is defined by the expression

$$Ph_{B_RA_R}P = h_{A_LB_L} = \vec{p}_{B_LA_L}\cdot\vec{S}_{A_LB_L}. \tag{5}$$

Due to the fact that valley momentum and pseudo-spin change signs: $\vec{p}_{A_LB_L} = -\vec{p}_{B_RA_R}$ and $\vec{S}_{A_LB_L} = -\vec{S}_{B_RA_R}$, pseudo-helicity h_{AB} does not change its value.

The pseudo-helicity h_{AB} is expressed through the projection $\tilde{\mathcal{M}}_{AB} = \vec{\sigma}_{BA}\cdot\left(\vec{l}_{AB} + \hbar\vec{\sigma}_{BA}/2\right)$ of total angular momentum on the direction of spin $\vec{\sigma}_{BA}$ as [18, 19]:

$$\vec{\sigma}_{BA}\cdot\vec{p}_{AB} = \sigma^r_{BA}\left(p_{r,\,BA} + \imath\frac{\tilde{\mathcal{M}}_{AB}}{r} - \hbar/2\right) = \sigma^r_{BA}\left(p_{r,\,BA} + \imath\frac{\vec{\sigma}_{BA}\cdot\vec{l}_{AB}}{r}\right) \tag{6}$$

where σ^r_{BA} and $p_{r,\,BA}$ are radial components of the spin and the momentum respectively. According to (6) the preudo-spin-orbital scalar $\vec{\sigma}_{BA}\cdot\vec{l}_{AB}$ describes the coupling (interaction) of spin with valley currents flowing along a closed loop in a clock-wise or in opposite directions

as shown in Fig. 1a. Hence, there exists a preferred direction along which spin projection of bispinor (1) does not change after transition from one moving reference frame into another one. At this, the spin of a particle precesses and, respectively, the described transformation of the electrons and holes into each other in exciton is a pseudo-precession.

As a result, the coupling of pseudo-spin and valley currents stipulates the spin precession of exciton charge carriers in graphene. In our model the orientation of non-equilibrium spin of the states of monolayer graphene in electromagnetic fields may be retained for a long time due to prohibition of change for exciton pseudo-helicity. Pseudo-precession is possible, if spins of p_z -electrons are anti-ordered (pseudo-antiferromagnetic ordering). Therefore, the pseudo-spin precession of the exciton can be implemented through the exchange interaction. In what follows, we determine the operators $\vec{p}_{AB(BA)}$, $\vec{\sigma}_{BA(AB)}$ and describe effects of pseudo-spin and valley current coupling.

3 Effects of Coupling between Pseudo-Spin and Valley Current

In quasi-relativistic approximation (c^{-1} expansion) the eigenproblem for the equation of motion for the secondary quantized field $\widehat{\chi}^{\dagger}_{-\sigma_A}$ in the model shown in Fig. 1a has the form [20, 21]

$$\left\{\vec{\sigma}\cdot\vec{p}\;\hat{v}_F^{qu}-\frac{1}{c}\left(i\Sigma^x_{rel}\right)_{AB}\left(i\Sigma^x_{rel}\right)_{BA}\right\}\widehat{\chi^{\dagger}_{-\sigma_A}}(\vec{r})\,|0,-\sigma\rangle = E_{qu}(p)\widehat{\chi^{\dagger}_{-\sigma_A}}(\vec{r})\,|0,-\sigma\rangle\,. \tag{7}$$

Here a Fermi velocity operator $\hat{v}_F^{qu}$ is defined as

$$\hat{v}_F^{qu}=\left[\left(\Sigma^x_{rel}\right)_{BA}+c\hbar\vec{\sigma}\cdot(\vec{K}_A+\vec{K}_B)\right],$$

$\left(\Sigma^x_{rel}\right)_{BA}$, $\left(\Sigma^x_{rel}\right)_{AB}$ are determined through an ordinary exchange interaction contribution, for example:

$$\left(\Sigma^x_{rel}\right)_{AB}\widehat{\chi}^{\dagger}_{\sigma_B}(\vec{r})\,|0,\sigma\rangle=\sum_{i=1}^{N_v\,N}\int d\vec{r}_i\widehat{\chi}^{\dagger}_{\sigma_i{}_B}(\vec{r})\,|0,\sigma\rangle$$
$$\times\langle 0,-\sigma_i|\widehat{\chi}^{\dagger}_{-\sigma_i^A}(\vec{r}_i)V(\vec{r}_i-\vec{r})\widehat{\chi}_{-\sigma_B}(\vec{r}_i)|0,-\sigma_{i'}\rangle,$$

$N_v\,N$ is a total number of electrons in the system, N_v is a number of valence electrons. After applying to the wave function a non-unitary transformation as

$$\widehat{\tilde{\chi}^{\dagger}_{-\sigma_A}}\,|0,-\sigma\rangle=\left(\Sigma^x_{rel}\right)_{BA}\widehat{\chi^{\dagger}_{-\sigma_A}}\,|0,-\sigma\rangle\,.$$

we obtain (neglecting mixing of the states for the Dirac points) the equation similar to 2D QFT:

$$\left\{\vec{\sigma}^{AB}_{2d}\cdot\vec{p}_{BA}-c^{-1}\widetilde{\Sigma_{BA}\Sigma_{AB}}\right\}\widehat{\tilde{\chi}^{\dagger}_{-\sigma_A}}(\vec{r})\,|0,-\sigma\rangle=\tilde{E}_{qu}(p)\widehat{\tilde{\chi}^{\dagger}_{-\sigma_A}}(\vec{r})\,|0,-\sigma\rangle \tag{8}$$

with $\vec{p}_{BA}\widehat{\tilde{\chi}^{\dagger}_{-\sigma_A}}=\left(\Sigma^x_{rel}\right)_{BA}\vec{p}\,\left(\Sigma^x_{rel}\right)^{-1}_{BA}\widehat{\tilde{\chi}^{\dagger}_{-\sigma_A}}$, $\tilde{E}_{qu}=E_{qu}/\hat{v}_F^{BA}$, $\hat{v}_F^{BA}=\left(\Sigma^x_{rel}\right)_{BA}$, $c^{-1}\widetilde{\Sigma_{BA}\Sigma_{AB}}$ is a mass term.

The action of the transformation in the momentum-energy space is shown in Fig. 1b. As $\left(\Sigma^x_{rel}\right)_{BA}\neq\left(\Sigma^x_{rel}\right)_{AB}$, the vector $\vec{p}_{BA}$ is rotated in respect to $\vec{p}_{AB}$ and stretched. According to Fig. 1b, ellipses in momentum spaces of electrons and holes are rotated on 90° in respect

to each others. Therefore, the sequence of exchange interactions $(\Sigma^x_{rel})_{AB}(\Sigma^x_{rel})_{BA}(\Sigma^x_{rel})_{AB}$ for valley currents makes rotation initially the electron Brillouin zone and Dirac band into the hole Brillouin zone and Dirac band and then vice-versa. Thus, the exchange $(\Sigma^x_{rel})_{AB(AB)} \equiv \Sigma_{AB(BA)}$ changes the sublattices wave functions:

$$|\psi_{AB}\rangle = \Sigma_{AB}\,|\psi^*_{BA}\rangle\,.$$

Owing to it and neglecting a very small mass term $c^{-1}\widetilde{\Sigma_{BA}\Sigma_{AB}}$, the equation in which the operator Fermi velocity enters, can be rewritten as follows:

$$\vec{\sigma}^{BA}_{2D}\cdot\vec{p}_{AB}\,|\psi_{AB}\rangle = E_{qu}\,|\psi^*_{BA}\rangle\,. \tag{9}$$

Now, one can really see that the obtained equation is an equation of motion for a Majorana bispinor wave function of the graphene charge carriers. One has $\Sigma^2_{AB} = \Sigma_{AB}$ because $|\psi^*_{BA}\rangle = \Sigma_{BA}\,|\psi_{AB}\rangle$ and $|\psi^*_{BA}\rangle = \Sigma_{BA}\left|\tilde{\psi}_A\right\rangle$. Therefore, a following identity holds up to normalization constant $\langle 0|\,\hat{v}_F\,|0\rangle$:

$$|\psi_{AB}\rangle = \Sigma_{AB}\frac{\Sigma_{BA}}{\langle 0|\,\hat{v}_F\,|0\rangle}\,|\psi_{AB}\rangle = \frac{1}{\langle 0|\,\hat{v}_F\,|0\rangle}\left\{\Sigma_{BA}\,|\psi_{AB}\rangle + [\Sigma_{AB},\Sigma_{BA}]\,|\psi_{AB}\rangle\right\} \tag{10}$$

where $[\cdot,\cdot]$ denotes commutator. Substituting (10) into the right-hand side of the equation (9) one gets the following approximation [22]:

$$\vec{\sigma}^{BA}_{2D}\cdot\vec{p}_{AB}\,|\psi_{AB}\rangle = \tilde{E}_{qu}\left\{1 + (\Delta\Sigma + [\Sigma_{AB},\Sigma_{BA}])\,/\,\langle 0|\,\hat{v}_F\,|0\rangle\right\}|\psi_{AB}\rangle \tag{11}$$

where $\Delta\Sigma = \Sigma_{BA} - \Sigma_{AB}$, $\tilde{E}_{qu} = E_{qu}/\,\langle 0|\,\hat{v}_F\,|0\rangle$.

Neglecting the correction entering in eq. (11) one obtains a band structure in zero-order approximation shown in Fig. 2.

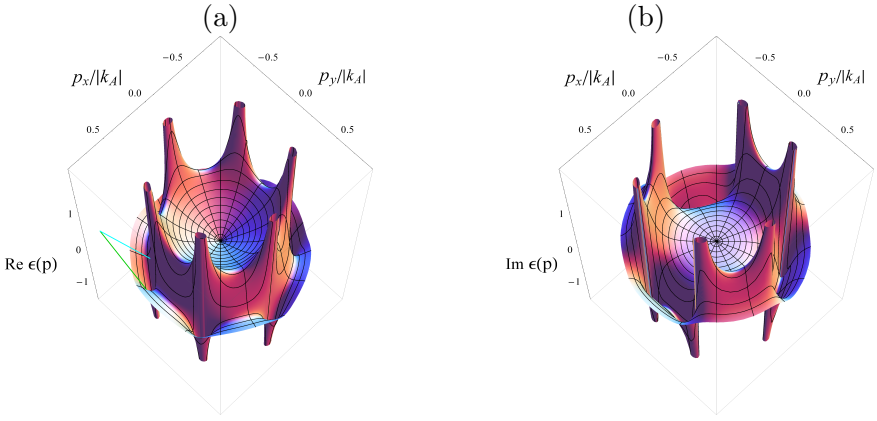


Figure 2: A spitting of Dirac cone replicas: real (a) and imaginary (b) parts. One of six pairs of Weyl-like nodes: source and sink is indicated by 2 rays (in color)

This zero-order approximation is a graphene model with a decay as the energy has a small imaginary part according simulations performed within this approximation.

But, the energy of the eigenproblem in the approximation (11) is real as Fig. 3 demonstrates $\Im m\ E_{qu} = 0$. In the vicinity of the Dirac points the degeneration of Dirac cone replicas is not removed, since the graphene bands remain Dirac cones in accordance with Fig. 4. Far from

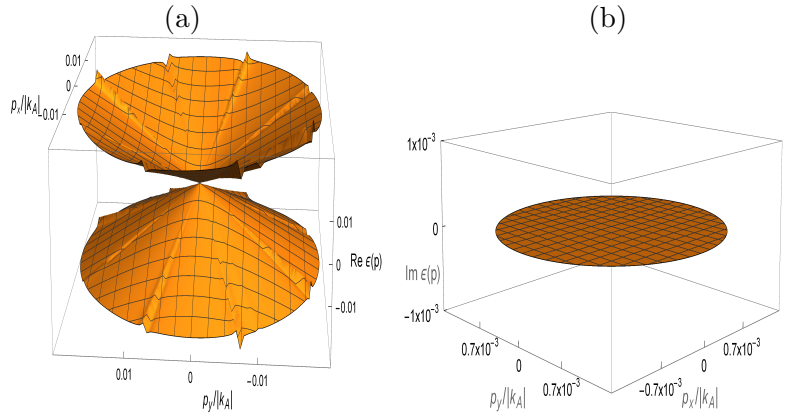


Figure 3: A band structure in the graphene model with partial unfolding of Dirac cone: real (a) and imaginary (b) parts; range of high momenta (in color)

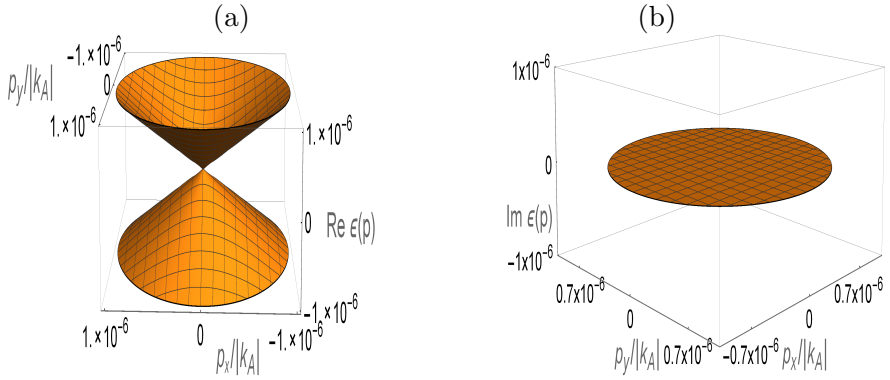


Figure 4: Band structure in the graphene model with partial unfolding of Dirac cone: real (a) and imaginary (b) parts; range of small momenta (in color)

the apexes of Weyl-like cones, Weyl nodes begin to overlapped forming the so-called Fermi-arcs Large number of Fermi-arcs is observed far from the apexes of Weyl cones in a contour plot of energy bands of graphene. When approaching to Weyl apexes the density of Fermi arcs increases rapidly as it is demonstrated in Fig. 5.

Accordingly to Fig. 2a, there appear hyperbolic points in the band structure of the graphene model. This, together with the fact that the Dirac cone base is rotated in respect of the replica's base on 30° due to hexagonal symmetry of the lattice, leads to destabilization of an electron (hole) configuration state in forbidden lattice sites. Such instability explains a phenomenon of commensurability - incommensurability lattices in heterostuctures graphene/ hexagonal boron [23].

4 Low Frequency Conductivity of Monolayer Graphene

To calculate complex conductivity one can use either phenomenological approach based on Boltzmann equation (see detail e.g., in [9, 11]) or Kubo formalism based on linear response theory [14, 13, 12]. We will use the last one adapted to the developed model and shortly outline the necessary steps to be performed.

To account of interaction of quasi-particle excitations in graphene with electromagnetic field the equation (8) should be modified by change for generalized momentum operator $\vec{p}_{AB} \rightarrow$

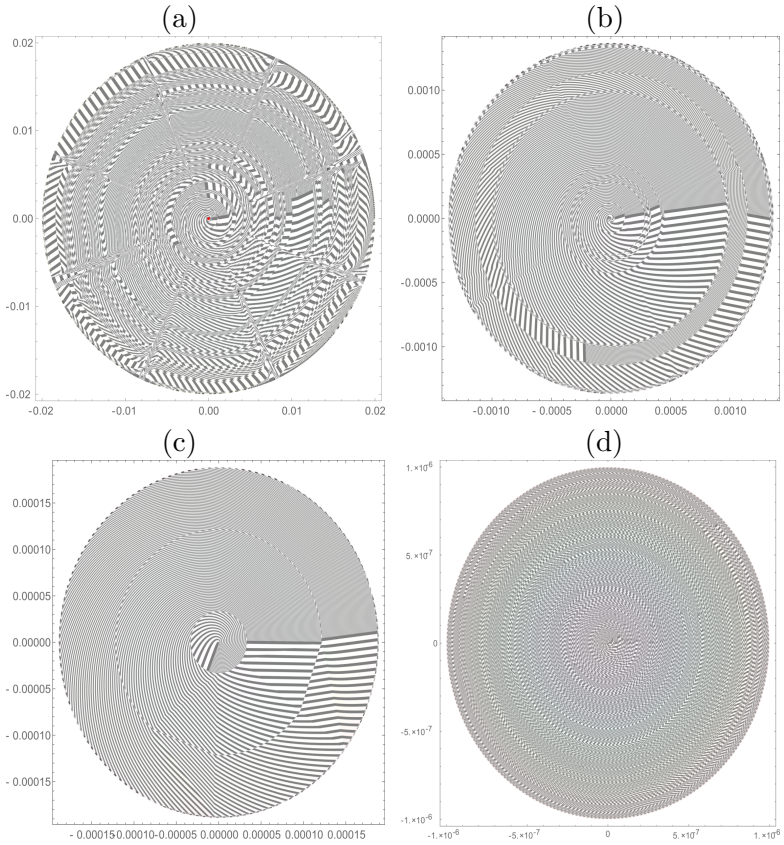


Figure 5: Density of Fermi arcs at different distance from the Weyl nodes apexes for momentum k= 10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}.

$\vec{p}_{AB} - \frac{e}{c}\vec{A}$.

Then one can follow [13] finding the operator of current in interaction representation, accounting linear in vector potential $\vec{A}$ terms only and performing averaging over the Gibbs ensemble. In this way a formula for Fourier-Laplace image of complex conductivity can be obtained which includes an ohmic term and contribution due to the process of pairs production (Zitterbewegung) and magneto-optic effects. The last two contribution will be the topic of subsequent publications. And for ohmic term we obtain the formula similar to that of [24]:

$$\sigma_{ij}(\omega,k) = \frac{ie^2}{\pi^2}\left\{\sum_{a=1,2}\int \frac{v^i{}_{AB(BA)}v^j{}_{AB(BA)}\left\{f\left[\varepsilon_a(p_-)\right]-f\left[\varepsilon_a(p_+)\right]\right\}}{\left[\varepsilon_a(p_+)-\varepsilon_a(p_-)\right]\left[\omega-\varepsilon_a(p_+)+\varepsilon_a(p_-)\right]}\,d^2p\right.$$
$$\left. +2\omega\int\frac{v^i_{12}v^j_{21}\left\{f\left[\varepsilon_1(p_-)\right]-f\left[\varepsilon_2(p_+)\right]\right\}}{\left[\varepsilon_2(p_+)-\varepsilon_1(p_-)\right]\left[\omega^2-\left(\varepsilon_2(p_+)-\varepsilon_1(p_-)\right)^2\right]}d^2p.\right. \tag{12}$$

Here $\varepsilon_a(p_\pm) = \Re e\ E_{qu}(p_\pm)_a$, $a = 1,2$ are eigenvalues of eq.(11) in zero-order approximation; $f(\varepsilon_a)$, $a = 1,2$ is a quasi-particle Fermi-Dirac distribution function; a-th bands are the valence ($a = 1$) and conductivity ($a = 2$) bands, $i(j) = x, y$, $\vec{p}_\pm = \vec{p} \pm \vec{k}$,

$$\vec{v}_{AB(BA)} = v_F\frac{\partial\left[\vec{\sigma}^{BA(AB)}_{2D}\cdot\vec{p}_{AB(BA)}\right]}{\partial\vec{p}},$$

v_F is the Fermi velocity. The last formula is velocity operator definition in the proposed model.

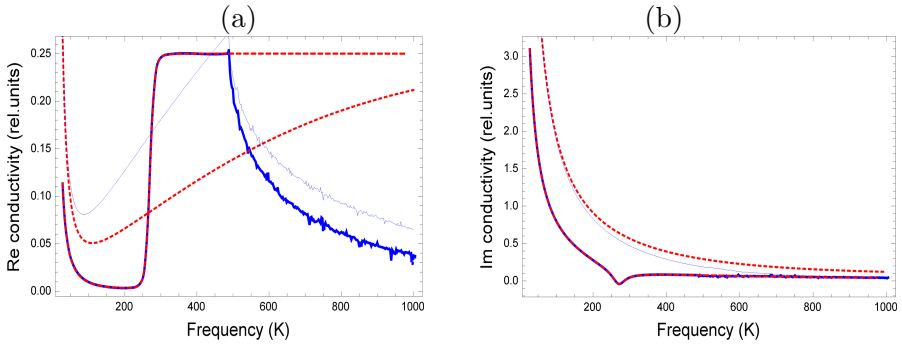


Figure 6: Real (a) and imaginary (b) parts of conductivity in $e^2/\hbar$ for doped graphene in the models $N = 3$ (thick and thin solid blue curves) and in a model $N = 2$ [25] (dashed and dotted red curves). The chemical potentials equal to 135 and 33 K at temperatures 3 (thick solid and dashed curves) and 200 K (thin solid and dotted curves), correspondingly (in color)

Let us examine the case of small spatial dispersion $k \to 0$ in zero-order approximation. The approximation $k \to 0$ describes optical conductivity [25, 24] and accordingly Fig. 6 is valid in frequency range above 300 K. Optical conductivities of doped graphene monolayer in models $N = 2$ and $N = 3$ are coincided except of a phenomenon of negative differential conductivity (negative slope in the plot) which is absent in the model $N = 2$. Frequency dependencies of conductivity for pure graphene are noticeably distinguished for the models $N = 2$ and $N = 3$. In the absence of doping, the model $N = 2$ predicts the conductivity value practically diminished by factor two. Increase of the conductivity as well as conductivity oscillations at high frequencies in the model $N = 3$ are stipulated by the Dirac cone replicas splitting in Fig. 2. The observed consequences of the splitting in Figs. 6a,b are similar with those related to appearance of Weyl node and antinode. The existence of the last also leads to oscillatory behavior [26, 27, 28, 29, 30, 31]. However, the equation (11) is an approximation of the equation of the Majorana type and the similarity is stipulated by the imaginary additive to the energy in Fig. 2, and, hence, by a chosen approximation allowing the decay of exciton pairs. Imaginary part of non-doped graphene conductivity for the model $N = 2$ is positive in accordance with Fig. 6b, that is the consequence of the absence of charge carriers at the Fermi level. Opposite to that the oscillations of the conductivity real part due to appearance of Weyl-like node pairs proceed at the background of trending to zero of the conductivity imaginary part. Hence, plasma oscillations take place in the sufficiently high value of the frequency and, respectively, the pure graphene is able to screen in optical range. Let us consider an influence of spatial dispersion on longitudinal conductivity at low frequencies (13.3 K) in Fig. 7. In the model $N = 2$ with spatial dispersion the dielectric permeability of non-doped graphene is positive as well as in the limit $k \to 0$, and, therefore the charge carrier density at Fermi level is zero. Contrary to that in the model $N = 3$ with the spatial dispersion there exist plasma modes at low frequency as dielectric permeability can gain zero and negative values. Hence, the pure graphene is able to screen in electrophysical range.

5 Conclusion

Summarizing the finding, partial unfolding of Dirac bands proceeds from coupling between anti-ordered pseudo-spins and valley currents. Accounting of the pseudo-spin and valley currents coupling turns out to be important for the description of optical and electrophysical properties of monolayer graphene. Splitting of Dirac cone replicas on Weyl-like node and anti-node leads

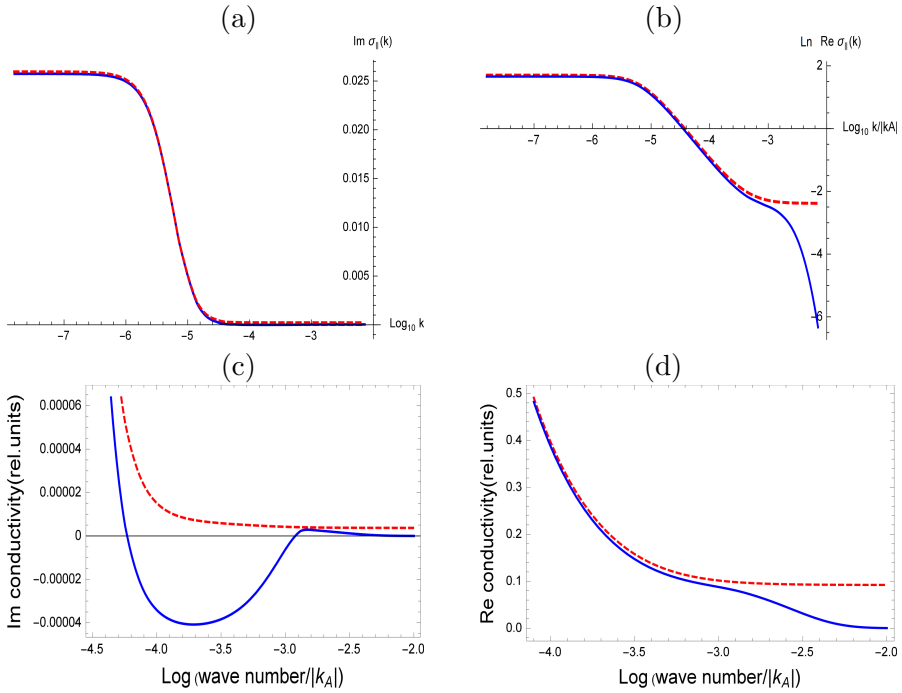


Figure 7: Imaginary (a) and real (b) parts of longitudinal conductivity vs Logarithm of wave number normalized on wave number distance $|k_A|$ from the Γ point to the Dirac one in the Brillouin zone in the models $N = 3$ (solid blue curves) and $N = 2$ [25] (dashed red curves). The chemical potential equals to 1 K at temperature 100 K and frequency 13.3 K. Insets (c) and (d) demonstrate the behaviour of longitudinal conductivity in the region of high wave numbers (in color)

to polarization effects similar to graphene doping.

References

[1] K.S. Novoselov, D. Jiang, F. Schedin, T.J. Booth, V.V. Khotkevich, S.V. Morozov, A.K. Geim, PNAS. **102**, 10451 (2005)

[2] G. Fiori, F. Bonaccorso, G. Iannaccone, T. Palacios, D. Neumaier, A. Seabaugh, S.K. Banerjee, L. Colombo, Nature Nanotechnology **9**, 768 (2014)

[3] S. Tarasenko, Nature Nanotechnology **9**, 752 (2014)

[4] Jing-Rong Wang, Guo-Zhu Liu , J. Phys.: Condens. Matter **23**, 155602 (2011)

[5] G.W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984)

[6] P. R. Wallace. Phys.Rev. **71**, 622 (1947).

[7] S. Reich, J. Maultzsch, C. Thomsen, Phys. Rev. B. **66**, 035412 (2002)

[8] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, Y. Zhang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos, A.A. Firsov, Nature **438**, 197 (2005).

[9] T. Ando, Y. Zheng and H. Suzuura, J. Phys. Soc. Jpn. **71**, 1318 (2002)

[10] S. Adam, E. H. Hwang, V. M. Galitski, and S. Das Sarma PNAS, **104** 18392 (2007)

[11] A.H. Castro Neto, F. Guinea, N.M. Peres, K.S. Novoselov, A.K. Geim, Rev. Mod. Phys. **81**, 109-162 (2009).

[12] N M R Peres J. Phys.: Condens. Matter **21**, 323201 (2009)

[13] L.A. Falkovsky, A.A. Varlamov, Eur. Phys. J. B. **56**, 281-284 (2007).

[14] V.P. Gusynin, S.G. Sharapov, J.P. Carbotte, Int. J. Mod. Phys. B. **21**, 4611 (2007)

[15] I. Fialkovsky, D.V. Vassilevich, European Phys. J. B **85**, 384 (2012)

[16] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory* (Addison–Wesley Publishing Company, 1995).

[17] V.N. Gribov, *Quantum Electrodynamics* (R & C Dynamics, Izhevsk, 2001)

[18] V.A. Fock, *Principles of quantum mhechanics.* (Science, Moscow, 1976)

[19] H. Krylova, L. Hursky, *Spin polarization in strong-correlated nanosystems* (LAP LAMBERT Academic Publishing, AV Akademikerverlag GmbH & Co., Germany, 2013).

[20] H.V. Grushevskaya, G.G. Krylov, Int. J. Nonlin.Phen.Com.Sys. **16**, 189 (2013)

[21] H.V. Grushevskaya, G.G. Krylov, in: *Nanotechnology in the Security Systems, NATO Science for Peace and Security Series C: Environmental Security*, ed. by J. Bonča, S. Kruchinin (Springer, Dordrecht, 2015), ch. 3

[22] H. V. Grushevskaya, G. Krylov, J. Nonlin. Phenom. in Complex Sys. **18**, 266 (2015)

[23] C. R. Woods, L. Britnell, A. Eckmann, R. S. Ma, J. C. Lu, H. M. Guo, X. Lin, G. L. Yu, Y. Cao, R. V. Gorbachev, A. V. Kretinin, J. Park, L. A. Ponomarenko, M. I. Katsnelson, Yu. N. Gornostyrev, K. Watanabe, T. Taniguchi, C. Casiraghi, H-J. Gao, A. K. Geim, K. S. Novoselov, Nature Physics **10**, 451 (2014)

[24] L.A. Falkovsky, Low Temp. Phys. **37**, 480 (2011).

[25] L.A. Falkovsky, J.Phys. CS **129**, 012004 (2008)

[26] L.V. Min, Zhang Shou-Cheng, Int. J. Mod. Phys. B **27**, 1350177 (2013).

[27] Pavan Hosur, Xiao-Liang Qi, Phys. Rev. B **91**, 081106 (2015)

[28] B. Rosenstein, B.Ya. Shapiro, D. Li, I. Shapiro, J. Phys.: Condensed Matter **27**, 025701 (2015)

[29] Jianhui Zhou, Hao-Ran Chang, Di Xiao , Phys. Rev. B **91**, 035114 (2015).

[30] R. Sachdeva, A. Thakur, G. Vignale, A. Agarwal, Phys. Rev. B **91**, 20 (2015)

[31] R. Lundgren, P. Laurell, G.A. Fiete, Phys. Rev. B **90**, 165115 (2014).

Интегральная гиперболическая динамика частиц в пространстве-времени Минковского

С.С. Кокарев
НИИ ГСГФ (Фрязино) — РНОЦ Логос (Ярославль)

Аннотация

На основе гиперболически сферически-симметричного решения волнового уравнения в пространстве-времени Минковского — пространственно-временного аналога закон Кулона — и принципа суперпозиции построено действие для системы взаимодействующих частиц. Показано, что соответствующие уравнения движения являются интегро-дифференциальными. Их запись в форме второго закона Ньютона в релятивистской форме выявляет динамическую природу массы: она получается как коллективный эффект части гиперболического взаимодействия рассматриваемой частицы с ее окружением (принцип Маха). Анализируются частные случаи гиперболического самодействия одиночной мировой линии и взаимодействия пары частиц с параллельными мировыми линиями.

1 Введение

Специальная теория относительности (СТО), построенная в начале XX века, послужила основой для нового понимания сущности пространства-времени и физических процессов, протекающих в нем. Одним из главных ингредиентов СТО является ее геометрическая интерпретация: в основе всей релятивистской физики лежит концепция 4-мерного пространства-времени Минковского $\mathcal{M}_{1,3}$ с псевдоевклидовой метрикой $(\eta) = \mathrm{diag}(1,-1,-1,-1)$. Геометрический язык СТО позволяет наиболее ясно и последовательно сформулировать ее сущность и основные положения, вывести всевозможные следствия и эффекты, проанализировать эксперименты в области релятивистской физики и сделать последовательный переход к общей теории относительности (ОТО). 4-мерный язык псевдоевклидовой геометрии уже давно стал обычным рабочим инструментом физиков. В прикладных и фундаментальных задачах он часто используется как удобный математический прием для ковариантной записи уравнений и соотношений. Между тем, СТО представляет собой не столько расширение и модификацию законов ньютоновской механики, сколько кардинально новый взгляд на физический мир — новую *релятивистскую парадигму*. Эта теория уже на уровне своих фундаментальных положений кардинально отличается от классической механики и вводит понятия, аналоги которых в нерелятивистской физике как правило, либо существенно отличаются, либо вообще отсутствуют. См. Таблицу:

Объекты-Свойства	Ньютон	Минковский
1. Пр-во событий	$E^1 \times E^3$ (евкл. время (1D) и пр-во (3D))	$\mathcal{M}_{1,3}$ — 4-мерное ПВ
2. Событие	Пара $(t,x) \in E^1 \times E^3$	радиус-вектор $X \in \mathcal{M}_{1,3}$
3. Метрика	η_1 для E^1, η_3 для E^3	η для $\mathcal{M}_{1,3}$
4. База топологии	евклидовы сферы	гиперболоиды (нехаусдорф.)
5. Свет	не выделен	изотропные векторы и конус
6. Временной порядок	абсолютный	относительный
7. Инварианты	Длина, форма, длительность,...	4-мерная длина
8. Элем. физ. объект	частица (мат. точка)	частица (мировая линия) — ?
9. Составные объекты	система частиц (тела)	4-тела (мировые трубки)
10. Динамика	отображение $E^1 \to E^3$ (эволюция)	статика в $\mathcal{M}_{1,3}$
11. Масса	количество материи	форма энергии
12. Законы сохранения	энергия и импульс	4-импульс
13. Основной оператор	Δ — оператор Лапласа	$\Box$ — оператор Даламбера

В качестве теоретической основы настоящей статьи лежит правая часть таблицы. Обоснование законов Ньютона в рамках концепции 4-мерной статики сильно напряженных струн (9-10 строки таблицы) были сделаны ранее в работе [1]. Чтобы продвигаться далее, следуя логике 4-мерной геометрии Минковского, необходимо отступить от традиционного толкования элементарной частицы (строка 8 и знак вопроса там): элементарным физическим объектом мы будем считать не частицу или, точнее, не мировую линию частицы, а истинный аналог материальной точки пространства $\mathcal{M}_{1,3}$ — метрическую сферу нулевого радиуса. В пространстве-времени ей соответствует световой конус с сосредоточенными на ней материальными характеристиками. Из общих соображений таким элементарным объектам-источникам соответствует пространственно-временное поле — 4-мерный гиперболический аналог кулоновского поля. Протяженные структуры типа мировых линий или мировых трубок, вытянутых во времениподобном направлении, получаются при выстраивании (конденсации) элементарных точек-событий в такие структуры, которое описывается в рамках некоторой обобщенной теории конденсированных сред в 4-мерном пространстве-времени[1]. При этом их коллективное поле соответствует наблюдаемым физическим полям, с которыми имеет дело стандартная классическая теория поля.

На самом деле, впервые гипотеза подобного поля была сделана в рамках псевдофинслерова пространства Бервальда-Моора, которое индуцируется алгеброй поличисел, аналогично тому, как евклидова геометрия на плоскости индуцируется алгеброй комплексных чисел [2]-[3]. В цитированных работах сделан набросок теории гиперболического поля в рамках пространств Бервальда-Моора и показано, что такие модели содержат в себе физику в пространстве-времени Минковского или даже в искривленных пространствах-временах ОТО. Любопытно однако, что некоторые существенные свойства гиперболических полей естественным образом переносятся в пространство-время Минковского и их обсуждение оказывается более простым без тех специфических особенностей, которые привносит в теорию финслерова геометрия.

2 Гиперболическое центрально-симметричное решение

В нашем рассмотрении мы будем отталкиваться от гиперболического аналога закона Кулона в пространстве Минковского. Этот аналог определяется как сферически-симметричное (в смысле псевдоевклидовой сферы) решение волнового уравнения в пустом пространстве-времени, окружающем центр гиперболической сферы:

$$\Box U = 0. \tag{1}$$

Напомним еще раз, что центром метрической гиперболической сферы в псевдоевклидовом пространстве Минковского с метрической точки зрения является не точка[2] $X_0 = (t_0, x_0, y_0, z_0)$, а множество точек — световой конус с вершиной в X_0, — удовлетворяющее уравнению:

$$\varrho^2(X, X_0) = (t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 = 0. \tag{2}$$

В дальнейшем мы везде полагаем $X_0 = 0$.

Напомним, что сферически-симметричное решение уравнения Лапласа в пустоте

$$\Delta\phi = 0 \tag{3}$$

[1] В такой теории будут присутствовать времениподобные силы и взаимодействия, не свойственные стандартной релятивистской физике.

[2] Здесь и далее, если не указано особо, мы работаем в системе единиц, в которой $c = 1$.

оказывается единственным (с точностью до константы) и, как показывает непосредственная проверка, содержит в себе и всю информацию о точечном источнике. Действительно, кулоновский потенциал $\phi = q/r$ как решение (3) на самом деле удовлетворяет уравнению

$$\Delta\phi = -4\pi q\delta(x)\delta(y)\delta(z) = -\frac{q\delta(r)}{r^2}. \tag{4}$$

во всем пространстве. В последнем равенстве был учтен закон преобразования дельта-функции при переходе к криволинейным координатам.

Аналогично случаю кулоновского поля, мы не задаемся вопросом о структуре источника в уравнении (1): сингулярные характеристики источника будут автоматически содержаться в самом решении. Для его получения перейдем к гиперболически сферически-симметричной 4-мерной системе координат с центром в нуле:

$$\begin{cases} t &= \varrho\cosh\chi; \\ x &= \varrho\sinh\chi\,\sin\theta\,\cos\varphi; \\ y &= \varrho\sinh\chi\,\sin\theta\,\sin\varphi; \\ z &= \varrho\sinh\chi\,\cos\theta. \end{cases} \tag{5}$$

Здесь ϱ — 4-радиус, χ — гиперболический угол, θ и φ — пара стандартных сферических углов. Формулы (5) справедливы для областей $t^2-x^2-y^2-z^2>0$. Метрика Минковского в этой системе координат получается с помощью обычных правил преобразования интервала и имеет вид:

$$ds^2 = d\varrho^2 - \varrho^2(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)). \tag{6}$$

В дифференциальной геометрии волновой оператор определяется инвариантным образом по формуле:

$$\Box \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial}{\partial\xi^\beta}\right), \tag{7}$$

где g — детерминант метрического тензора, $g^{\alpha\beta}$ — компоненты контравариантной метрики, матрица которой обратна к $(g_{\alpha\beta})$. Из (6) следует, что $g = -\varrho^6\sinh^4\chi\sin^2\theta$, а обратная метрика имеет вид:

$$(g^{\alpha\beta}) = \mathrm{diag}\left(1, -\frac{1}{\varrho^2}, -\frac{1}{\varrho^2\sinh^2\chi}, -\frac{1}{\varrho^2\sinh^2\chi\sin^2\theta}\right). \tag{8}$$

Подставляя эти выражения в общую формулу (7), приходим к выражению для волнового оператора в 4-мерной сферической системе координат:

$$\Box = \frac{1}{\varrho^3}\frac{\partial}{\partial\varrho}\left(\varrho^3\frac{\partial}{\partial\varrho}\right) - \frac{1}{\varrho^2}\Delta_{\chi,\theta,\varphi}, \tag{9}$$

где введено обозначение для угловой части волнового оператора[3]:

$$\Delta_{\chi,\theta,\varphi} \equiv \frac{1}{\sinh^2\chi}\left(\frac{\partial}{\partial\chi}\sinh^2\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{10}$$

Подставляя общий вид сферически-симметричного решения $U = U(\varrho)$ в оператор (9), приходим к уравнению:

$$\Box U = \frac{1}{\varrho^3}\frac{\partial}{\partial\varrho}\left(\varrho^3\frac{\partial U(\rho)}{\partial\varrho}\right) = 0. \tag{11}$$

[3]При малых гиперболических углах $\sinh\chi \approx \chi$ и выражение для $\Delta_{\chi,\theta,\varphi}$ переходит в оператор Лапласа в 3-мерной сферической системе координат с $r = \chi$.

Его общее решение имеет вид:

$$U(\rho) = \frac{\mathcal{Q}}{\varrho^2} + C, \tag{12}$$

где $\mathcal{Q}$ и C — константы интегрирования. Будем рассматривать полученное решение как аналог фундаментального решения для гиперболического поля, источником которого являются *материальные события* — конус с распределенной характеристикой $\mathcal{Q}$, которую мы будем называть *гиперболическим зарядом*. Вариант физической интерпретации гиперболического заряда мы дадим чуть позже. Непосредственной проверкой можно убедиться, что полученное решение удовлетворяет 4-мерному аналогу уравнения[4] (4):

$$\Box U = -\frac{2\mathcal{Q}}{\varrho^3}\delta(\varrho). \tag{13}$$

Выше мы употребили термин "аналог фундаментального решения", поскольку в отличие от классического фундаментального решения математической физики, особенность (12) сосредоточена не в точке, а на световом конусе. Будем в дальнейшем называть решение (12) *гиперболическим фундаментальным решением* волнового уравнения, в отличие от хорошо известного фундаментального решения (причинной функции Грина) классической теории:

$$G = \theta(t)\frac{\delta(t-r)}{4\pi r}, \tag{14}$$

где $\theta(t)$ — ступенчатая функция Хевисайда. Нетрудно убедиться[5], что решение (12) удовлетворяет уравнению (13) в обобщенном смысле во всех причинных областях.

3 Гиперболическая вселенная

В этом разделе мы рассмотрим общую теорию системы частиц, взаимодействующих посредством пространственно-временного потенциала (12). Мы предполагаем, что частицы характеризуются протяженными времениподобными мировыми линиями конечной или бесконечной длины, и взаимодействуют посредством универсального гиперболического закона Кулона (12), распространенного на протяженные источники с помощью принципа суперпозиции, аналогично тому, как это делается в электростатике.

Запишем гиперболическое действие системы N частиц, мировые линии которых имеют непрерывно распределенный гиперболический заряд:

$$\mathcal{S}_N \equiv \mathcal{S}[X_1(\tau_1), \ldots, X_N(\tau_N)] = A\sum_{i\le j=1}^{N}\int\limits_{\Gamma_i}\int\limits_{\Gamma_j}\frac{\lambda_i(\tau_i)\lambda_j(\tau_j)}{\varrho_{ij}^2}ds_i ds_j, \tag{15}$$

где $\varrho_{ij} = (t(\tau_i) - t(\tau_j))^2 - (\vec{r}(\tau_i) - \vec{r}(\tau_j))^2$ — 4-интервал между элементами мировых линий ds_i и ds_j, $ds_i = \sqrt{\dot{X}_i\dot{X}_i}\,d\tau_i$, τ_i — параметр на мировой линии i-ой частицы, λ_i — линейная плотность гиперболического заряда i-ой мировой линии. Отметим, что действие (15) по существу является действием взаимодействия (включая также и самодействие мировых линий) в пространстве-времени.

[4]Отметим, что правую часть в (13) технически проще сразу записать в сферически симметричной системе координат, поскольку в такой форме она не содержит бесконечного множителя Ω_H — аналога множителя 4π в (4) определяющего меру множества всех направлений в $\mathcal{M}_{1,3}$.

[5]По существу, в процессе выкладок требуется два соотношения из теории обобщенных функций: $d\theta(\pm x)/dx = \pm\delta(x)$.

Чтобы компактно записать уравнения движения:

$$\delta_{X_i}\mathcal{S} = 0, \tag{16}$$

получаемые посредством вариационной процедуры, введем следующие полезные обозначения. Определим *дифференциальный вектор влияния* j-ого элемента на i-ый (в натуральной параметризации):

$$\overrightarrow{Q}_{ij}(s_i, s_j) \equiv \epsilon_{ij}\left(\frac{2\lambda_i(s_i)\lambda_j(s_j)}{\varrho_{ij}^4(s_i, s_j)}(\overrightarrow{X}_i(s_i) - \overrightarrow{X}_j(s_j)) + \frac{d}{ds_i}\left(\frac{\lambda_i(s_i)\lambda_j(s_j)}{\varrho_{ij}^2(s_i, s_j)}\dot{\overrightarrow{X}}_i(s_i)\right)\right), \tag{17}$$

где

$$\epsilon_{ij} = \begin{cases} 1, & i \neq j; \\ 2, & i = j \end{cases}$$

— фактор, учитывающий двойной вклад самодействия элементов мировой линии. Величина $\overrightarrow{Q}_{ij}(s_i, s_j)$ характеризует влияние элемента ds_j на элемент ds_i, отнесенных к положениям s_i и s_j на мировых линиях Γ_i и Γ_j соответственно. Отметим, что $\overrightarrow{Q}_{ij} \neq \overrightarrow{Q}_{ji}$. Далее определим *интегральный вектор влияния* j-ой мировой линии на элемент i-ой:

$$\overrightarrow{Q}_{ij}^R(s_i) \equiv \int\limits_{\Gamma_j} \overrightarrow{Q}_{ij}(s_i, s_j)\, ds_j. \tag{18}$$

Величина $\overrightarrow{Q}_{ij}^R(s_i)$ по своему определению зависит от индексов i, j и переменной s_i, но теперь уже не зависит от переменной s_j. Определим, наконец, *интегральный вектор влияния* окружения на i-ую мировую линию:

$$\langle\overrightarrow{Q}_i^R\rangle(s_i) \equiv \sum_{j=1}^{N} \overrightarrow{Q}_{ij}^R(s_i). \tag{19}$$

Тогда непосредственной проверкой можно убедиться, что имеет место логическая эквивалентность:

$$\delta_{X_i}\mathcal{S} = 0 \Leftrightarrow \langle\overrightarrow{Q}_i^R\rangle(s_i) = 0, \quad i = 1, \ldots, N. \tag{20}$$

Уравнения движения в форме (20) представляют собой систему нелинейных интегро-дифференциальных уравнений, определяющих полную историю частиц. Для прояснения физического смысла уравнений (20) и выявления их связи с классической ньютоновской формой уравнений механики следует подробно расписать производную в правой части определения (17) и выделить коэффициенты при производных $\dddot{\overrightarrow{X}}_i, \ddot{\overrightarrow{X}}_i$ и свободную часть. Выполнение этой процедуры приводит к удобным определениям: *мгновенной массы* (индуцированной окружением):

$$\mathcal{M}_i(s_i) \equiv \sum_{j=1}^{N} \mathcal{M}_{ij}, \quad \mathcal{M}_{ij}(s_i) \equiv \epsilon_{ij}\int\limits_{\Gamma_j} \frac{\lambda_j(s_j)}{\varrho^2(s_i, s_j)} ds_j, \tag{21}$$

и *мгновенной равнодействующей силы*:

$$\overrightarrow{\mathcal{F}}_i(s_i) \equiv \sum_{j=1}^{N} \overrightarrow{\mathcal{F}}_{ij}, \quad \overrightarrow{\mathcal{F}}_{ij}(s_i) \equiv -2\epsilon_{ij}\hat{\mathcal{P}}_i \cdot \int\limits_{\Gamma_j} \frac{\lambda_j(s_j)}{\varrho^4(s_i, s_j)}(\overrightarrow{X}_i - \overrightarrow{X}_j) ds_j, \tag{22}$$

где $\hat{\mathcal{P}}_i \equiv (\hat{\mathrm{Id}} - \overrightarrow{X}_i \otimes \overleftarrow{X}_i)$ — оператор проектирования векторов и форм на 3-пространство $\overrightarrow{X}_i^{\perp}(s_i)$. С учетом этих определений, мы приходим к эквивалентной (квазиньютоновской) форме уравнений (20):

$$\mathcal{M}_i \ddot{\overrightarrow{X}}_i = \overrightarrow{\mathcal{F}}_i - \frac{d \ln \lambda_i}{ds_i} \mathcal{M}_i \dot{\overrightarrow{X}}_i. \tag{23}$$

Проектируя это уравнение на $\dot{\overrightarrow{X}}_i$ с учетом соотношений $\overrightarrow{\mathcal{F}}_i \cdot \dot{\overrightarrow{X}}_i = 0$ и $\ddot{\overrightarrow{X}}_i \cdot \dot{\overrightarrow{X}}_i = 0$ приходим к соотношениям:

$$\lambda_i = \mathrm{const}_i \tag{24}$$

выражающим закон сохранения линейной плотности гиперболического заряда и приводящим к ньютоновским уравнениям динамики:

$$\mathcal{M}_i \ddot{\overrightarrow{X}}_i = \overrightarrow{\mathcal{F}}_i, \quad i = 1, \ldots, N. \tag{25}$$

Сделаем несколько замечаний по поводу уравнений (20) и (25).

1. На языке векторов влияния закон движения частицы можно выразить следующим образом: *средний по всем частицам вектор влияния, отнесенный к любой частице системы, равен нулю.* Очевидно, что вселенная, состоящая из N частиц рассматриваемого типа, имеет существенно самосогласованный характер, при этом процедура самосогласования имеет при $N \gg 1$ явно статистическую природу.

2. Исходное действие (15) было чистым действием гиперболического взаимодействия, в то время как уравнения (23) имеют вид уравнений движения частицы, выведенных из традиционного принципа наименьшего действия, содержащего специфические кинетический и потенциальный члены. Другими словами, рассматриваемый подход позволяет интерпретировать кинетическую энергию частиц как специфическую форму (разновидность) потенциальной энергии пространственно-временного взаимодействия.

3. Теория, основанная на действии (15), проливает новый свет на физическую природу массы. В соответствии с формулами (21) масса является интегральным результатом части гиперболического взаимодействия, зависящим от всей истории движения окружающих частиц (принцип Маха). В соответствии с выражением (21) следует ожидать, что в динамике пары частиц их массы могут заметно флуктуировать. Стационарные массы получаются, когда мы рассматриваем ансамбль большого числа частиц, в котором массовая часть гиперболического взаимодействия усредняется и флуктуации сглаживаются. Такое усреднение подразумевает решение полной самосогласованной задачи о статике ансамбля гиперболически взаимодействующих мировых нитей, что представляет собой технически сложную задачу.

4. Сохранение линейной плотности гиперболического заряда λ вдоль мировой линии у каждой из частиц, как это будет показано далее, эквивалентно закону сохранения обобщенного 3-мерного заряда (массы и (или) электрического заряда). Из уравнения (23) ясно, что несохранение этой величины на ньютоновском языке эквивалентно наличию времениподобной "силы трения" . Такого рода силы отсутствуют в релятивистской физике. Наличие подобной силы можно было бы интерпретировать как изменение темпа хода собственного времени [3].

5. Интересно, что граничные условия для уравнений (20), получаемые в процессе их вывода посредством вариационной процедуры, с помощью (21) можно записать на языке масс или импульсов на концах A и B мировых линий:

$$\delta\overrightarrow{X}_A \cdot \overrightarrow{P}(A) = \delta\overrightarrow{X}_B \cdot \overrightarrow{P}(B), \tag{26}$$

где $\overrightarrow{P} = \mathcal{M}\dot{\overrightarrow{X}}$ — 4-импульс. Помимо стандартных граничных условий $\delta\overrightarrow{X}_A = \delta\overrightarrow{X}_B = 0$, используемых в теоретической механике, соотношения (26) допускают произвольные пространственно-подобные вариации концов мировых линий (релятивистский аналог скользящих концов).

6. В динамике, выведенной на основе действия (15), вообще говоря не выполняется соотношение $\overrightarrow{\mathcal{F}}_{ij} = -\overrightarrow{\mathcal{F}}_{ji}$ (третий закон Ньютона), хотя оно может выполняться при специальных предположениях относительно распределения и динамики частиц в пространстве-времени. Как следствие этого обстоятельства, в общем случае отсутствует и аналог закона сохранения полного импульса системы и, следовательно, возможно самоускорение. Отметим, что отсутствие самоускорения и сохранение полного импульса $\overrightarrow{P} = \sum\limits_{i=1}^{N} \overrightarrow{P}_i$ является более слабым ограничением на систему, чем выполнение в ней 3-его закона Ньютона [4].

4 Вселенная одной частицы

Рассмотрим самый простой случай динамики частиц, рассмотренной в предыдущем разделе, при $N = 1$. Единственное интегро-дифференциальное уравнение, определяющее глобальное равновесие этой мировой линии, сводится к равенству нулю интегрального вектора самодействия:

$$\langle \overrightarrow{Q}_1^R \rangle = \int\limits_{\Gamma_1} \overrightarrow{Q}_{11}\, ds_1 = 0. \tag{27}$$

Уравнение (27) вообще говоря не исключает существование многократных временных петель и нетривиального запутывания мировой линии в пространстве-времени. Для более наглядного представления характера уравнений и потенциальных проблем, связанных с его решением, рассмотрим частный случай одномерного движения, для которого закон представляется 4-мерным радиус вектором следующего специального вида: $\overrightarrow{X} = (t, \varphi(t), 0, 0)$, где $\varphi(t)$ — некоторая гладкая функция, характеризующая пространственный закон движения в координатной параметризации. Для этого случая уравнения (27) сводятся к одному единственному независимому интегро-дифференциальному уравнению для $\varphi(t)$ вида:

$$2\int\limits_{T_1}^{T_2} \frac{(t_1 - t_2)D_2}{\varrho^4}\, dt_2 + \frac{\dot\varphi_1\ddot\varphi_1}{D_1^4}\int\limits_{T_1}^{T_2} \frac{D_2}{\varrho^2}\, dt_2 + \frac{1}{D_1^2}\frac{\partial}{\partial t_1}\int\limits_{T_1}^{T_2} \frac{D_2}{\varrho^2}\, dt_2 = 0, \tag{28}$$

где $\varphi_i \equiv \varphi(t_i)$; $\quad D_i \equiv \sqrt{1 - \dot\varphi_i^2}$; $\quad \varrho^2 \equiv (t_1 - t_2)^2 - (\varphi_1 - \varphi_2)^2$. Ввиду сильной нелинейности уравнения (28) и отсутствия регулярных методов его анализа и решения, вопрос об отыскании общего решения и условиях его единственности остается открытым. Между тем, вопрос о существовании частного решения имеет положительный ответ: непосредственной проверкой нетрудно убедиться, что при $\varphi(t) = x_0 + vt$ величина $Q_{11} \equiv 0$. Другими словами, *инерциальные движения являются решениями уравнений (28) в сильном смысле, так как*

на них обращается в нуль дифференциальный вектор самодействия, а не его среднее значение. Этот результат, вполне естественный и ожидаемый с позиций законов классической механики Ньютона, с одной стороны обнаруживает факт существования решений уравнения (28), но не закрывает вопроса о наличии других движений, удовлетворяющих этому уравнению (и тем более общему уравнению (27)), которые мы в рассматриваемой картине также с полным основанием могли бы называть инерциальными.

Простой анализ обнаруживает, что нерелятивистское ускоренное движение с $\varphi(t) = x_0 + vt + at^2/2$ является решением в сильном смысле с точностью до слагаемых $O(a^2)$, и в слабом с точностью до $O(a^3)$, если все встречающиеся в вычислениях расходящиеся интегралы понимать в смысле их главного значения.

Остановимся более подробно на физическом смысле параметров модели для случае прямолинейной мировой линии, которую всегда можно интерпретировать как мировую линию покоящейся частицы за счет надлежащего выбора системы отсчета. Оказывается, что самодействие такой мировой линии самосогласованно определяет некоторые ее важнейшие характеристики! Действительно энергию самодействия мировой линии длины T можно представить в виде:

$$U_{\text{self-act}} = \frac{1}{2}\int\limits_{-T/2}^{T/2}\int\limits_{-T/2}^{T/2}\frac{\lambda ds_1 \cdot \lambda ds_2}{(s_1 - s_2)^2} = \frac{\lambda^2}{2}\int\limits_{Q_T(0)}\frac{ds_1 \wedge ds_2}{(s_1 - s_2)^2} \tag{29}$$

Последнее выражение представляет результат самодействия как интеграл от 2-формы по квадрату со стороной T, причем 2-форма имеет сингулярность на диагонали $s_1 = s_2$ (самодействие одиночных элементов мировой линии). Для придания физического смысла расходящемуся интегралу, сделаем его регуляризацию, схема которой иллюстрируется на рис. 1.

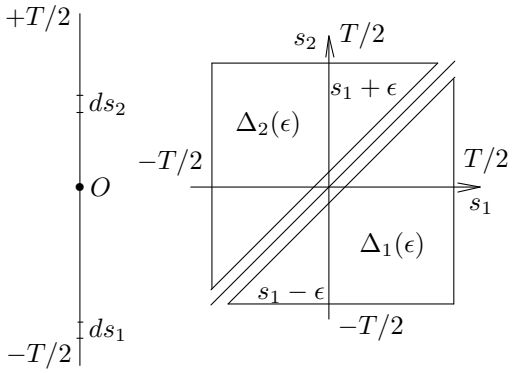


Рис. 1. Самодействие мировой линии и регуляризация интеграла

Согласно этой схеме мы выбрасываем из квадрата ϵ-окрестность диагонали и включаем наш интеграл в однопараметрическое семейство интегралов, зависящих от величины параметра ϵ:

$$\int\limits_{Q_T(0)} = \lim_{\epsilon\to 0}\int\limits_{\Delta_1(\epsilon)} + \int\limits_{\Delta_2(\epsilon)}, \tag{30}$$

где каждый из интегралов справа берется по треугольной области, на которой 2-форма регулярна. Результат вычисления можно представить в следующем виде:

$$U_{\text{self-act.reg.}} = \lambda^2(N - \ln N - 1) \overset{N\gg 1}{=} \lambda^2 N, \quad N \equiv \frac{T}{\epsilon}, \tag{31}$$

имеющем при $\epsilon \ll T$ физический смысл суммы большого числа квантов энергии самодействия λ^2. Другими словами, *регуляризация интеграла энергии самодействия приводит*

к необходимости введения в теорию фундаментального параметра временной длительности ϵ, при этом на каждый элемент мировой линии длины ϵ (его можно назвать квантом истории частицы) приходится энергия самодействия λ^2.

На самом деле, с позиций классической механики энергию $U_{\text{self-act}}$ следует интерпретировать как действие свободной частицы. Для последовательной интерпретации необходимо выполнить одно из двух интегрирований в (29) в явном виде:

$$U_{\text{self-act}} = \mathcal{S}_0 = -\frac{\lambda^2 T}{2} \int\limits_{-T/2}^{T/2} \frac{ds}{(T/2-s)(T/2+s)} \tag{32}$$

График подынтегральной функции в (32) представлен на рис. 2.

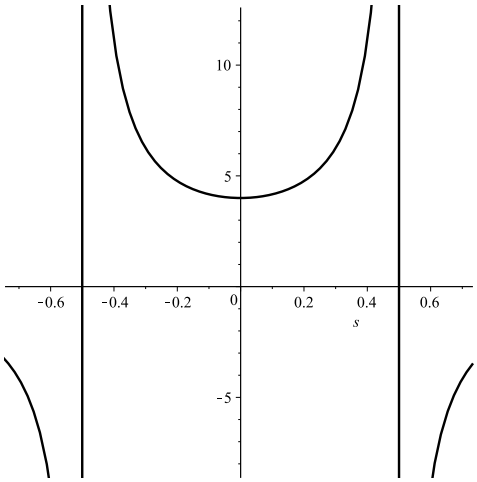


Рис. 2. Функция $f = 1/(T/2-s)(T/2+s)$ в (32) (на графике $T = 1$.)

Вдали от концов мировой линии (т.е. для $-T/2 \ll s \ll T/2$) подинтегральная функция меняется очень медленно и потому часть энергии самодействия в этой области принимает вид классического действия свободной частицы:

$$\mathcal{S}_0(s_1, s_2) \approx -\frac{\lambda^2 T}{2} \int\limits_{s_1}^{s_2} f(0)\, ds = -m_0 c \int\limits_{s_1}^{s_2} ds, \tag{33}$$

где $m_0 = 2\lambda^2/T$. Таким образом, *гиперболическое самодействие свободной частицы определяет вдали от концов ее мировой линии параметр, имеющий смысл массы покоя частицы.* Ее конечный характер обеспечивается как наличием гиперболического заряда мировой линии ($\lambda \neq 0$), так и конечностью ее истории ($T < \infty$). Если предположить, что $\lambda^2 \sim \hbar$ и $T \gtrsim 13$ млрд. св. лет, то получим оценку для величины характерной энергии покоя, индуцируемой гиперболическим самодействием: $m_0 c^2 \lesssim 10^{-32}$эВ!

4.1 Статическое взаимодействие частиц

Чтобы выявить внутреннюю связь между классической теорией поля и гиперболическим полем, рассмотрим очень простую ситуацию — пару покоящихся в некоторой инерциальной системе отсчета классических частиц-источников. В 4-мерной системе координат, согласованной с этой системой отсчета, рассматриваемая пара частиц будет представляться в $\mathcal{M}_{1,3}$ парой мировых линий, параллельных оси времени и разделенных пространственным расстоянием r. Эти линии "сотканы" из материальных событий. Энергию самодействия

каждой частицы из пары мы подробно рассмотрели в предыдущем разделе. Поэтому сразу перейдем к энергии их гиперболического взаимодействия:

$$\phi_{12}(r) = \frac{1}{2}\int\limits_{-T/2}^{T/2}\int\limits_{-T/2}^{T/2} \frac{\lambda_1\lambda_2}{(t_2-t_1)^2 - r^2} dt_1 dt_2 \tag{34}$$

(множитель $1/2$ появился из-за того, что двойное интегрирование двукратно учитывает пары элементов на мировых линиях).

Поскольку вычисление интеграла связано с двумя регуляризациями, имеющими определенный физический смысл, ниже мы приводим подробные вычисления. Выполняя замену переменных $\xi_1 = t_1/r$, $\xi_2 = t_2/r$, приходим к факторизации размерных и безразмерных выражений в (34):

$$\phi_{12}(r) = \frac{\lambda_1\lambda_2}{2} I(a), \tag{35}$$

где безразмерный интеграл $I(a)$, зависящий только от безразмерного параметра $a = T/2r$ выражается формулой:

$$I(a) = \int\limits_{-a}^{a}\int\limits_{-a}^{a} \frac{d\xi_1\, d\xi_2}{(\xi_2-\xi_1)^2 - 1}. \tag{36}$$

Для эффективного вычисления этого интеграла заметим, что геометрически он является интегралом точной 2-формы по квадратной области Q_{2a} ($2a$ — сторона квадрата) на плоскости переменных (ξ_1, ξ_2):

$$I(a) = \int\limits_{Q_{2a}} \frac{d\xi_1 \wedge d\xi_2}{(\xi_2-\xi_1)^2 - 1}. \tag{37}$$

Перейдем к новым координатам:

$$u = \xi_1 - \xi_2; \quad v = \xi_1 + \xi_2. \tag{38}$$

Элемент площади $d\xi_1 \wedge d\xi_2 = (du \wedge dv)/2$, а область интегрирования на плоскости переменных (u, v) будет представлять собой квадрат $\bar{Q}_{2a}$ (рис. 3) с вершинами, лежащими на осях в точках с координатами $\pm 2a$.

В новых переменных интеграл (37) примет вид:

$$I(a) = \frac{1}{2}\int\limits_{\bar{Q}_{2a}} \frac{du \wedge dv}{u^2 - 1}. \tag{39}$$

Поскольку подынтегральное выражение имеет сингулярность на прямых $u = \pm 1$, необходима регуляризация. Суть применяемой ниже регуляризации заключается в одновременном выбрасывании вкладов в интеграл ϵ-окрестностей отрезков сингулярных прямых (ϵ-полос $B_{\epsilon 1}$ и B_{ϵ_2}) с последующим предельным переходом $\epsilon \to 0$. Границы ϵ-полос и прямая $u = 0$ (на ней 2-форма регулярна) задают следующее разбиение области интегрирования:

$$\bar{Q}_{2a} = \Delta_{\epsilon 1} \cup B_{\epsilon 1} \cup T_{\epsilon 1} \cup T_{\epsilon 2} \cup B_{\epsilon 2} \cup \Delta_{\epsilon 2}, \tag{40}$$

где треугольные области $\Delta_{\epsilon 1}, \Delta_{\epsilon 2}$ задаются неравенствами:

$$\Delta_{\epsilon 1}: \ -2a - u \le v \le 2a + u, \quad -2a \le u \le -1 - \epsilon; \tag{41}$$

$$\Delta_{\epsilon 2}: \ -2a + u \le v \le 2a - u, \quad 1 + \epsilon \le u \le 2a,$$

а трапецидальные $T_{\epsilon 1}, T_{\epsilon 2}$ — неравенствами:

$$T_{\epsilon 1}: \ -2a - u \le v \le 2a + u, \quad -1 + \epsilon \le u \le 0; \tag{42}$$

$T_{\epsilon 2}:\ -2a+u \le v \le 2a-u,\ \ 0 \le u \le 1-\epsilon.$

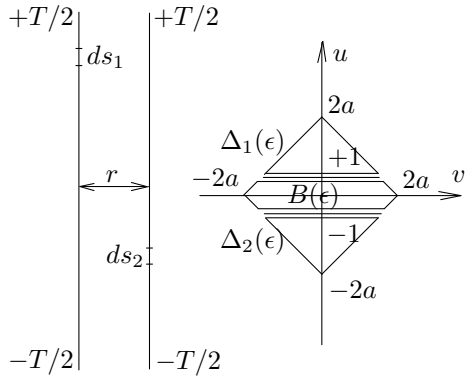


Рис. 3. Самодействие мировой линии и регуляризация интеграла

Интеграл (39) в регуляризованной форме теперь принимает вид:

$$I(a,\epsilon) = \frac{1}{2} \int\limits_{\bar{Q}_{2a}\setminus(B_{\epsilon 1}\cup B_{\epsilon 2})} \frac{du\wedge dv}{u^2-1}. \tag{43}$$

В каждой из областей регулярности интеграл сводится к повторному и вычисляется элементарно:

$$I_1 = \int\limits_{\Delta_{\epsilon 1}} = (2a-1)\ln(2a-1) - (2a+1)\ln(2a+1) + (2a+1)\ln(2+\epsilon) - (2a-1)\ln\epsilon; \tag{44}$$

$$I_2 = \int\limits_{T_{\epsilon 1}} = (2a-1)\ln\epsilon - (2a+1)\ln(2-\epsilon); \tag{45}$$

$$I_3 = \int\limits_{T_{\epsilon 2}} = I_2; \quad I_4 = \int\limits_{\Delta_{\epsilon 2}} = I_1. \tag{46}$$

Собирая все вместе, находим:

$$I(a,\epsilon) = (2a-1)\ln(2a-1) - (2a+1)\ln(2a+1) + (2a+1)(\ln(2+\epsilon) - \ln(2-\epsilon)). \tag{47}$$

Переходя в (47) к физическим обозначениям $2a = T/r$, получаем:

$$I(a,\epsilon) = \left(\frac{T}{r}-1\right)\ln\left(\frac{T}{r}-1\right) - (\ln(2+\epsilon)-\ln(2-\epsilon))\left(\frac{T}{r}+1\right)\ln\left(\frac{T}{r}+1\right) + \left(\frac{T}{r}+1\right). \tag{48}$$

Если теперь в этом выражении перейти к точным пределам $T\to\infty$, $\epsilon\to 0$, мы получим расходимость, независимо от порядка выполнения предельных переходов. Рассмотрим мир, в котором параметры T и ϵ отличны от своих идеальных предельных значений. Эти параметры имеют различный физический смысл: значение T отвечает за "длительность истории" частиц-источников, значение ϵ — за причинность. При $\epsilon = 0$ взаимодействие посредством гиперболического поля распространяется строго со скоростью света вдоль конусов. Малые отклонения ϵ от нуля соответствуют картине, в которой конуса слегка "размазаны" . При этом параметр ϵ в этой картине приобретает смысл дополнительной "фундаментальной постоянной" , которую формально можно представить в виде

$$\epsilon = \delta c/c, \tag{49}$$

где δc — абсолютная вариация скорости света c ("фундаментальная постоянная"). Для описания такого мира с несколько более общими свойствами, чем пространство-время Минковского в СТО, естественно рассмотреть не предел выражения (48) при $T \to \infty$, $\epsilon \to 0$, а его асимптотический вид при этих условиях. Удерживая несколько первых членов соответствующих разложений, мы получим (отбрасывая несущественные вещественные константы):

$$I(T,\epsilon) \overset{\text{as}}{=} \frac{\epsilon T}{r} + 2\ln(r/T) + O(\epsilon) + O((r/T)^2). \tag{50}$$

В выражении (50) выделяется кулоновская часть — она сохраняется благодаря конечному значению комбинации ϵT, отвечающей за своеобразный баланс между длительностью истории и причинностью, и логарифмическая часть.

С учетом (35) окончательное выражение для энергии взаимодействия пары покоящихся частиц-источников в асимптотическом приближении можно записать следующим образом:

$$\phi_{12}(r) \overset{\text{as}}{=} \frac{\alpha_1\alpha_2}{r} + \lambda_1\lambda_2 \ln r, \tag{51}$$

где $\alpha_i = \lambda_i\sqrt{\epsilon T/2}$ — кулоновские заряды, λ_i — логарифмические заряды, которые совпадают с линейной плотностью исходного гиперболического заряда.

График функции (51) представлен на рис. 4.

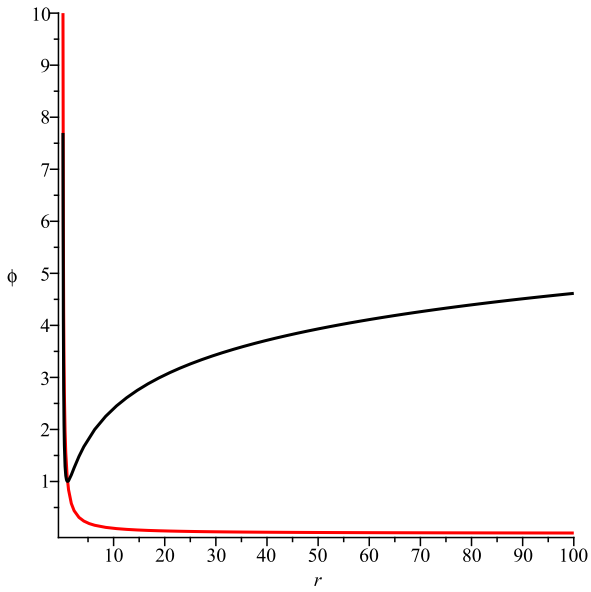


Рис. 4. Энергия гиперболического взаимодействия частиц. Черная кривая — потенциал (51) при $\lambda_1 = \lambda_2 = 1$, $\alpha_1 = \alpha_2 = 1$, красная кривая — кулоновская часть.

В потенциале (51) на малых расстояниях доминирует кулоновская часть, а на больших — логарифмическая. При этом "малые расстояния" в рассматриваемой асимптотической теории определяются естественным условием $r \ll \epsilon T$. Для грубого согласования с наблюдениями, следует положить $\epsilon \lesssim 10^{-10}$ (современная точность измерения скорости света), $T \gtrsim 10^{24}$м (масштаб времени существования Вселенной по современным представлениям), тогда кулоновская область определяется неравенством: $r \ll 10^{14}$м, что с запасом покрывает размеры Солнечной системы. С другой стороны, на космологических масштабах в энергии взаимодействия заведомо доминирует логарифмическая часть. Нетрудно показать, что логарифмический потенциал обеспечивает плоский характер кривых вращения частиц, вращающихся вокруг общего массивного центра, без привлечения концепции темной материи. Действительно, полагая во втором законе Ньютона для вращательного движения силу

притяжения $\sim 1/r$, получаем:

$$\frac{v^2}{R} \sim \frac{A}{R} \Rightarrow v \sim \text{const.} \tag{52}$$

Отметим, что полученный результат в принципиальном отношении позволяет обойтись без концепции темной материи, как это делается, например, в теориях типа MOND (Modified Newton Dynamics) [5]. Из предыдущего изложения очевидно, что в отличие от теорий MOND, мы модифицируем не вид второго закона Ньютона, а фундаментальный закон взаимодействия.

Список литературы

[1] S.S.Kokarev, Classical solids dynamics as 4D statics of elastic strings, *Nuovo Cimento B* **116**, 915 (2001), gr-qc/0108007.

[2] D.G. Pavlov, S.S. Kokarev, Essentials of Polynumbers Field Theory, In *Advances in General Relativity Research* (ed. C. Williams) 2015, ch.7, pp. 157-266, Nova Science Publishers, NY,USA

[3] D.G. Pavlov, S.S. Kokarev, Algebra, Geometry and Physics of Hyperland, In *Advances in General Relativity Research* (ed. C. Williams) 2015, ch.8, pp. 267-342, Nova Science Publishers, NY,USA

[4] С.С.Кокарев, Три лекции о законах Ньютона, *Сборник трудов РНОЦ "Логос"* (2006) **1**, с.45-72, arxiv: 0905.3285

[5] M. Milgrom, MOND theory, arxiv: 1404.7661 [astro-ph.CO]

Spinor realization of Skyrme - Faddeev chiral model of particles

Yu. P. Rybakov
soliton4@mail.ru; Department of Theoretical Physics and Mechanics,
Peoples Friendship University of Russia,
117198 Moscow, 6, Miklukho-Maklay street, RUSSIA

Abstract

We discuss the 16-spinor field realization of Skyrme - Faddeev chiral model of baryons and leptons as topological solitons. The main idea behind this paper consists in unifying the approaches, suggested by Skyrme and Faddeev for description of baryons and leptons respectively, by using the special 8-semispinor identity invented by the Italian geometer F. Brioschi. The peculiar property of this unified model is the necessity of generalizing the Einstein gravitational theory by including Kraichnan invariant (i. e. Riemann curvature tensor squared) in the Lagrangian through special structure of the Higgs potential implying the spontaneous breaking of symmetry. This fact reveals the two consequences. The first one concerns the essential role of higher derivatives of the metric tensor at small distances, and the second one concerns the behavior of the model at large distances implying the correspondence with Quantum Mechanics.

Keywords: topological solitons, Skyrme—Faddeev model, baryon charge, lepton charge, Hopf invariant, Kraichnan invariant, Brioschi identity, nonlinear spinor field

1 Introduction. Brioschi identity and 16-Spinor field model

We should first emphasize the crucial role behind this research played by Gustav Mie ideas concerning field theory of matter and stress the necessity of their revival to date. G. Mie made an important contribution to the modern physics. His ingenious idea of representing particles as localized concentrations of electromagnetic energy [1] prompted Einstein to search for unitary field theory of matter, with particles being described by regular particle-like solutions to nonlinear field equations ("bunched fields"). G. Mie new fascinating idea concerned using 8-spinors as special constructive elements for description of internal structure of particles as extended objects [2]. It is worthwhile to stress that to date these ideas are anew brought into play within the scope of chiral models of baryons and leptons.

In the popular Skyrme model [3] baryons are considered as topological solitons, the baryon number $\mathbb{B}$ being identified with the degree type topological charge: $\mathbb{B} = \deg(S^3 \to S^3)$ serving as generator of the homotopy group $\pi_3(S^3) = \mathbb{Z}$. Similarly, in the Faddeev model [4] leptons are considered as topological solitons endowed with the Hopf invariant Q_H, the latter one playing the role of the lepton number $\mathbb{L} = Q_H$ serving as generator of the homotopy group $\pi_3(S^2) = \mathbb{Z}$. To unify these two approaches we introduce 16-spinor field $\Psi = \Psi_1 \oplus \Psi_2$, Ψ_i, $i = 1, 2$, being 8-spinors, and the two kinds of internal Pauli matrices: $\vec{\lambda} = I_4 \otimes \vec{\sigma} \otimes I_2$, $\vec{\Lambda} = I_8 \otimes \vec{\sigma}$, where $\vec{\sigma}$ stands for usual Pauli matrices and I_n- for $n-$dimensional unit matrix. As was shown in [5], for each 8-spinor ψ the following Brioschi identity [6] holds:

$$j_\mu j^\mu - \tilde{j}_\mu \tilde{j}^\mu = s^2 + p^2 + \vec{v}^2 + \vec{a}^2, \tag{1}$$

where standard bilinear spinor quantities are introduced: $s = \bar{\psi}\psi$, $p = \imath\overline{\psi}\gamma_5\psi$, $\vec{v} = \overline{\psi}\,\vec{\lambda}\psi$, $\vec{a} = \imath\overline{\psi}\gamma_5\vec{\lambda}\psi$, $j_\mu = \overline{\psi}\gamma_\mu\psi$, $\tilde{j}_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$, $\overline{\psi} = \psi^+\gamma_0$, with γ_μ, $\mu = 0, 1, 2, 3$, being Dirac matrices. In view of the identity (1), one can use the special structure of the Higgs potential V implying the spontaneous symmetry breaking:

$$V = \frac{\sigma^2}{8}\left(j_\mu j^\mu - \varkappa_0^2\right)^2, \tag{2}$$

with σ and $\varkappa_0$ being parameters of the model and the natural boundary condition at space infinity being imposed:

$$\lim_{r\to\infty} j_\mu j^\mu = \varkappa_0^2, \quad r = |\vec{x}|. \tag{3}$$

As follows from (1) and (3), the two kinds of topological solitons are possible [5] due to the choice of S^2- or S^3-manifolds as phase spaces, with the vacuum fixed point being determined by (3). In fact, S^2-manifold is determined by the $O(3)-$invariant $\vec{v}^2$ and corresponds to Faddeev model or lepton sector. Similarly, S^3-manifold is determined by the chiral invariant $s^2+\vec{a}^2$ and corresponds to Skyrme model or baryon sector.

However, the unification of Skyrme and Faddeev models supposes that the vacuum state Ψ_0 should be universal, that can be realized within the scope of 16-spinor field model, if one supposes that $\Lambda\Psi_0 = 0$, $\Lambda = (1-\Lambda_3)/2$, and S^2-manifold is determined by the invariant $\vec{V}^2$, $\vec{V} = \overline{\Psi}\vec{\Lambda}\Psi$, S^3-manifold being unchanged. To choose lepton sector, one remarks that condition $\vec{a} = \imath(\overline{\Psi}\gamma_5\vec{\lambda}\Psi) = 0$ implies $\mathbb{B} = 0$ and can be satisfied if one supposes the mirror symmetry of the lepton sector:

$$\Psi \to \gamma_0\Psi. \tag{4}$$

Introducing the 16-spinor of the form

$$\Psi = \oplus_{j=1}^2(\varphi_j \oplus \chi_j \oplus \xi_j \oplus \zeta_j), \tag{5}$$

where φ_j, χ_j, ξ_j, ζ_j stand for 2-spinors, one finds from invariance condition (4) that $\varphi_j = \chi_j$, $\xi_j = \zeta_j$ for the Weyl representation of $\gamma-$matrices, i. e. one gets an effective 8-spinor. In this case $\vec{V} = \overline{\Psi}\vec{\Lambda}\Psi \neq 0$ and in accordance with the Brioschi identity

$$\left(j_\mu j^\mu - \vec{V}^2\right)/16 = \left(|\varphi_2|^2 + |\zeta_2|^2\right)\left(|\varphi_1|^2 + |\zeta_1|^2\right) - |\varphi_1^+\varphi_2 + \zeta_1^+\zeta_2|^2 \geq 0$$

due to the Schwartz inequality.

Now let us choose baryon sector, taking into account the charge independence of strong interactions, or mirror symmetry of the isotopic space:

$$\Psi \to \gamma_0\gamma_5\gamma_2\lambda_2\Psi^*. \tag{6}$$

Inserting (5) into the invariance condition (6) one finds for the baryon sector $\xi_j = \imath\sigma_2\varphi{*_j}$, $\zeta_j = \imath\sigma_2\chi{*_j}$, i. e. again one gets an effective 8-spinor. In this case $p = \vec{v} = V_2 = 0$, i. e. $\mathbb{L} = 0$, but $s \neq 0$, $\vec{a} \neq 0$ and in accordance with the Brioschi identity

$$(s^2 + \vec{a}^2)/16 = |\varphi_1^+\chi_1 + \varphi_2^+\chi_2|^2 + |\varphi_1^T\sigma_2\chi_1 + \varphi_2^T\sigma_2\chi_2|^2 \leq (j_\mu j^\mu)/16.$$

Thus, the structure (2) of the Higgs potential is consistent with the topology constraints of lepton or baryon sectors.

In view of these arguments, let us consider the following Lagrangian density [5]:

$$\mathcal{L}_{\text{spin}} = \frac{1}{2\lambda^2}\overline{\partial_\mu\Psi}\gamma^\nu j_\nu\partial^\mu\Psi + \frac{\epsilon^2}{4}f_{\mu\nu}f^{\mu\nu} - V, \tag{7}$$

where $f_{\mu\nu} = (\overline{\Psi}\gamma^\alpha\partial_{[\mu}\Psi)(\overline{\partial_{\nu]}\Psi}\gamma_\alpha\Psi)$ is the anti-symmetric tensor of Skyrme—Faddeev type and λ, ϵ are constant parameters. The first term in (5) is similar to the sigma-model one and includes the projector $P = \gamma^0\gamma^\nu j_\nu$ on the positive energy states.

2 Interaction with physical vector fields

Due to the general gauge invariance principle, the group of phase transformations of Ψ with some charge generator Γ_e and left, right isotopic rotations with generators $P_{L,R}\lambda^a\Lambda/2$, $a = 1, 2, 3$, $P_{L,R} = (1 \pm \gamma_5)/2$, give rise to the interactions with the electromagnetic field A_μ and left, right Yang—Mills fields $A_\mu^{aL,R}$ respectively through the extention of the derivative $\partial_\mu\Psi \to D_\mu\Psi = \partial_\mu\Psi - \imath e_0\Gamma_e A_\mu\Psi + (A_\mu^L + A_\mu^R)\Psi$, where we adopt $\Gamma_e = P_3\Lambda$, $P_3 = (1 - \lambda_3)/2$, $A_\mu^{L,R} = P_{L,R}e_{1L,R}A_\mu^{aL,R}\lambda^a\Lambda/(2\imath)$. Here e_0, e_{1L}, e_{1R} stand for corresponding coupling constants. However, the Yang—Mills fields being responsible for strong interactions, they should not give any contribution to the lepton sector. In view of mirror symmetry (4), this is possible only if the sum $A_\mu^{aL} + A_\mu^{aR}$ is proportional to γ_5, or the following constraint holds:

$$e_{1L}A_\mu^{aL} + e_{1R}A_\mu^{aR} = 0. \tag{8}$$

In view of (8) the extended derivative reads:

$$D_\mu\Psi = \partial_\mu\Psi - \imath e_0 P_3\Lambda A_\mu\Psi + g_0\gamma_5\mathbb{A}_\mu\Psi, \tag{9}$$

where $g_0 = e_{1L} = e_{1R}$, $\mathbb{A}_\mu = \lambda^a\Lambda A_\mu^{aL}/(2\imath)$, with the standard Yang—Mills Lagrangian taking the form:

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{32\pi}\mathrm{Sp}\left\{(\partial_\mu\mathbb{A}_\nu - \partial_\nu\mathbb{A}_\mu)(\partial^\mu\mathbb{A}^\nu - \partial^\mu\mathbb{A}^\nu) + g_0^2[\mathbb{A}_\mu, \mathbb{A}_\nu][\mathbb{A}^\mu, \mathbb{A}^\nu]\right\}. \tag{10}$$

It is worth-while to stress that our spinor model admits a new possibility to derive usually adopted hypothesis of the electric charge Q_e quantization, i. e. $Q_e = ne_0$, $n \in \mathbb{Z}$. To this end let us introduce the modified Maxwell Lagrangian density:

$$\mathcal{L}_{\mathrm{em}} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}f(u), \tag{11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f(u)$ is some unknown function of the special Lorentz and dilatation invariant

$$u = (n_\mu A^\mu)^2(-E_\nu E^\nu)^{-1/2}. \tag{12}$$

Here $E_\nu = n^\mu F_{\mu\nu}$, the unit time-like vector n^μ being identified with the normalized Dirac current: $n^\mu = j^\mu/j$, $j = (j_\nu j^\nu)^{1/2}$. In fact, for the static soliton with the charge q one finds the equation for the scalar potential $\Phi = A^0$:

$$\Phi\,\mathrm{div}\,[\vec{E}(2f - uf'(u))] = 2uf'(u)(\vec{E})^2, \quad \vec{E} = -\vec{\bigtriangledown}\Phi. \tag{13}$$

At large distances $r \to \infty$ one gets $\Phi = (q/r)[1 + \mathcal{O}(1/r)]$, $u = q[1 + \mathcal{O}(1/r^2)]$ and the left hand side of (13) behaves as $\mathcal{O}(1/r^5)$, or $f'(q) = 0$. The latter equation being considered as the quantization condition for Q_e, one can suggest the simplest choice of the function $f(u) = 1 + \mu_0\sin^2[\pi u/(2e_0)]$, where μ_0 stands for some small constant.

Finally, let us consider small excitations of our soliton configuration near the vacuum: $\Psi = \Psi_0 + \xi$ as $|\vec{x}| \to \infty$. Then the linearized equation for ξ after the substitution $\xi = k\Psi_0$ takes the form:

$$\partial_\mu\partial^\mu k + \frac{1}{2}M_0^2(k^* + k) = 0, \quad M_0 \equiv 2\sigma\lambda\varkappa_0. \tag{14}$$

Thus, from (14) one derives the equations for real and imaginary parts of $k = k_1 + \imath k_2$:

$$\partial_\mu\partial^\mu k_1 + M_0^2 k_1 = 0, \quad \partial_\mu\partial^\mu k_2 = 0. \tag{15}$$

According to (15), our model admits two types of vacuum excitations: massive and massless ones. For massive soliton the excitation k_1 could be interpreted as the wave function if the parameter M_0 were replaced with the real mass M of the soliton. To satisfy this condition, we first introduce the interaction with the gravitational field via the new extended derivative $\bigtriangledown_\mu \Psi = (D_\mu - \Gamma_\mu)\Psi$, where Γ_μ stands for spinor affine connection. Then we generalize the Higgs potential (2) through replacing σ^2 with the new invariant:

$$\sigma^2 = -\frac{16D^3}{\lambda^2 G^2 K^2 \varkappa_0^2}, \tag{16}$$

where G is the Newton gravitational constant, $D = \overline{\bigtriangledown_\mu \Psi}\gamma^\alpha j_\alpha \bigtriangledown^\mu \Psi$ and K is the Kraichnan invariant constructed with the help of the Riemannian curvature tensor: $K = R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}/48$. The validity of this choice of the invariants in the Higgs potential becomes evident at large distances, since in view of the Schwarzschild metric $D = -r_g^2\varkappa_0^2/(4r^4)$, $K = r_g^2/r^6$, where $r_g = GM$ and natural units $\hbar = c = 1$ are used.

3 Axially-symmetric states and strong gravity

Let us consider static axially-symmetric states in geometric toroidal coordinates $x^1 = r$, $x^2 = \vartheta$, $x^3 = \phi$, $r \in [0, \infty]$, $\vartheta \in [-\pi, \pi]$, $\phi \in [0, 2\pi]$, where the scale parameter a is inserted in standard cylindrical coordinates: $z = r\sin\vartheta$, $\rho = a + r\cos\vartheta$. Therefore the squared interval takes the form:

$$ds^2 = e^{2\eta}dt^2 - e^{2\alpha}dr^2 - e^{2\beta}d\vartheta^2 - e^{2\gamma}d\phi^2, \tag{17}$$

where the metric coefficients η, α, β, γ depend only on r, ϑ. Let us now calculate the Kraichnan invariant for the metric (17), the result being:

$$K = \frac{1}{12}\left[\sum_{i=1}^{6} k_i^2 + 2e^{-2\alpha-2\beta}(k_7^2 + k_8^2)\right], \tag{18}$$

where the following quantities are introduced:

$$k_1 = e^{-2\alpha}[\partial_1^2\eta + (\partial_1\eta)^2 - \partial_1\eta\partial_1\alpha] + e^{-2\beta}\partial_2\eta\partial_2\alpha;$$
$$k_2 = k_1(\alpha \Leftrightarrow \beta); \quad k_5 = k_1(\eta \Rightarrow \gamma); \quad k_6 = k_2(\eta \Rightarrow \gamma);$$
$$k_3 = e^{-2\alpha}\partial_1\eta\partial_1\gamma + e^{-2\beta}\partial_2\eta\partial_2\gamma;$$
$$k_4 = e^{-2\alpha}[\partial_1^2\beta + (\partial_1\beta)^2 - \partial_1\beta\partial_1\alpha] + (\alpha \Leftrightarrow \beta, 1 \Rightarrow 2);$$
$$k_7 = \partial_{12}\eta + \partial_1\eta\partial_2\eta - \partial_1\eta\partial_2\alpha - \partial_2\eta\partial_1\beta;$$
$$k_8 = k_7(\eta \Rightarrow \gamma).$$

Using invariant (18), one can find the structure of the metric at small distances $r \ll a$, where the higher derrivatives give the most important contribution to the gravitational equations, which can be obtained through minimizing K. To this end we first remark the symmetry of K with respect to transposition $\eta \Leftrightarrow \gamma$. The solution to the corresponding equations reads:

$$\alpha = 0, \quad \beta = \log r, \quad \eta = \gamma = \log(a + r\cos\vartheta), \tag{19}$$

with the Kraichnan invariant being

$$K = \frac{1}{12(a + r\cos\vartheta)^4}. \tag{20}$$

Thus we conclude that the strong gravity approximation (19) works well in the domain $r \ll a$, with the field configuration appearing as closed string of the large closure radius $a \gg r$. Let us show that such states appear in the lepton sector.

4 Axially-symmetric states in lepton sector

To find the structure of leptonic axially-symmetric configurations at small distances, we suppose that the corresponding spinor fields are invariant under the combined rotations around third axis in coordinate and $\vec{\Lambda}$-space, with the symmetry group being

$$G = \mathrm{diag}\,[SO(2)_s \otimes SO(2)_\Lambda]\,. \tag{21}$$

In view of mirror symmetry (4) and taking into account that the generator of the group (21) reads $J_3 - \Lambda_3/2$, where $J_3 = -\imath\partial_\phi + \sigma_3/2$, one gets the following equations for invariant fields:

$$J_3\Psi_1 = \frac{1}{2}\Psi_1, \quad J_3\Psi_2 = -\frac{1}{2}\Psi_2,$$

with the evident solutions:

$$\Psi_1 = \mathrm{col}\{f_1, f_1, u_1, u_1\} \oplus \mathrm{col}\{g_1, g_1, v_1, v_1\}e^{\imath\phi}, \tag{22}$$

$$\Psi_2 = \mathrm{col}\{f_2, f_2, u_2, u_2\}e^{-\imath\phi} \oplus \mathrm{col}\{g_2, g_2, v_2, v_2\}, \tag{23}$$

where the new functions f_i, g_i, u_i, v_i, $i = 1, 2$, depend only on cylindrical coordinates ρ and z.

Inserting (22) and (23) into the vector field $\vec{V} = \overline{\Psi}\vec{\Lambda}\Psi$, one finds:

$$V_1 + \imath V_2 = {\Psi_1}^+\Psi_2 = 2Re^{-\imath\phi}, \quad R = f_1^* f_2 + u_1^* u_2 + g_1^* g_2 + v_1^* v_2;$$
$$V_3/2 = |f_1|^2 + |u_1|^2 + |g_1|^2 + |v_1|^2 - |f_2|^2 - |u_2|^2 - |g_2|^2 - |v_2|^2.$$

It means that

$$V_2/V_1 = \tan(\mu - \phi), \quad \tan\mu = \mathrm{Im}R/\mathrm{Re}R, \quad \mu = \mu(\rho, z). \tag{24}$$

We remark now that the structure (22) and (23) permits one to calculate the Hopf index Q_H identified with the lepton charge $\mathbb{L}$ of the configuration. To this end one can use the Whitehead integral representation for the Hopf invariant:

$$Q_H = \frac{1}{(4\pi)^2}\int d^3x\,(\vec{a}\,\mathrm{rot}\,\vec{a}), \tag{25}$$

where

$$\partial_i a_k - \partial_k a_i = \varepsilon_{abc}\partial_j n^a \partial_k n^b n^c$$

and $\vec{n}$ stands for the unit 3-vector $\vec{n} = \vec{V}/|\vec{V}|$.

To perform the integration in (25), let us use the inverse Hopf mapping $\vec{n} = \chi^+\vec{\sigma}\chi$, where

$$\chi = \mathrm{col}\left\{\cos A + \imath\sin A\cos B,\, \sin A\sin Be^{\imath C}\right\}. \tag{26}$$

Taking into account that

$$\vec{a} = 2\mathrm{Im}(\chi^+\vec{\bigtriangledown}\chi), \quad \mathrm{rot}\,\vec{a} = -2\imath[\vec{\bigtriangledown}\chi^+\vec{\bigtriangledown}\chi],$$

one can represent (25) as follows:

$$Q_H = \frac{1}{8\pi^2}\int d^3x\,\left(\left[\vec{\bigtriangledown}n_3\vec{\bigtriangledown}\varphi\right]\vec{\bigtriangledown}C\right), \tag{27}$$

where $\tan\varphi(\rho, z) = \tan A\cos B$. On the other hand, one easily finds from (26) that

$$(n_1 + \imath n_2)\cos\varphi = \sin 2A\sin B\exp[\imath(C - \varphi)],$$

whence

$$n_2/n_1 = V_2/V_1 = \tan(C - \varphi). \tag{28}$$

Comparing (24) and (28), one concludes that

$$\varphi = -\mu(\rho, z), \quad C = -\phi. \tag{29}$$

Inserting (29) into (27), one gets through Stokes theorem:

$$Q_H = -\frac{1}{4\pi} \int d\rho dz \, \mathrm{rot}\, (\mu \vec{\bigtriangledown} n_3)|_\phi = -\frac{1}{4\pi} \int_L [\mu] \, dn_3, \tag{30}$$

where the function $\mu(\rho, z)$ can admit jump of the form $[\mu] = 2\pi n$, $n \in \mathbb{Z}$, on the line $L = \{z = 0, 0 \leq \rho \leq a\}$.

Finally one obtains the value of the Hopf invariant:

$$Q_H = \frac{1}{2} n \left[n_3(0) - n_3(\rho = a, z = 0) \right] = n,$$

with the natural boundary conditions being:

$$n_3(0) = 1, \quad n_3(\rho = a, z = 0) = -1,$$

the latter point corresponding to the south pole of the sphere S^2.

5 Conclusions

The resulting Lagrangian of our 16-spinor model reads: $\mathcal{L} = \mathcal{L}_{\text{spin}} + \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{Mie}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_g$, with $\mathcal{L}_g$ being the Einstein gravitational Lagrangian and $\mathcal{L}_{\text{Mie}} = -(A_\mu A^\mu)^3 \nu_0/6$ standing for Mie correction term providing regular behavior of A_μ at small distances. It is worth-while to stress that the gravitational field plays an important role in our model, since the wave-particle duality principle of Quantum Mechanics has the gravitational origin and the vacuum excitation ξ can be considered as the wave function for the soliton center in the special stochastic representation [7, 8]. On the other hand, at small distances the gravity becomes strong and provides the existence of topologically nontrivial field configurations with closed string structure [4].

References

[1] G. Mie. Grundlagen einer Theorie der Materie.// Ann. d. Physik. 1912. B. 39, H. 1. S. 1–40.

[2] G. Mie. Die Geometrie der Spinoren.// Ann. d. Physik. 1933. Folge 5. B. 17, H. 5. S. 465–500.

[3] T. H. R. Skyrme. A unified field theory of mesons and baryons.// Nucl. Phys. 1962. Vol. 31, no 4. P. 556–569 .

[4] L. D. Faddeev. Gauge-invariant model of electromagnetic and weak interactions of leptons.// Rep. of Ac. Sci. USSR. 1973. Vol. 210, no 4. P. 807–810.

[5] Yu. P. Rybakov. Soliton configurations in generalized Mie electrodynamics. Phys. of Atomic Nuclei.// 2011. Vol. 74, no 7. P. 1073–1076.

[6] E. Cartan. The Theory of Spinors. - Hermann: Paris, 1966.

[7] Yu. P. Rybakov. On the causal interpretation of Quantum Mechanics.// Found. of Physics. 1974. Vol. 4, no 4. P. 149–161.

[8] Yu. P. Rybakov. Solitons and Quantum Mechanics.// Dynamics of Complex Systems. 2009. Vol. 3, no 4. P. 3–15.

Connection between electromagnetism and gravity

A.J. Silenko
alsilenko@mail.ru; Research Institute for Nuclear Problems,
Belarusian State University, Minsk, Belarus and
BLTP, Joint Institute for Nuclear Research, Dubna, Russia

Abstract

We introduce gravitoelectric and gravitomagnetic fields and compare equations of motion for momentum and spin of relativistic particles in local Lorentz frames with corresponding equations in classical electrodynamics. Unlike electromagnetic fields, gravitoelectromagnetic ones depend on the particle four-momentum. Explicit formulas for the gravitoelectromagnetic fields in some important special cases are obtained and the local Lorentz transformations are considered. The Mathisson force and the Thomas precession in general relativity are shortly discussed.

1 Introduction

Great success achieved in description of electromagnetic and gravitational phenomena stimulated a search for direct analogies between electrodynamics and gravity. This search is hampered by the nonlinearity of equations of motion in gravity. Probably, the best approach to overcome this difficulty has been proposed by Pomeransky and Khriplovich [1]. They have introduced the gravitoelectromagnetic fields in coframes which define local Lorentz frames. These fields are not irrespective of a considered particle because they depend on its four-velocity. Owing to this circumstance, the Pomeransky-Khriplovich (PK) fields are only effective. Nevertheless, the equations of motion in Riemannian spacetimes become similar to those in electrodynamics after the introduction of the PK fields.

We consider Lorentz transformations in coframes in Sec. 2. Then, we explain and develop the conception of gravitoelectromagnetism proposed by Pomeransky and Khriplovich (Sec. 3). In Sec. 4, we calculate the gravitoelectromagnetic fields in some important special cases. Local Lorentz transformations of gravitoelectromagnetic fields are discussed in Sec. 5. The Mathisson force and the Thomas precession in general relativity (GR) are determined in Sec. 6. The results obtained are summarized in Sec. 7.

We denote world and spatial indices by Greek and Latin letters $\alpha, \mu, \nu, \ldots = 0, 1, 2, 3$, $i, j, k, \ldots = 1, 2, 3$, respectively. Tetrad indices are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \ldots = 0, 1, 2, 3$. Temporal and spatial tetrad indices are distinguished by hats. The signature is $(+---)$. Commas before indices denote partial derivatives. The system of units with $c = 1$ is used. We include c into some equations when this inclusion clarifies the problem.

2 Lorentz transformations in coframes

Let us consider Lorentz transformations in coframes (local Lorentz frames). Our explanation partially follows Refs. [2, 3].

One of the most powerful methods of GR is an introduction of tetrads. They define the coframe characterized by the Minkowski metric $ds^2 = \eta_{ab}dx^a dx^b$, $\eta_{ab} = \mathrm{diag}\,(1,-1,-1,-1)$. The metric tensor of a given spacetime can be split into tetrads e^a_μ satisfying the relations

$$e^a_\mu e_{a\nu} = g_{\mu\nu}, \quad e_{a\mu}e^\mu_b = \eta_{ab}, \quad e^a_\mu e^\mu_b = \delta^a_b, \quad e^a_\mu e^\nu_a = \delta^\nu_\mu. \tag{1}$$

As usual, the world and tetrad indices (which all run from 0 to 3) are raised and lowered with the metric and Minkowski tensors, $g_{\mu\nu}$ and η_{ab}, respectively.

Any tetrad can be attributed to an observer. Observers carrying different tetrads may move relative to each other. A locality of a Lorentz frame defined by some tetrad is caused by nonzero derivatives of the metric tensor. First derivatives define forces like the Newton force while second derivatives define the spacetime curvature and tidal forces. These forces are felt by the observer or can be detected in the observer's lab. In particular, the velocity of light is equal to c near the observer but the light can be accelerated (due to the Newton-like force) and can undergo a deflection.

Let us consider two observers and two local Lorentz frames in the same area, i.e., in the vicinity of some point $(x^0_{(0)}, x^1_{(0)}, x^2_{(0)}, x^3_{(0)})$. Since $dx^a = e^a_\mu dx^\mu$, $dx'^a = e'^a_{\ \mu} dx^\mu$, the connection between coordinates in the two frames is given by [2]

$$dx^a = T^a_b dx'^b, \tag{2}$$

where

$$T^a_b = e^a_\mu e'^\mu_{\ b}. \tag{3}$$

The connection between the four-velocities in the two frames has the form

$$u^a \equiv \frac{dx^a}{d\tau} = T^a_b u'^b, \tag{4}$$

where $\tau = s/c$ is the proper time.

We should underline that these relations are not valid beyond the local area.

Certainly, the connection between two local Lorentz frames is realized by local Lorentz transformations. If the axes in the two frames are parallel but the direction of the velocity $\boldsymbol{V}$ of the primed frame in the unprimed frame is arbitrary, this connection is given by

$$\begin{gathered} dx'^{\widehat{0}} = \gamma(dx^{\widehat{0}} - \boldsymbol{\beta}\cdot d\widehat{\boldsymbol{r}}), \quad d\widehat{\boldsymbol{r}}\,' = d\widehat{\boldsymbol{r}} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot d\boldsymbol{r}) - \boldsymbol{\beta}\gamma dx^{\widehat{0}}, \\ dx^{\widehat{0}} = \gamma(dx'^{\widehat{0}} + \boldsymbol{\beta}\cdot d\widehat{\boldsymbol{r}}\,'), \quad d\widehat{\boldsymbol{r}} = d\widehat{\boldsymbol{r}}\,' + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot d\widehat{\boldsymbol{r}}\,') + \boldsymbol{\beta}\gamma dx'^{\widehat{0}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{V}}{c}, \end{gathered} \tag{5}$$

where $\gamma = (1-\boldsymbol{\beta}^2)^{-1/2}$ is the Lorentz factor. We need to specify that the axes of the two frames remain to be parallel. Unlike the usual Lorentz transformations, the relative motion of the local Lorentz frames may be accelerated ($\dot{\boldsymbol{V}} \neq 0$).

The present analysis demonstrates that the quantities dx^a and u^a are four-vectors relative to the local Lorentz transformations. The four-momentum $p_a = -\partial S/(\partial x^a)$ (S is an action) and other four-vectors with tetrad components possess the same property. The tetrad e^μ_a is a world four-vector when a is fixed and is a four-vector relative to the local Lorentz transformations when μ is fixed.

However, we should note that the quantity $e^\mu_a A^a$ may not be a covariant four-vector even if A^a is a four-vector relative to the local Lorentz transformations. Examples of such a situation are given in Sec. IVD of Ref. [4].

It is important to consider the case when the unprimed frame is at rest relative the world one. In this case, the unprimed tetrad satisfies the Schwinger gauge [2, 3, 5, 6] ($e^{\widehat{0}}_i = 0$, $e^0_{\widehat{i}} = 0$).

We can mention that the Schwinger gauge defines an infinite set of tetrads carrying by observers immobile in the world frame. These tetrads are connected by *spatial* transformations. We should underline that different tetrads (satisfying the Schwinger gauge) lead to different equations of motion [7]. The corresponding Hamiltonians also differ [7, 8]. However, an agreement can be restored with appropriate coordinate transformations.

Equation (3) reduces in the weak-field approximation when gravitational and noninertial fields are weak [$|g_{\mu\nu} - \eta_{\mu\nu}| \ll 1$ $(\mu, \nu = 0, 1, 2, 3)$]. Since the quantities $e^{\widehat{0}}_{0}$ and $e^{\widehat{i}}_{i}$ are close to 1, the equation $e^{a}_{\mu} e^{\nu}_{a} = \delta^{\nu}_{\mu} = 0$ $(\mu \neq \nu)$ results in $e^{\widehat{\nu}}_{\mu} = -e^{\nu}_{\widehat{\mu}}$. Here hats point out the tetrad indices. The following relations are valid:

$$g_{0i} = e^{\widehat{0}}_{i} - e^{\widehat{i}}_{0}, \quad g_{0i} = e'^{\widehat{0}}_{i} - e'^{\widehat{i}}_{0} = e'^{i}_{\widehat{0}} - e'^{0}_{\widehat{i}}, \quad T^{\widehat{0}}_{\widehat{i}} = e^{\widehat{0}}_{i} + e'^{0}_{\widehat{i}}, \quad T^{\widehat{i}}_{\widehat{0}} = e^{\widehat{i}}_{0} + e'^{i}_{\widehat{0}}. \tag{6}$$

Evidently, $T^{\widehat{0}}_{\widehat{i}} - T^{\widehat{i}}_{\widehat{0}} = 0$.

Equation (5) can be presented in the form

$$dx'^{a} = L^{a}_{b} dx^{b}. \tag{7}$$

Therefore,

$$T^{a}_{b} = L^{a}_{b}, \tag{8}$$

where

$$L^{\widehat{0}}_{\widehat{0}} = \gamma, \quad L^{\widehat{0}}_{\widehat{i}} = L^{\widehat{i}}_{\widehat{0}} = -\beta^{i}\gamma, \quad L^{\widehat{j}}_{\widehat{i}} = \delta^{j}_{i} + \frac{\gamma^{2}}{\gamma + 1}\beta^{i}\beta^{j}. \tag{9}$$

Equations (3), (5), (8), and (9) define the dependence of the relative motion of observers on the tetrads carrying by them.

The connection between the two tetrads can also be obtained in an explicit form. Equation (7) and the definition of tetrads result in

$$e'^{a}_{\mu} = L^{a}_{b} e^{b}_{\mu}. \tag{10}$$

The tetrads can be used even for two observers moving in the Minkowski spacetime. If we suppose the first observer to be at rest, the tetrad carried by him has only trivial components. In this case, $dx^{\widehat{\mu}} = dx^{\mu}$ and Eq. (10) takes the form

$$e'^{a}_{\mu} = L^{a}_{\mu}, \quad e'^{\widehat{0}}_{0} = \gamma, \quad e'^{\widehat{0}}_{i} = e'^{\widehat{i}}_{0} = -\beta^{i}\gamma, \quad e'^{\widehat{i}}_{j} = \delta^{i}_{j} + \frac{\gamma^{2}}{\gamma + 1}\beta^{i}\beta^{j}. \tag{11}$$

The tetrad (11) satisfies the requirements (1).

Let us also consider an accelerated frame. In this case, $e^{\widehat{0}}_{0} = 1 + \boldsymbol{a} \cdot \boldsymbol{r}/c^{2}$ and the nontrivial tetrad components are

$$e'^{\widehat{0}}_{0} = \gamma e^{\widehat{0}}_{0}, \quad e'^{\widehat{1}}_{0} = -\beta\gamma e^{\widehat{0}}_{0}, \quad e'^{\widehat{0}}_{1} = -\beta\gamma, \quad e'^{\widehat{1}}_{1} = \gamma. \tag{12}$$

3 Equations of motion in coframes

In the present work, we explain and develop the conception of gravitoelectromagnetism first proposed by Pomeransky and Khriplovich [1]. This conception has been advanced in several works [2, 3, 4, 9, 10, 11, 12].

Certainly, there are other conceptions of gravitoelectromagnetism. The conventional conception of gravitoelectromagnetism (see Refs. [13, 14, 15]) is based on a four-potential of gravitoelectromagnetic field expressed in terms of components of the metric tensor. This conception

can be applied for a nonrelativistic particle in the weak-field approximation and does not work in the relativistic case. Of course, this is nothing but the simplest way to introduce the gravitoelectromagnetic fields. Some other approaches provide a description of dynamics of spinning particles beyond the nonrelativistic approximation and the weak-field one. We can mention the known papers by Bailey and Israel [16] and by Yee and Bander [17]. Important results have been obtained [18] with the canonical formulation of GR based on the Arnowitt-Deser-Misner parametrization. In this connection, Ref. [19] can also be noticed. The results obtained allow one to draw a parallel between electromagnetism and gravity. In particular, some approaches connecting this parallel have been proposed in Refs. [20, 21, 22]. Established relations between electromagnetism and gravity are of interest. Nevertheless, only the gravitoelectromagnetic fields introduced by Pomeransky and Khriplovich allow one to reveal a deep analogy between spinning particles in electromagnetic and gravitational/inertial fields.

We can note that the canonical method has also been applied in Refs. [3, 4, 11, 12, 23, 24] for a derivation of quantum mechanical equations of motion. In Ref. [12], this method has also been used to obtain the corresponding classical equations. In the present work, we basically follow the original method by Pomeransky and Khriplovich [1]. The distinctive feature of the Pomeransky-Khriplovich fields is their introduction in the coframes.

The equation of motion of a pointlike spinless particle in GR is given by

$$Du^{\mu} = 0. \tag{13}$$

This equation defines the particle motion on a geodesic line which is perturbed by the Mathisson force for spinning particles and by tidal forces for extended ones. As a rule, these forces are relatively small and can be neglected in the present study. The motion of spinning particles with allowance for a particle deflection from the geodesic line is defined by the Mathisson-Papapetrou equations [25, 26].

The orthogonality condition interconnects the spin four-vector a^{μ} with either the four-velocity or the four-momentum. The three-component spin $\boldsymbol{\zeta}$ is defined in the particle rest frame, when $a^{\mu} = (0, \boldsymbol{\zeta})$. The Mathisson-Pirani [25, 27] condition connects the spin with the four-velocity,

$$u^{\mu} a_{\mu} = 0, \tag{14}$$

while the Tulczyjev condition [28] joins the spin with the four-momentum,

$$p^{\mu} a_{\mu} = 0. \tag{15}$$

A choice of specific condition does not influence next derivations. However, we can note that there exist situations when the results following from the Mathisson-Papapetrou equations with the supplementary condition (15) are not satisfactory from the physical point of view [29].

When Eq. (13) is satisfied,

$$Du^{a} = D(u^{\mu} e^{a}_{\mu}) = u^{\mu} e^{a}_{\mu;\nu} dx^{\nu}. \tag{16}$$

Therefore,

$$\frac{Du^{a}}{d\tau} = e^{a}_{\mu;\nu} u^{\mu} u^{\nu} = e^{\mu}_{b} e^{a}_{\mu;\nu} e^{\nu}_{c} u^{b} u^{c} = \Gamma^{a}{}_{bc} u^{b} u^{c}. \tag{17}$$

Here $\Gamma_{abc} = -\Gamma_{bac} = e^{\mu}_{b} e^{\nu}_{c} e_{a\mu;\nu}$ are the Lorentz connection coefficients (Ricci rotation coefficients). They can also be presented in the form

$$\Gamma_{abc} = \frac{1}{2} \left(\lambda_{abc} + \lambda_{bca} - \lambda_{cab} \right), \quad \lambda_{abc} = -\lambda_{acb} = e^{\mu}_{b} e^{\nu}_{c} (e_{a\mu,\nu} - e_{a\nu,\mu}). \tag{18}$$

Since $u^a = u^\mu e^a_\mu$ is a world scalar, $Du^a = du^a$ and Eq. (17) takes the form [1]

$$\frac{du^a}{d\tau} = \Gamma^a{}_{bc} u^b u^c. \tag{19}$$

Since $D(u^\mu a_\mu) = 0$, Eqs. (13) and (14) result in

$$Da^\mu = 0. \tag{20}$$

Equations (13) and (20) can be considered as a mathematical formulation of the equivalence principle for the particle and spin.

The same derivation as above leads to the equation of spin motion in local Lorentz frames. Since

$$\frac{Da^a}{d\tau} = e^a_{\mu;\nu} u^\mu u^\nu = e^\mu_b e^a_{\mu;\nu} e^\nu_c u^b u^c = \Gamma^a{}_{bc} s^b u^c, \tag{21}$$

we finally obtain:

$$\frac{da^a}{d\tau} = \Gamma^a{}_{bc} a^b u^c, \tag{22}$$

Equations (19) and (22) have been first obtained in Ref. [1]. We can note that the use of local Lorentz frames for a description of spin dynamics is quite natural. The three-component spin which evolution is commonly described is defined in the particle rest frame. This frame can be attributed to some observer and the local Lorentz transformation of the spin (pseudo)vector coincides with its transformation in electrodynamics (see, e.g., Ref. [30]):

$$a^a = (a^{\widehat{0}},\ \widehat{\boldsymbol{a}}), \quad \widehat{\boldsymbol{a}} = \boldsymbol{\zeta} + \frac{\gamma^2 \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{\zeta})}{\gamma+1}, \quad a^{\widehat{0}} = \boldsymbol{\beta}\cdot\widehat{\boldsymbol{a}} = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\zeta}. \tag{23}$$

In this case, $u^a = (\gamma, \gamma\boldsymbol{\beta})$. Let us remember that $\boldsymbol{V} = \boldsymbol{\beta}c$ is the velocity of relative motion of two local Lorentz frames. Therefore, the quantity $\boldsymbol{\zeta}$ defines here the three-component spin in the instantaneously accompanying frame. The spin motion in the nonrotating instantaneously accompanying frame and in the particle rest frame differs due to the Thomas effect [31, 32]. The angular velocity of the Thomas precession in electrodynamics is given by

$$\boldsymbol{\omega}_T = -\frac{1}{\gamma+1}\left(\boldsymbol{u}\times\frac{d\boldsymbol{u}}{dt}\right). \tag{24}$$

Equations (19) and (22) are similar to the corresponding equations of motion in electrodynamics:

$$\frac{du_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} u^\nu, \tag{25}$$

$$\frac{ds_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} s^\nu. \tag{26}$$

Equation (26) describes the spin motion of a particle with the *Dirac magnetic moment* ($g = 2$).

The electromagnetic field tensor can be expressed in terms of the electric and magnetic fields: $F_{\mu\nu} = (\boldsymbol{E}, \boldsymbol{B})$. A similarity between $(e/m)F_{\mu\nu}$ and $\Gamma_{abc}u^c$ has allowed Pomeransky and Khriplovich to introduce the gravitoelectric and gravitomagnetic fields: $c\Gamma_{abc}u^c = (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}})$. Explicitly [1]

$$\mathcal{E}_{\widehat{i}} = c\Gamma_{0ic}u^c, \quad \mathcal{B}_{\widehat{i}} = -\frac{c}{2} e_{ikl}\Gamma_{klc}u^c. \tag{27}$$

For the gravitoelectromagnetic fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, we do not make a difference between upper and lower indices.

As a result of comparison of the foregoing equations of motions in gravitational/inertial and electromagnetic fields, Pomeransky and Khriplovich [1] have obtained the following equation of motion for the three-component spin

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{s}, \quad \boldsymbol{\Omega} = \frac{1}{u^0}\left[-\boldsymbol{\mathcal{B}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{E}}}{u^{\widehat{0}} + 1}\right]. \tag{28}$$

The validity of Eq. (28) has been substantiated in Refs. [2, 3, 10]. It has been demonstrated [2, 3] that the local Lorentz frame which is at rest relative to the world one satisfies the Schwinger gauge. Other gauges define the equations of motion in frames moving relative to the world frame. This important property was not taken into account in Refs. [1, 9, 10].

The equations of particle motion can be presented as follow [1]:

$$\frac{d\widehat{\boldsymbol{u}}}{dt} = \frac{u^{\widehat{0}}}{u^0}\left(\boldsymbol{\mathcal{E}} + \frac{\widehat{\boldsymbol{u}} \times \boldsymbol{\mathcal{B}}}{u^{\widehat{0}}}\right), \quad \frac{du^{\widehat{0}}}{dt} = \frac{\boldsymbol{\mathcal{E}} \cdot \widehat{\boldsymbol{u}}}{u^0}. \tag{29}$$

We can underline that the equation of motion (19) is essentially nonlinear and Eq. (22) contains the four-velocity. These properties differ the equations of motion in GR from the corresponding equations in electrodynamics (25) and (26). Therefore, the gravitoelectromagnetic fields depend on the four-velocity and they are effective fields. In particular, one cannot use these fields to construct a free gravitoelectromagnetic field.

We should take into account that Eqs. (22), (28), and (29) are not equally useful. Equations (22) and (28) are perfect for a description of the spin motion. It is often admissible to disregard a nonuniformity of the gravitoelectromagnetic fields and their dependence on the four-velocity. Alternatively, one can take into account the above-mentioned nonuniformity and the evolution of the four-velocity in the local Lorentz frame. As a contrary, Eq. (29) does not define measurable dynamics of the four-velocity. First of all, one cannot ignore the nonlinearity of Eq. (28) caused by the dependence of the gravitoelectromagnetic fields on the four-velocity. Second, an observer needs a description of the four-velocity dynamics in the world frame. In particular, the conventional force which governs the particle motion is defined in this frame and has the form $f^i = dp^i/dt = mc(du^i/dt)$.

Thus, the introduction of the gravitoelectromagnetic fields *defined in local Lorentz frames* simplifies the description of motion of spinning particles in GR. Equation (27) shows that the important specific feature of the gravitoelectromagnetic fields is their dependence on the particle four-velocity. However, the results presented do not define some other fundamental properties of gravitoelectromagnetic fields. It is very important to find a transformation law of these fields. This problem will be solved in Sec. 5.

4 Gravitoelectromagnetic fields in some important special cases

It is instructive to calculate the gravitoelectromagnetic fields in the most important special cases. Certainly, we will determine these fields in the local Lorentz frames which are at rest relative to the corresponding world frames. We will consider, among others, several examples of a nonstationary (time-dependent) metric which importance for analyzing fundamental problems of GR has been recently affirmed in Ref. [7].

The general form of the line element of an arbitrary gravitational field can be given by [11, 33]

$$ds^2 = V^2c^2dt^2 - \delta_{\widehat{i}\widehat{j}}W^{\widehat{i}}{}_k W^{\widehat{j}}{}_l\,(dx^k - K^k cdt)\,(dx^l - K^l cdt). \tag{30}$$

An analysis of the most important metrics can be simplified with the use of isotropic coordinates. In this case, the line element takes the form (see Ref. [3])

$$ds^2 = V^2c^2dt^2 - W^2\,\delta_{ij}\,(dx^i - K^i cdt)\,(dx^j - K^j cdt). \tag{31}$$

4.1 General noninertial frame

The accelerated and rotating noninertial frame presents the general case of inertial fields in a flat spacetime. The acceleration $\boldsymbol{a}$ and the angular velocity of rotation $\boldsymbol{\omega}$ of an observer are independent of the spatial coordinates but may depend arbitrarily on time. The *exact* metric of the general noninertial frame has the form (31), where

$$V = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}, \quad W = 1, \quad \boldsymbol{K} = -\frac{1}{c}\,(\boldsymbol{\omega}\times\boldsymbol{r}). \tag{32}$$

Explicitly, the metric is given by [34]

$$ds^2 = \left[\left(1 + \frac{\boldsymbol{a}(t)\cdot\boldsymbol{r}}{c^2}\right)^2 - \frac{[\boldsymbol{\omega}(t)\times\boldsymbol{r}]^2}{c^2}\right] c^2dt^2 - 2[\boldsymbol{\omega}(t)\times\boldsymbol{r}]\cdot d\boldsymbol{r}dt - \delta_{ij}dx^idx^j. \tag{33}$$

The nonzero Lorentz connection coefficients have the same forms for time-dependent and time-independent inertial fields:

$$\Gamma_{\widehat{i}\widehat{j}\widehat{0}} = -\frac{ce_{ijk}\omega^k(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}, \quad \Gamma_{\widehat{0}\widehat{i}\widehat{0}} = -\Gamma_{\widehat{i}\widehat{0}\widehat{0}} = -\frac{a^i(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}. \tag{34}$$

The gravitoelectromagnetic fields are given by

$$\boldsymbol{\mathcal{E}} = -\frac{c\boldsymbol{a}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\widehat{0}}, \quad \boldsymbol{\mathcal{B}} = \frac{c^2\boldsymbol{\omega}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\widehat{0}}. \tag{35}$$

There is only the gravitoelectric field in the uniformly accelerated frame:

$$\boldsymbol{\mathcal{E}} = -\frac{c\boldsymbol{a}(t)}{c^2 + \boldsymbol{a}(t)\cdot\boldsymbol{r}}u^{\widehat{0}}, \quad \boldsymbol{\mathcal{B}} = 0. \tag{36}$$

In the rotating frame, there is only the gravitomagnetic field:

$$\boldsymbol{\mathcal{E}} = 0, \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\omega}(t)u^{\widehat{0}}. \tag{37}$$

It is important that the gravitoelectric field does not depend on $\widehat{\boldsymbol{u}}$. All equations presented in this subsection are *exact*.

The use of Eqs. (28) and (29) allows one to reproduce known formulas for the particle motion and the spin rotation in the general noninertial frame.

4.2 Gravitoelectromagnetic fields in weak-field approximation

The weak-field approximation can often be used. To find gravitoelectromagnetic fields in this approximation, we use the Schwinger gauge and suppose that the tetrad components $e_\mu^{\widehat{\mu}}$ ($\mu = \widehat{\mu},\ \mu = 0,1,2,3$) are close to unit. Under these conditions, some important general relations can be obtained:

$$\begin{gathered} e_{\widehat{\mu}}^{\nu} + e_{\mu}^{\widehat{\nu}} = \delta_\mu^\nu, \quad g_{\mu\nu} = e_{\widehat{\mu}\nu} + e_{\widehat{\nu}\mu}, \\ \Gamma_{abc} = \frac{1}{2}\left(e_{\widehat{a}b,c} - e_{\widehat{b}a,c} + g_{bc,a} - g_{ac,b}\right). \end{gathered} \tag{38}$$

As a result, nonzero Lorentz connection coefficients are given by

$$\begin{gathered} \Gamma_{\widehat{0}\widehat{i}\widehat{0}} = -\frac{1}{2}g_{00,i}, \quad \Gamma_{\widehat{0}\widehat{i}\widehat{j}} = -\frac{1}{2}\left(g_{0i,j} + g_{0j,i} - g_{ij,0}\right), \quad \Gamma_{\widehat{i}\widehat{j}\widehat{0}} = \frac{1}{2}\left(g_{0j,i} - g_{0i,j}\right), \\ \Gamma_{\widehat{i}\widehat{j}\widehat{k}} = \frac{1}{2}\left(g_{jk,i} - g_{ik,j}\right). \end{gathered} \tag{39}$$

The gravitoelectromagnetic fields are equal to

$$\begin{aligned}\mathcal{E}_{\widehat{i}} &= -\frac{c}{2}\left[g_{00,i}u^{\widehat{0}} + \left(g_{0i,j} + g_{0j,i} - g_{ij,0}\right)u^{\widehat{j}}\right],\\ \mathcal{B}_{\widehat{i}} &= \frac{c}{4}e_{ijk}\left[\left(g_{0j,k} - g_{0k,j}\right)u^{\widehat{0}} + \left(g_{jl,k} - g_{kl,j}\right)u^{\widehat{l}}\right].\end{aligned} \tag{40}$$

In Ref. [2], Eqs. (39) and (40) have been obtained for a stationary metric. Only the gravitoelectric field explicitly depends on the time derivative.

We can conclude that the equations of motion in coframes (28) and (29) become very simple in the weak-field approximation. In this case, they contain only first derivatives of the metric tensor and can be easily derived. The equations obtained are relativistic.

4.3 Lense-Thirring metric

Lense and Thirring have discovered in 1918 that rotating bodies "drag" the spacetime around themselves (frame dragging [35]). In other words, they have demonstrated the similarity between rotating frames and spacetimes created by rotating bodies.

The Lense-Thirring (LT) metric [35] defines a gravitational field of a rotating source in the weak-field approximation. It can be obtained from the Kerr metric when the distance from the source is much large than the gravitational radius. The static part of the LT metric characterizes the Schwarzschild field of a distant source. It is convenient to transform the LT metric to the isotropic coordinates [3]:

$$V = 1 - \frac{GM}{c^2 r}, \quad W = 1 + \frac{GM}{c^2 r}, \quad \boldsymbol{K} = \frac{\boldsymbol{\omega}\times\boldsymbol{r}}{c}, \quad \boldsymbol{\omega} = \frac{2G}{c^2 r^3}\boldsymbol{J} = \left(0,\ 0,\ \frac{2GMa}{c\,r^3}\right). \tag{41}$$

Here $\boldsymbol{J} = Mca\boldsymbol{e}_z$ is the total angular momentum of the source and M is its mass.

For this metric, the gravitoelectromagnetic fields read

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= -\frac{Gm}{cr^3}\boldsymbol{r}u^{\widehat{0}} + \frac{3G}{c^2 r^5}\left[\boldsymbol{r}(\boldsymbol{J}\cdot(\boldsymbol{r}\times\widehat{\boldsymbol{u}})) - (\boldsymbol{r}\times\boldsymbol{J})(\widehat{\boldsymbol{u}}\cdot\boldsymbol{r})\right],\\ \boldsymbol{\mathcal{B}} &= -\frac{Gm}{cr^3}\boldsymbol{r}\times\widehat{\boldsymbol{u}} - \frac{G}{c^2 r^3}\left[\frac{3(\boldsymbol{r}\cdot\boldsymbol{J})\boldsymbol{r}}{r^2} - \boldsymbol{J}\right]u^{\widehat{0}}.\end{aligned} \tag{42}$$

Equations (28) and (42) lead to the following equivalent equations of spin motion [3]:

$$\begin{aligned}\boldsymbol{\Omega} = \frac{1}{u^0}\Bigg\{&\frac{GM}{cr^3}\cdot\frac{2u^{\widehat{0}}+1}{u^{\widehat{0}}+1}\boldsymbol{r}\times\widehat{\boldsymbol{u}} + \frac{G}{c^2 r^5}\left[3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{J}) - r^2\boldsymbol{J}\right]u^{\widehat{0}}\\ &-\frac{3G}{c^2 r^5}\cdot\frac{1}{u^{\widehat{0}}+1}\left[(\boldsymbol{r}\times\widehat{\boldsymbol{u}})\,(\boldsymbol{J}\cdot(\boldsymbol{r}\times\widehat{\boldsymbol{u}})) + (\boldsymbol{r}\cdot\widehat{\boldsymbol{u}})\,(\widehat{\boldsymbol{u}}\times(\boldsymbol{r}\times\boldsymbol{J}))\right]\Bigg\},\end{aligned} \tag{43}$$

$$\begin{aligned}\boldsymbol{\Omega} = \frac{1}{u^0}\Bigg\{&\frac{GM}{cr^3}\cdot\frac{2u^{\widehat{0}}+1}{u^{\widehat{0}}+1}\boldsymbol{r}\times\widehat{\boldsymbol{u}} + \frac{G}{c^2 r^5}\left[3\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{J}) - r^2\boldsymbol{J}\right]u^{\widehat{0}}\\ &-\frac{3G}{c^2 r^5}\cdot\frac{1}{u^{\widehat{0}}+1}\left[2(\boldsymbol{r}\times\widehat{\boldsymbol{u}})\,(\boldsymbol{J}\cdot(\boldsymbol{r}\times\widehat{\boldsymbol{u}})) + (\widehat{\boldsymbol{u}}\times(\boldsymbol{r}\times\widehat{\boldsymbol{u}}))(\boldsymbol{r}\cdot\boldsymbol{J}) + r^2(\widehat{\boldsymbol{u}}\times(\widehat{\boldsymbol{u}}\times\boldsymbol{J}))\right]\Bigg\}.\end{aligned} \tag{44}$$

These equations describe the geodetic precession and the LT one for the *relativistic* particle.

Equations (43) and (44) agree with the approximate formula by Lense and Thirring [35]. However, one of important preferences of the relativistic approach is the discovery of additional dependence of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ on the nondiagonal and diagonal components of the metric tensor, respectively. The corresponding contributions to the angular velocity of the spin precession do not follow from the nonrelativistic approximation.

For the Schwarzschild metric in the isotropic coordinates far from the source, V and W are given by Eq. (41), $\boldsymbol{K} = 0$, and the gravitoelectromagnetic fields take the form

$$\boldsymbol{\mathcal{E}} = -\frac{GM\boldsymbol{r}}{cr^3}u^{\widehat{0}} = \frac{\boldsymbol{g}u^{\widehat{0}}}{c}, \quad \boldsymbol{\mathcal{B}} = -\frac{GM}{cr^3}\boldsymbol{r}\times\widehat{\boldsymbol{u}} = \frac{\boldsymbol{g}\times\widehat{\boldsymbol{u}}}{c}, \tag{45}$$

where $\boldsymbol{g}$ is the Newtonian acceleration. We can state the significant difference between the gravitoelectromagnetic fields in the uniformly accelerated frame and in the Schwarzschild spacetime. The gravitomagnetic field in the Schwarzschild spacetime, contrary to the uniformly accelerated frame, is nonzero. This field is not weak at least for relativistic particles.

5 Local Lorentz transformations of gravitoelectromagnetic fields

The properties of the local Lorentz transformations allow us to determine the general dependence of the gravitoelectromagnetic fields from the choice of a tetrad. The results obtained in Sec. 2 and the explicit expression of $\Gamma_{abc}u^c$ in terms of tetrads (18) lead to the conclusion that this quantity is an antisymmetric tensor relative to the local Lorentz transformations. Therefore, the gravitoelectric and gravitomagnetic fields, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, transform like the electric and magnetic ones:

$$\boldsymbol{\mathcal{E}}' = \gamma\left[\boldsymbol{\mathcal{E}} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{E}}) + \boldsymbol{\beta}\times\boldsymbol{\mathcal{B}}\right], \quad \boldsymbol{\mathcal{B}}' = \gamma\left[\boldsymbol{\mathcal{B}} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{B}}) - \boldsymbol{\beta}\times\boldsymbol{\mathcal{E}}\right]. \tag{46}$$

This property also establishes a great similarity between electromagnetism and gravity.

When the local Lorentz transformations (46) are used, one needs to take into account the dependence of the gravitoelectromagnetic fields on the four-velocity. To express $\boldsymbol{\mathcal{E}}'$ and $\boldsymbol{\mathcal{B}}'$ in terms of the four-velocity in the primed frame, one needs to transform the components of u^c entering into Eq. (27) as follows:

$$u^{\widehat{0}} = \gamma(u'^{\widehat{0}} + \boldsymbol{\beta}\cdot\widehat{\boldsymbol{u}}'), \quad \widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}' + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\widehat{\boldsymbol{u}}') + \boldsymbol{\beta}\gamma u'^{\widehat{0}}. \tag{47}$$

6 Mathisson force and Thomas precession

One of great achievements of Myron Mathisson was the discovery of an additional force acting on a spinning particle in a curved spacetime. The Mathisson force is similar to the Stern-Gerlach one in electrodynamics. The gravitoelectromagnetism allows us to explain the main difference between the two forces as a result of the specific dependence of the gravitoelectromagnetic fields on the particle four-velocity. An analysis of the Mathisson force can be fulfilled in the general case (see, e.g., Refs. [12, 29, 36]). Nevertheless, a derivation of a simple expression in the weak-field approximation seems to be rather important. In particular, this expression shows that a similarity between the gravity and electromagnetism exists even for effects depending on curvature.

To take into account an influence of the spin on the particle motion, we may use the Hamiltonian method and may add the Hamiltonian of a spinless particle in a gravitational field by the term $\boldsymbol{s}\cdot\boldsymbol{\Omega}$:

$$\mathcal{H} = \mathcal{H}_0 + \boldsymbol{s}\cdot\boldsymbol{\Omega}. \tag{48}$$

The possibility of this addition has been mentioned in Ref. [1]. In Refs. [11, 12], this addition has been rigorously substantiated not only in framework of classical gravity but also for Dirac particles.

It follows from Eqs. (28) and (48) that the additional force acting on the spinning particle is given by

$$\boldsymbol{f}_M = -\nabla(\mathcal{H} - \mathcal{H}_0) = \nabla\left(\frac{1}{u^0}\boldsymbol{s}\cdot\left[\boldsymbol{\mathcal{B}} - \frac{\widehat{\boldsymbol{u}}\times\boldsymbol{\mathcal{E}}}{u^{\widehat{0}}+1}\right]\right). \tag{49}$$

In electrodynamics, the Stern-Gerlach force *acting on a Dirac particle* has the similar form:

$$\boldsymbol{f}_{SG} = \frac{e}{mc}\nabla\left(\boldsymbol{s}\cdot\left[\boldsymbol{B} - \frac{\boldsymbol{u}\times\boldsymbol{E}}{u^0+1}\right]\right). \tag{50}$$

Equation (50) is obtained in the classical limit. The most important difference between Eqs. (49) and (50) consists in the dependence of the gravitoelectromagnetic fields on the four-velocity.

Since the Stern-Gerlach and Mathisson forces are proportional to gradients of scalars, they bring the additions only to the electric and gravitoelectric forces, respectively. This property has been proved in Ref. [12] for arbitrarily strong gravitational fields. As a result, $\nabla\times\boldsymbol{f}_{SG}=0$, $\nabla\times\boldsymbol{f}_M=0$.

The Mathisson force violates the weak equivalence principle [36] because particles with different spin orientations move on different trajectories. Since the gravitoelectromagnetic fields are proportional to derivatives of the metric, the Mathisson force is proportional to the curvature (see Ref. [12]).

The resulting force acting on a spinning particle in a local Lorentz frame is given by

$$\boldsymbol{F} = \frac{mcu^{\widehat{0}}}{u^0}\left(\boldsymbol{\mathcal{E}} + \frac{\widehat{\boldsymbol{u}}\times\boldsymbol{\mathcal{B}}}{u^{\widehat{0}}}\right) + \boldsymbol{f}_M. \tag{51}$$

In Ref. [12], the Mathisson force has been obtained in an explicit form. Evidently, Eq. (51) presents the next order approximation as compared with the PK equations.

To derive the general equation for the Thomas precession, we can follow the method developed in electrodynamics and presented in Refs. [32, 37]. The angular velocity of the Thomas precession in GR is very similar to that in electrodynamics and is given by

$$\boldsymbol{\omega}_T = -\frac{1}{\gamma+1}\left(\widehat{\boldsymbol{u}}\times\frac{d\widehat{\boldsymbol{u}}}{d\widehat{t}}\right). \tag{52}$$

7 Discussion and summary

We have considered in detail the Lorentz transformations in coframes and have shown that four-vectors with tetrad components (including e^a_μ with fixed μ) satisfy the local Lorentz transformations.

We have derived the equations of motion in coframes and have analyzed the introduction of the gravitoelectromagnetic fields (27) being the fields in the anholonomic tetrad frame. This introduction significantly simplifies a description of motion of spinning particles in GR. The gravitoelectromagnetic fields are only effective because they depend on the four-velocity of the test particle. However, their introduction allows one to define dynamics of the four-velocity and spin by Eqs. (28) and (29) which are similar to the corresponding equations in electrodynamics. Equations (28) and (29) are valid when the mutual influence of particle and spin motion is neglected. In this approximation, the equation of spin motion is linear in spin and Eq. (28) is perfect. As a contrary, Eq. (29) overcomes the nonlinearity of GR *only formally* because of the dependence of the gravitoelectromagnetic fields on the four-velocity. The use of Eq. (29) for a description of the particle motion is less convenient because this motion is defined by the

evolution of the world four-velocity. However, the transition to the world coordinates in Eq. (29) is not difficult because

$$u^{\mu} = e^{\mu}_{a} u^{a}, \quad \frac{du^{\mu}}{dt} = e^{\mu}_{a} \frac{du^{a}}{dt} + \left(\frac{\partial e^{\mu}_{a}}{\partial t} + v^{i} e^{\mu}_{a,\,i} \right) u^{a}.$$

Moreover, the investigation of the particle motion is simplified when the trajectory is infinite. In this case, one can apply the fact that the quantities u^{μ} and u^{a} coincide at the initial and final parts of such a trajectory of the particle because of the very large distance to the field source [11].

We have calculated the gravitoelectromagnetic fields in the most important special cases. In particular, we have obtained the general formula (40) for the gravitoelectromagnetic fields in the weak-field approximation and have demonstrated the significant difference between the gravitoelectromagnetic fields in the uniformly accelerated frame and in the Schwarzschild spacetime far from the source. The gravitomagnetic field in the Schwarzschild spacetime, contrary to the uniformly accelerated frame, is nonzero.

The developed conception of gravitoelectromagnetism is based on the PK fields and on the preferable use of the Schwinger gauge. The conception is applicable for relativistic spinning particles in arbitrarily strong gravitational fields and in noninertial frames. Different tetrads belonging the Schwinger gauge may lead to different equations of motion [7] and to different gravitoelectromagnetic fields. The gravitoelectromagnetic fields satisfy the local Lorentz transformations (46) similar to the Lorentz transformations in electrodynamics.

We should underline that the results presented demonstrate the equivalence of all tetrads. While tetrads belonging the Schwinger gauge are much more convenient, the use of other tetrads is also possible.

In Sec. 6, we have obtained the general formulas (49) and (52) for the Mathisson force and the Thomas precession, respectively.

Finally, we can conclude that the introduction of the gravitoelectromagnetic fields applicable for an exact description of relativistic inertial and gravitational effects can be regarded as a important powerful method in general relativity.

Acknowledgements

The author is grateful to E. A. Tolkachev for valuable discussions. The work was supported in part by the Belarusian Republican Foundation for Fundamental Research (grant No. Φ14D-007) and by the Heisenberg-Landau program of the German Ministry for Science and Technology (Bundesministerium für Bildung und Forschung).

References

[1] A.A. Pomeransky and I.B. Khriplovich, Zh. Eksp. Teor. Fiz. **113**, 1537 (1998) [J. Exp. Theor. Phys. **86**, 839 (1998)].

[2] A.J. Silenko, Acta Phys. Polon. B Proc. Suppl. **1**, 87 (2008).

[3] Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **80**, 064044 (2009).

[4] Yu. N. Obukhov, A. J. Silenko, and O. V. Teryaev, Phys. Rev. D **90**, 124068 (2014).

[5] P.A.M. Dirac, Interacting gravitational and spinor fields, in: *Recent developments in general relativity*, Festschrift for Infeld. Pergamon Press, Oxford and PWN, Warsaw (1962), pp. 191-200.

[6] J. Schwinger, Phys. Rev. **130**, 800 (1963); 1253 (1963).

[7] M. Arminjon, Int. J. Theor. Phys. **52**, 4032 (2013); **53**, 1993 (2014); **54**, 2218 (2014).
[8] M. Arminjon, J. Phys.: Conf. Ser. **626**, 012030 (2015).
[9] A.A. Pomeransky, R.A. Senkov, and I.B. Khriplovich, Usp. Fiz. Nauk **43**, 1129 (2000) [Phys. Usp. **43**, 1055 (2000)].
[10] M. Dvornikov, Int. J. Mod. Phys. D **15**, 1017 (2006).
[11] Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **84**, 024025 (2011).
[12] Y.N. Obukhov, A.J. Silenko, O.V. Teryaev, Phys. Rev. D **88**, 084014 (2013).
[13] B. Mashhoon, Gravitoelectromagnetism: A brief review, in: *The Measurement of Gravitomagnetism: A Challenging Enterprise*. Edited by L. Iorio, Nova Science, New York (2007), pp. 29-39.
[14] A. Tartaglia and M.L. Ruggiero, Eur. J. Phys. **25**, 203 (2004); A. Tartaglia and M.L. Ruggiero, Analogies and Differences between Gravito-Electromagnetism and Electromagnetism, in: *The Measurement of Gravitomagnetism: A Challenging Enterprise*. Edited by L. Iorio, Nova Science, New York (2007), pp. 41-50.
[15] L. Ryder, *Introduction to General Relativity*. Cambridge Univ. Press, New York (2009).
[16] I. Bailey, W. Israel, Commun. Math. Phys. **42**, 65 (1975).
[17] K. Yee, M. Bander, Phys. Rev. D **48**, 2797 (1993).
[18] J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **77**, 081501(R) (2008); J. Steinhoff, G. Schäfer, and S. Hergt, Phys. Rev. D **77**, 104018 (2008); J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **78**, 101503(R) (2008); J. Steinhoff, G. Schäfer, Europhys. Lett. **87**, 50004 (2009).
[19] E. Barausse, E. Racine, and A. Buonanno, Phys. Rew. D **80**, 104025 (2009).
[20] G. Schäfer, Gen. Rel. Grav. **36**, 2223 (2014).
[21] R.T. Jantzen, P. Carini and D. Bini, Ann. Phys. (N.Y.) **215**, l (1992).
[22] L.F.O. Costa, Int. J. Mod. Phys. A **24**, 1695 (2009).
[23] A.J. Silenko and O.V. Teryaev, Phys. Rev. D **71**, 064016 (2005).
[24] A.J. Silenko and O.V. Teryaev, Phys. Rev. D **76**, 061101(R) (2007).
[25] M. Mathisson, Acta Phys. Polon. **6**, 163 (1937).
[26] A. Papapetrou, Proc. Roy. Soc. Lond. A **209**, 248 (1951).
[27] F. Pirani, Acta Phys. Polon. **15**, 389 (1956).
[28] W. Tulczyjev, Acta Phys. Polon. **18**, 393 (1959).
[29] R. Plyatsko, M. Fenyk, Phys. Rew. D **91**, 064033 (2015).
[30] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, *Quantum electrodynamics. Course of Theoretical Physics*. Vol. 4, 2nd ed. Butterworth-Heinemann, Oxford (1982).
[31] L.H. Thomas, Nature (London), **117**, 514 (1926); Philos. Mag. **3**, 1 (1927).
[32] J.D. Jackson, *Classical Electrodynamics*. 3rd ed. New York, John Wiley & Sons (1998).
[33] S. Hergt and G. Schäfer, Phys. Rev. D **77**, 104001 (2008).
[34] F.W. Hehl and W.T. Ni, Phys. Rev. D **42**, 2045 (1990).
[35] H. Thirring, Phys. Z. **19** (1918), 33 [Gen. Rel. Grav. 16 (1984), 712]; Phys. Z. **22**, 29 (1921) [Gen. Rel. Grav. **16**, 725 (1984)]; J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918) [Gen. Rel. Grav. **16**, 727 (1984)].
[36] R. Plyatsko, Phys. Rev. D **58**, 084031 (1998); R. Plyatsko and M. Fenyk, **85**, 104023 (2012); **87**, 044019 (2013).
[37] A.J. Silenko, Phys. Scr. **90**, 065303 (2015).

Casimir effect in the background of a fiber bundle

Yu.A. Sitenko
yusitenko@bitp.kiev.ua; Bogolyubov Institute for Theoretical Physics,
National Academy of Sciences of Ukraine, Kyiv, Ukraine

Abstract

The quantized massive matter field is described in terms of a spinor section of a fiber bundle, and an influence of a background field given by the bundle curvature on the vacuum of the quantized matter in the base space bounded by two parallel material plates is studied. Employing the most general set of boundary conditions at the plates in the case of the background uniform magnetic field directed orthogonally to the plates, we find the pressure from the vacuum onto the plates. In physically plausible situations, the Casimir effect in the background of the bundle curvature is repulsive, independently of a choice of boundary conditions and of a distance between the plates.

1 Introduction

Almost seven decades ago, Casimir [1] predicted an attraction between grounded metal plates as a macroscopic effect of vacuum fluctuations in quantum field theory. Since then, his prediction has been confirmed experimentally with great precision, opening prospects for its application in modern nanotechnology, see review in [2].

The detected Casimir force (or pressure) between parallel plates separated by distance a,

$$F_{EM} = -\frac{\pi^2}{240a^4}, \tag{1}$$

is due to the vacuum fluctuations of the quantized electromagnetic field only [1]. As to the vacuum fluctuations of other quantized fields, their contribution to the Casimir effect was theoretically considered erstwhile, see, e.g., [2]. It suffices to note here that this contribution is of order of a^{-4} at $a \ll \lambda_C$ and of order of $a^{-4}(a/\lambda_C)^{\nu}\exp(-2a/\lambda_C)$ at $a \gg \lambda_C$, where $\lambda_C = m^{-1}$ is the Compton wavelength of the matter field of mass m; the sign of this contribution, as well as exponent ν, depends on a boundary condition and the spin of the matter field. Usually, the Casimir effect is validated experimentally for the macroscopic separation of plates: $a > 10^{-8}\,\mathrm{m}$. So, even if one takes the lightest massive particle, electron ($\lambda_C = 3.86 \times 10^{-13}\,\mathrm{m}$), then it becomes clear that the case of $a \ll \lambda_C$ has no relation to physics reality. Whereas, in the realistic case of $a \gg \lambda_C$, the contribution of the vacuum fluctuations of quantized massive matter fields to the Casimir effect is vanishing.

However, quantized massive matter fields can be described in terms of sections of fiber bundles. In the present work, I study an impact of the bundle curvature considered as a bacground field on the vacuum of the quantized massive matter field described in terms of a section of a fiber bundle; both the quantized and bacground fields are confined to a bounded base space. A crucial issue for my analysis is a choice of boundary conditions, and I adhere to the most general one. Namely, the principles of comprehensibility and mathematical consistency require that operators of physical observables in quantum mechanics be self-adjoint, see, e.g., [3]. To put

it simply, a multiple action is well defined for a self-adjoint operator only, allowing for the construction of functions of the operator, such as resolvent, evolution, heat kernel and zeta-function operators, with further implications upon second quantization.

The mathematical demand for the self-adjointness of a differential operator acting on sections of fiber bundles in the case of a bounded base space is a somewhat more general than the physical demand for the confinement of appropriate quantized matter fields within this space. The concept of confined matter fields is quite familiar in the context of condensed matter physics: collective excitations (e.g., spin waves and phonons) exist only inside material objects and do not spread outside. Nonetheless, a quest for boundary conditions ensuring the confinement of the quantized matter was initiated in particle physics in the context of a model description of hadrons as composites of quarks and gluons [4, 5]. If an hadron is an extended object occupying spatial region Ω bounded by surface $\partial\Omega$, then the quark matter field, $\psi(\mathbf{r})$, is subject to the MIT bag boundary condition [6]:

$$[I + \mathrm{i}\beta(\mathbf{n}\cdot\boldsymbol{\alpha})]\psi(\mathbf{r})|_{\mathbf{r}\in\partial\Omega} = 0, \tag{2}$$

where α^1, α^2, α^3 and β are the generating elements of the Dirac-Clifford algebra. However, the point is that this boundary condition is not the only one. The most general set of boundary conditions in the case of a simply-connected boundary involves four arbitrary parameters [7], and the explicit form for this set has been given [8]; the set is compatible with the self-adjointness of the Dirac hamiltonian operator, and its four parameters can be interpreted as the self-adjoint extension parameters. In the present work, I follow the lines of works [7, 8] by proposing a different, embracing more cases, form of the four-parameter generalization of the MIT bag boundary condition.

Thus, let us consider a spinor section of a fiber bundle with an Abelian bundle connection; the bundle curvature can be interpreted as the magnetic field strength. Both the bundle curvature and the spinor section are confined to three-dimensional spatial region Ω bounded by two-dimensional surface $\partial\Omega$. To study a response of the vacuum of the spinor matter to the background bundle curvature, we restrict ourselves to the case of a boundary consisting of two parallel planes; the background magnetic field strength is assumed to be uniform and orthogonal to the planes. As was already explained, we start from the most general set of mathematically acceptable (i.e. compatible with the self-adjointness) boundary conditions. Further follow physical constraints that the spinor matter be confined within the planes and that the spectrum of the wave number vector in the direction which is orthogonal to the planes be unambiguously (although implicitly) determined. Employing these mathematical and physical restrictions, I consider the generalized Casimir effect which is due to vacuum fluctuations of the quantized spinor matter field in the presence of the background uniform magnetic field; the pressure from the vacuum onto the bounding planes will be found.

2 Self-adjointness and boundary conditions

Defining a scalar product as $(\tilde{\chi}, \chi) = \int\limits_{\Omega} \mathrm{d}^3 r\, \tilde{\chi}^\dagger \chi$, we get, using integration by parts,

$$(\tilde{\chi}, H\chi) = (H^\dagger\tilde{\chi}, \chi) - \mathrm{i}\int\limits_{\partial\Omega} \mathrm{d}\mathbf{s}\cdot\tilde{\chi}^\dagger\boldsymbol{\alpha}\chi, \tag{3}$$

where

$$H = H^\dagger = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m \tag{4}$$

is the formal expression for the Dirac hamiltonian operator and $\boldsymbol{\nabla}$ is the covariant derivative involving both the affine and bundle connections (natural units $\hbar = c = 1$ are used). Operator H is Hermitian (or symmetric in mathematical parlance),

$$(\tilde{\chi}, H\chi) = (H^{\dagger}\tilde{\chi}, \chi), \tag{5}$$

if

$$\int\limits_{\partial\Omega} \mathrm{d}\mathbf{s} \cdot \tilde{\chi}^{\dagger}\boldsymbol{\alpha}\chi = 0. \tag{6}$$

The latter condition can be satisfied in various ways by imposing different boundary conditions for χ and $\tilde{\chi}$. However, among the whole variety, there may exist a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ; then the domain of definition of $H^{\dagger}$ (set of functions $\tilde{\chi}$) coincides with that of H (set of functions χ), and operator H is self-adjoint. The action of a self-adjoint operator results in functions belonging to its domain of definition only, and, therefore, a multiple action and functions of such an operator can be consistently defined.

Condition (6) is certainly fulfilled when the integrand in (6) vanishes, i.e.

$$\tilde{\chi}^{\dagger}(\mathbf{n} \cdot \boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{7}$$

To fulfill the latter condition, we impose the same boundary condition for χ and $\tilde{\chi}$ in the form

$$\chi|_{\mathbf{r}\in\partial\Omega} = K\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = K\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{8}$$

where K is a matrix (element of the Dirac-Clifford algebra) which is determined by conditions

$$K^2 = I, \quad K^{\dagger}(\mathbf{n} \cdot \boldsymbol{\alpha})K = -\mathbf{n} \cdot \boldsymbol{\alpha}. \tag{9}$$

It should be noted that, in addition to (7), the following combination of χ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^{\dagger}(\mathbf{n} \cdot \boldsymbol{\alpha})K\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{\chi}^{\dagger}K^{\dagger}(\mathbf{n} \cdot \boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{10}$$

Using the standard representation for the Dirac matrices, one can get matrix K in the off-diagonal form [8]

$$K = \frac{(1 + u^2 - v^2 - \mathbf{t}^2)\beta + (1 - u^2 + v^2 + \mathbf{t}^2)I}{2\mathrm{i}(u^2 - v^2 - \mathbf{t}^2)}(u\mathbf{n} \cdot \boldsymbol{\alpha} + v\beta\gamma^5 - \mathrm{i}\mathbf{t} \cdot \boldsymbol{\alpha}), \tag{11}$$

where $\mathbf{t} = (t^1, t^2)$ is a two-dimensional vector which is tangential to the boundary, $\mathbf{t} \cdot \mathbf{n} = 0$, and $\gamma^5 = \mathrm{i}\alpha^1\alpha^2\alpha^3$. Matrix K is Hermitian in two cases only when it takes forms

$$K_+ = -\mathrm{i}\beta(\mathbf{n} \cdot \boldsymbol{\alpha}) \quad (u = 1, \quad v = 0, \quad \mathbf{t} = 0) \tag{12}$$

and

$$K_- = \mathrm{i}v\beta\gamma^5 + \mathbf{t} \cdot \boldsymbol{\alpha} \quad (u = 0, \quad v^2 + \mathbf{t}^2 = 1). \tag{13}$$

Matrix K_+ (12) corresponds to the choice of the standard MIT bag boundary condition [6], cf. (2),

$$(I - K_+)\chi|_{\mathbf{r}\in\partial\Omega} = (I - K_+)\tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = 0, \tag{14}$$

when relation (10) takes form

$$\tilde{\chi}^{\dagger}\beta\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{15}$$

It is instructive to go over from off-diagonal matrix K (11) to Hermitian matrix $\tilde{K}$, presenting boundary condition (8) as

$$\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{16}$$

with $\tilde{K} = \tilde{K}^\dagger$ determined by conditions

$$\tilde{K}^2 = I, \quad [\tilde{K}, \mathbf{n}\cdot\boldsymbol{\alpha}]_+ = 0. \tag{17}$$

This transition is implemented with the use of the block-diagonal Hermitian matrix, N, obeying condition

$$(I-N)K = K^\dagger(I-N); \tag{18}$$

namely, the result is

$$\tilde{K} = (I-N)K + N. \tag{19}$$

Using parametrization

$$u = -\frac{\sin\tilde{\varphi}}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad v = \frac{\sin\varphi\cos\theta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}},$$

$$t^1 = \frac{\sin\theta\cos\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad t^2 = \frac{\sin\theta\sin\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}},$$

$$-\pi/2 < \varphi \le \pi/2, \quad -\pi/2 \le \tilde{\varphi} < \pi/2, \quad 0 \le \theta < \pi, \quad 0 \le \eta < 2\pi, \tag{20}$$

one gets

$$K = \mathrm{i}\frac{\beta\cos\varphi\cos\theta + I\cos\tilde{\varphi}}{\cos^2\varphi\cos^2\theta - \cos^2\tilde{\varphi}}[\mathbf{n}\cdot\boldsymbol{\alpha}\sin\tilde{\varphi} - \beta\gamma^5\sin\varphi\cos\theta + \mathrm{i}(\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta)], \tag{21}$$

where

$$[\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^1]_+ = [\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^2]_+ = [\alpha^1, \alpha^2]_+ = 0. \tag{22}$$

Then matrix N takes form

$$N = \beta\cos\varphi\cos\tilde{\varphi}\cos\theta - \beta\gamma^5(\mathbf{n}\cdot\boldsymbol{\alpha})\sin\varphi\sin\tilde{\varphi}\cos\theta +$$

$$+\mathrm{i}(\alpha^1\cos\eta + \alpha^2\sin\eta)(\mathbf{n}\cdot\boldsymbol{\alpha})\sin\tilde{\varphi}\sin\theta, \tag{23}$$

and one gets

$$\tilde{K} = [\beta\mathrm{e}^{\mathrm{i}\varphi\gamma^5}\cos\theta + (\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta]\mathrm{e}^{\mathrm{i}\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}; \tag{24}$$

in particular,

$$K_+ = \tilde{K}|_{\varphi=0,\,\tilde{\varphi}=-\pi/2,\,\theta=0}, \quad K_- = \tilde{K}|_{\varphi=\pi/2,\,\tilde{\varphi}=0}. \tag{25}$$

Thus, the explicit form of the boundary condition ensuring the self-adjointness of operator H (4) is

$$\left\{I - [\beta\mathrm{e}^{\mathrm{i}\varphi\gamma^5}\cos\theta + (\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta]\mathrm{e}^{\mathrm{i}\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}\right\}\chi|_{\mathbf{r}\in\partial\Omega} = 0 \tag{26}$$

(the same condition is for $\tilde{\chi}$). Four parameters of boundary condition (26), $\varphi, \tilde{\varphi}, \theta$ and η, are interpreted as the self-adjoint extension parameters.

In the context of the Casimir effect, one usually considers spatial region Ω with a disconnected boundary consisting of two connected components, $\partial\Omega = \partial\Omega^{(+)}\bigcup\partial\Omega^{(-)}$. Choosing coordinates $\mathbf{r} = (x, y, z)$ in such a way that x and y are tangential to the boundary, while z is normal to it, we identify the position of $\partial\Omega^{(\pm)}$ with, say, $z = \pm a/2$. In general, there are 8 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, θ_+ and η_+ corresponding to $\partial\Omega^{(+)}$ and φ_-, $\tilde{\varphi}_-$, θ_- and η_- corresponding to $\partial\Omega^{(-)}$. However, if some symmetry is present, then the number of self-adjoint extension parameters is diminished. For instance, if the boundary consists of two parallel planes, then the cases differing by the values of η_+ or η_- are physically indistinguishable, since they are

related by a rotation around a normal to the boundary. To avoid this unphysical degeneracy, one has to fix

$$\theta_+ = \theta_- = 0, \tag{27}$$

and there remains 4 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, φ_- and $\tilde{\varphi}_-$. Operator H (4) acting on functions which are defined in the region bounded by two parallel planes is self-adjoint, if the following condition holds:

$$\left\{ I - \beta \exp[\mathrm{i}(\varphi_\pm \gamma^5 \pm \tilde{\varphi}_\pm \alpha^z)] \right\} \chi|_{z=\pm a/2} = 0 \tag{28}$$

(the same condition holds for $\tilde{\chi}$). The latter ensures the fulfilment of constraints

$$\tilde{\chi}^\dagger \alpha^z \chi|_{z=\pm a/2} = 0 \tag{29}$$

and

$$\tilde{\chi}^\dagger \beta \exp\left\{ \mathrm{i}[\varphi_\pm \gamma^5 \pm (\tilde{\varphi}_\pm + \pi/2)\alpha^z] \right\} \chi|_{z=\pm a/2} = 0. \tag{30}$$

3 Induced vacuum energy in the bundle curvature background

Let us consider the quantized charged massive spinor field in the background of a static uniform magnetic field; then $\boldsymbol{\nabla} = \boldsymbol{\partial} - \mathrm{i}e\mathbf{A}$ and the connection can be chosen as $\mathbf{A} = (-yB, 0, 0)$, where B is the value of the magnetic field strength which is directed along the z-axis in Cartesian coordinates $\mathbf{r} = (x, y, z)$, $\mathbf{B} = (0, 0, B)$. The one-particle energy spectrum is

$$E_{nk} = \pm\omega_{nk}, \quad \omega_{nk} = \sqrt{2n|eB| + k^2 + m^2}, \quad -\infty < k < \infty, \quad n = 0, 1, 2, \ldots, \tag{31}$$

k is the value of the wave number vector along the z-axis, and n labels the Landau levels. Using the explicit form of the complete set of solutions to the Dirac equation, one can obtain the formal expression for the vacuum expectation value of the energy density

$$\varepsilon^\infty = -\frac{|eB|}{2\pi^2} \int\limits_{-\infty}^{\infty} \mathrm{d}k \sum_{n=0}^{\infty} \iota_n \omega_{nk}, \tag{32}$$

where the superscript on the left-hand side indicates that the magnetic field fills the whole (infinite) space; the appearance of factor $\iota_n = 1 - \frac{1}{2}\delta_{n0}$ on the right-hand side is due to the fact that there is one solution for the lowest Landau level, $\psi^{(0)}_{q0k}(\mathbf{r})$ (q is the value of the wave number vector along the x-axis, $-\infty < q < \infty$), and there are two solutions otherwise, $\psi^{(j)}_{qnk}(\mathbf{r})$ ($j = 1, 2$), $n \geq 1$. The integral and the sum in (32) are divergent and require regularization and renormalization. This problem has been solved long ago by Heisenberg and Euler [9] (see also [10]), and we just list here their result

$$\varepsilon^\infty_{\mathrm{ren}} = \frac{1}{8\pi^2} \int\limits_0^\infty \frac{\mathrm{d}\tau}{\tau} \mathrm{e}^{-\tau} \left[\frac{eBm^2}{\tau} \coth\left(\frac{eB\tau}{m^2}\right) - \frac{m^4}{\tau^2} - \frac{1}{3}e^2B^2 \right]; \tag{33}$$

note that the renormalization procedure involves subtraction at $B = 0$ and renormalization of the charge.

Let us turn now to the quantized charged massive spinor field in the background of a static uniform magnetic field in spatial region Ω bounded by two parallel planes $\partial\Omega^{(+)}$ and $\partial\Omega^{(-)}$; the position of $\partial\Omega^{(\pm)}$ is identified with $z = \pm a/2$, and the magnetic field is orthogonal to the planes.

The solution to the Dirac equation in region Ω is chosen as a superposition of two plane waves propagating in opposite directions along the z-axis,

$$\psi^{(j)}_{qnl}(\mathbf{r}) = \psi^{(j)}_{qnk_l}(\mathbf{r}) + \psi^{(j)}_{qn-k_l}(\mathbf{r}), \quad j = 0, 1, 2, \tag{34}$$

where the values of wave number vector k_l ($l = 0, \pm 1, \pm 2, ...$) are determined from the boundary condition, see (28):

$$\Big\{ I - \beta \exp[\mathrm{i}(\varphi_\pm \gamma^5 \pm \tilde{\varphi}_\pm \alpha^z)] \Big\} \psi^{(j)}_{qnl}(\mathbf{r})|_{z=\pm a/2} = 0, \quad (j = 1, 2) \quad n \geq 1 \tag{35}$$

and

$$\Big[I + \frac{\beta}{2} \Big(\pm \alpha^z \gamma^5 - 1 \Big) \mathrm{e}^{\mathrm{i}(\varphi_\pm - \tilde{\varphi}_\pm)\gamma^5} \Theta(\pm eB) -$$

$$- \frac{\beta}{2} \Big(\pm \alpha^z \gamma^5 + 1 \Big) \mathrm{e}^{\mathrm{i}(\varphi_\pm + \tilde{\varphi}_\pm)\gamma^5} \Theta(\mp eB) \Big] \psi^{(0)}_{q0l}(\mathbf{r})|_{z=\pm a/2} = 0; \tag{36}$$

the step function is introduced as $\Theta(u) = 1$ at $u > 0$ and $\Theta(u) = 0$ at $u < 0$. This boundary condition ensures that the normal component of current

$$\mathbf{J}_{qnlj}(\mathbf{r}) = \psi^{(j)\dagger}_{qnl}(\mathbf{r}) \boldsymbol{\alpha} \psi^{(j)}_{qnl}(\mathbf{r})$$

($j = 0, 1, 2$) vanishes at the planes, see (29),

$$J^z_{qnlj}(\mathbf{r})|_{z=\pm a/2} = 0, \tag{37}$$

which signifies that the quantized matter is confined within the planes.

The boundary condition depends on four self-adjoint extension parameters, $\varphi_+, \tilde{\varphi}_+, \varphi_-$ and $\tilde{\varphi}_-$, in the case of $n \geq 1$, see (35), and on two self-adjoint extension parameters, $\varphi_+ - \tilde{\varphi}_+$ and $\varphi_- + \tilde{\varphi}_-$ ($eB > 0$), or $\varphi_+ + \tilde{\varphi}_+$ and $\varphi_- - \tilde{\varphi}_-$ ($eB < 0$), in the case of $n = 0$, see (36). It should be noted that value $k_l = 0$ is allowed for special cases only, when the following condition holds:

$$\sin \frac{\varphi_+ - \varphi_- + \tilde{\varphi}_+ + \tilde{\varphi}_-}{2} \sin \frac{\varphi_+ - \varphi_- - \tilde{\varphi}_+ - \tilde{\varphi}_-}{2} = 0. \tag{38}$$

The spectrum of k_l is determined from a transcendental equation which in general possesses two branches and allows for complex values of k_l (details will be published elsewhere). It is not clear which of the branches should be chosen, and, therefore, we restrict ourselves to boundary conditions corresponding to the case of a single branch. The latter is ensured by imposing constraint

$$\varphi_+ = \varphi_- = \varphi, \quad \tilde{\varphi}_+ = \tilde{\varphi}_- = \tilde{\varphi}. \tag{39}$$

Then relations (28) and (30) take forms

$$\Big\{ I - \beta \exp[\mathrm{i}(\varphi \gamma^5 \pm \tilde{\varphi} \alpha^z)] \Big\} \chi|_{z=\pm a/2} = 0 \tag{40}$$

and

$$\tilde{\chi}^\dagger \beta \exp \Big\{ \mathrm{i}[\varphi \gamma^5 \pm (\tilde{\varphi} + \pi/2) \alpha^z] \Big\} \chi|_{z=\pm a/2} = 0. \tag{41}$$

respectively, while the equation determining the spectrum of k_l takes form

$$\cos(k_l a) + \frac{\omega_{nk_l} \operatorname{sgn}(E_{nk_l}) \cos\tilde{\varphi} - m \cos\varphi}{k_l \sin\tilde{\varphi}} \sin(k_l a) = 0, \tag{42}$$

where $\operatorname{sgn}(u) = \Theta(u) - \Theta(-u)$ is the sign function; note that the spectrum is real, consisting of values of the same sign, say, $k_l > 0$ (values of the opposite sign ($k_l < 0$) should be excluded to

avoid double counting). In the case of $\tilde{\varphi} = 0$, the spectrum of k_l is independent of the number of the Landau level, n, and of the sign of the one-particle energy, $\mathrm{sgn}(E_{nk_l})$; moreover, it is independent of φ, since the determining equation takes form

$$\sin(k_l a) = 0; \tag{43}$$

note that value $k_l = 0$ is admissible in this case, see (38)-(39). In what follows, we shall consider the most general case of two self-adjoint extension parameters, φ and $\tilde{\varphi}$, when the k_l-spectrum depends on n and on $\mathrm{sgn}(E_{nk_l})$, see (42).

Using the explicit form of the complete set of wave functions $\psi^{(j)}_{qnl}(\mathbf{r})\,(j = 0, 1, 2)$, we obtain the following expression for the vacuum expectation value of the energy per unit area of the boundary surface

$$\frac{E}{S} \equiv \int\limits_{-a/2}^{a/2} \mathrm{d}z\, \varepsilon = -\frac{|eB|}{2\pi} \sum_{\mathrm{sgn}(E_{nk_l})} \sum_l \sum_{n=0}^{\infty} \iota_n \omega_{nk_l}. \tag{44}$$

4 Casimir energy and force

Expression (44) can be regarded as purely formal, since it is ill-defined due to the divergence of infinite sums over l and n. To tame the divergence, a factor containing a regularization parameter should be inserted in (44). A summation over values $k_l > 0$, which are determined by (42), can be performed with the use of the following version of the Abel-Plana formula:

$$\sum_{\mathrm{sgn}(E_{nk_l})} \sum_{k_l>0} f(k_l^2) = \frac{a}{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}k f(k^2) + \frac{2\mathrm{i}a}{\pi} \int\limits_0^{\infty} \mathrm{d}\kappa \Lambda(\kappa) \{f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2]\}$$

$$-f(0) - \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}k f(k^2) \frac{m \cos\varphi \sin\tilde{\varphi}[k^2 - \mu_n(\varphi, \tilde{\varphi})]}{[k^2 + \mu_n(\varphi, \tilde{\varphi})]^2 + 4k^2 m^2 \cos^2\varphi \sin^2\tilde{\varphi}}, \tag{45}$$

where

$$\Lambda(\kappa) = \Bigg(-[\kappa^2 \cos 2\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})]\mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})$$

$$+ \frac{\sin\tilde{\varphi}}{a} \Big\{ -\kappa^2 m \cos\varphi (\cos 2\tilde{\varphi} \mathrm{e}^{2\kappa a} - 1) + [(2\kappa \sin\tilde{\varphi} - m\cos\varphi)\mathrm{e}^{2\kappa a}$$

$$+ m\cos\varphi]\mu_n(\varphi, \tilde{\varphi})\} \left[\kappa^2 - 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})\right]^{-1} \Bigg)$$

$$\times \Big\{ [\kappa^2 - 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})]\mathrm{e}^{4\kappa a}$$

$$-2[\kappa^2 \cos 2\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi})]\mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m \cos\varphi \sin\tilde{\varphi} - \mu_n(\varphi, \tilde{\varphi}) \Big\}^{-1} \tag{46}$$

and

$$\mu_n(\varphi, \tilde{\varphi}) = 2n|eB| \cos^2\tilde{\varphi} + m^2 \sin(\varphi + \tilde{\varphi}) \sin(\varphi - \tilde{\varphi}). \tag{47}$$

In (45), $f(u^2)$ as a function of complex variable u is assumed to decrease sufficiently fast at large distances from the origin of the complex u-plane, and this decrease is due to the use of

some kind of regularization for (44). However, the regularization in the second integral on the right-hand side of (45) can be removed; then

$$\mathrm{i}\{f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2]\} = -\frac{|eB|}{\pi}\sum_{n=0}^{\infty}\iota_n\sqrt{\kappa^2-\omega_{n0}^2} \tag{48}$$

with the range of κ restricted to $\kappa > \omega_{n0}$ for the corresponding terms; here, recalling (31), $\omega_{n0} = \sqrt{2n|eB|+m^2}$. As to the first integral on the right-hand side of (45), one immediately recognizes that it is equal to ε^{∞} (32) multiplied by a. Hence, if one ignores for a moment the terms in the last line of (45), then the problem of regularization and removal of the divergency in expression (44) is the same as that in the case of no boundaries, when the magnetic field fills the whole space. Defining the Casimir energy as the vacuum energy per unit area of the boundary surface, which is renormalized in the same way as in the case of no boundaries, we obtain

$$\frac{E_{\mathrm{ren}}}{S} = a\varepsilon^{\infty}_{\mathrm{ren}} - \frac{2|eB|}{\pi^2}a\sum_{n=0}^{\infty}\iota_n\int\limits_{\omega_{n0}}^{\infty}\mathrm{d}\kappa\Lambda(\kappa)\sqrt{\kappa^2-\omega_{n0}^2}$$

$$+\frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\iota_n\omega_{n0} + \frac{|eB|}{2\pi^2}\int\limits_{-\infty}^{\infty}\mathrm{d}k\sum_{n=0}^{\infty}\iota_n\sqrt{k^2+\omega_{n0}^2}\frac{m\cos\varphi\sin\tilde{\varphi}[k^2-\mu_n(\varphi,\tilde{\varphi})]}{[k^2+\mu_n(\varphi,\tilde{\varphi})]^2+4k^2m^2\cos^2\varphi\sin^2\tilde{\varphi}}, \tag{49}$$

$\varepsilon^{\infty}_{\mathrm{ren}}$ is given by (33). The sums and the integral in the last line of (49) (which are due to the terms in the last line of (45) and which can be interpreted as describing the proper energies of the boundary planes containing the sources of the magnetic field) are divergent, but this divergency is of no concern for us, because it has no physical consequences. Rather than the Casimir energy, a physically relevant quantity is the Casimir force per unit area of the boundary surface, i.e. pressure, which is defined as

$$F = -\frac{\partial}{\partial a}\frac{E_{\mathrm{ren}}}{S}, \tag{50}$$

and which is free from divergencies. We obtain

$$F = -\varepsilon^{\infty}_{\mathrm{ren}} - \frac{2|eB|}{\pi^2}\sum_{n=0}^{\infty}\iota_n\int\limits_{\omega_{n0}}^{\infty}\mathrm{d}\kappa\Upsilon(\kappa)\sqrt{\kappa^2-\omega_{n0}^2}, \tag{51}$$

where

$$\Upsilon(\kappa) \equiv -\frac{\partial}{\partial a}a\Lambda(\kappa) = \Big[\upsilon_1(\kappa)\mathrm{e}^{6\kappa a} + \upsilon_2(\kappa)\mathrm{e}^{4\kappa a} + \upsilon_3(\kappa)\mathrm{e}^{2\kappa a} + \upsilon_4(\kappa)\Big]$$

$$\times\Big\{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{4\kappa a}$$

$$-2[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})\Big\}^{-2} \tag{52}$$

and

$$\upsilon_1(\kappa) = -(2\kappa a-1)[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})][\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]$$

$$-2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} - (2\kappa\sin\tilde{\varphi} - m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi}, \tag{53}$$

$$\upsilon_2(\kappa) = (4\kappa a - 3)\left\{[\kappa^2 - \mu_n(\varphi,\tilde{\varphi})]^2 - 4\kappa^2 m^2 \cos^2\varphi \sin^2\tilde{\varphi}\right\}$$

$$+8\kappa^2[\kappa^2\cos^2\tilde{\varphi} - m^2\cos^2\varphi - \mu_n(\varphi,\tilde{\varphi})]\sin^2\tilde{\varphi} + 4[\kappa^2 + \mu_n(\varphi,\tilde{\varphi})]\kappa m\cos\varphi\sin\tilde{\varphi}, \qquad (54)$$

$$\upsilon_3(\kappa) = -(2\kappa a - 3)[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})][\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]$$

$$-2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} + (2\kappa\sin\tilde{\varphi} + m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi}, \qquad (55)$$

$$\upsilon_4(\kappa) = -[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]^2. \qquad (56)$$

5 Summary and discussion

The influence of a background uniform magnetic field and boundary conditions on the vacuum of the quantized charged spinor matter field (of mass m) confined between two parallel plates has been comprehensively analyzed, and the Casimir force acting onto the plates is found to take the form, see (51):

$$F = -\varepsilon^{\infty}_{\rm ren} + \Delta_{\varphi,\tilde{\varphi}}(a), \qquad (57)$$

where all dependence on the distance between the plates, a, and the choice of boundary conditions, parametrized by φ and $\tilde{\varphi}$, is contained in the second term, $\Delta_{\varphi,\tilde{\varphi}}(a)$. In the physically meaningful case, $ma \gg 1$, this term is exponentially damped as $\exp(-2ma)$, and the Casimir force is given by the first term, $F = -\varepsilon^{\infty}_{\rm ren}$. It should be noted that the Heisenberg-Euler vacuum energy density, $\varepsilon^{\infty}_{\rm ren}$, see (33), is negative (vanishing at $B = 0$ only), hence, the Casimir effect is repulsive, i.e. the pressure from the vacuum onto the plates is positive. Defining the critical value of the magnetic field as $B_{\rm crit} = m^2|e|^{-1}$, one can obtain in the limit of a supercritical magnetic field, $|B| \gg B_{\rm crit}$, from (33):

$$F = \frac{1}{24\pi^2}\frac{1}{\lambda_C^4}\left(\frac{B}{B_{\rm crit}}\right)^2 \ln\frac{2|B|}{B_{\rm crit}} \qquad (58)$$

(recall that $\lambda_C = m^{-1}$ is the Compton wavelength of the matter field). Note that the critical value is the lowest one, $B_{\rm crit} = 4.41\times 10^{13}\,{\rm G}$, for the case of quantized electron-positron matter, and supercritical magnetic fields with $|B| \gg 10^{13}\,{\rm G}$ may be attainable in some astrophysical objects, such as neutron stars and magnetars [11], and also gamma-ray bursters in scenarios involving protomagnetars [12]. A proper account for the influence of Casimir pressure (58) on physical processes in these objects should be taken.

Supercritical magnetic fields are not feasible in terrestrial laboratories where the maximal values of steady magnetic fields are of order of $10^5\,{\rm G}$, see, e.g., [13]. In the case of a subcritical magnetic field, $|B| \ll B_{\rm crit}$, one obtains from (33):

$$F = \frac{1}{360\pi^2}\frac{1}{\lambda_C^4}\left(\frac{B}{B_{\rm crit}}\right)^4. \qquad (59)$$

Let us compare this with the attractive Casimir force which is due to the quantized electromagnetic field, see F_{EM} (1), and define ratio

$$\frac{F}{F_{EM}} = -\frac{2}{3\pi^4}\left(\frac{a}{\lambda_C}\right)^4\left(\frac{B}{B_{\rm crit}}\right)^4. \qquad (60)$$

At $a = 10^{-6}\,\mathrm{m}$ and $B = 10^5\,\mathrm{G}$ the attraction is prevailing over the repulsion by six orders of magnitude, $F_{EM}/F \approx -10^6$, and the Casimir force is $F_{EM} \approx -1.3\,\mathrm{mPa}$. However, at $a = 10^{-5}\,\mathrm{m}$ and $B = 10^6\,\mathrm{G}$ the repulsion becomes dominant over the attraction by two orders of magnitude, $F/F_{EM} \approx -10^2$ and the Casimir force case takes value $F \approx 0.009\,\mathrm{mPa}$. Otherwise, the same value of the Casimir force is achieved at $a = 10^{-6}\,\mathrm{m}$ and $B = 10^7\,\mathrm{G}$. Thus, an experimental observation of the influence of the background magnetic field on the Casimir pressure seems to be possible in some future in terrestrial laboratories.

Acknowledgments

I would like to thank the Organizers of the IXth International Bolyai-Gauss-Lobachevsky Conference "Methods of Non-Euclidean Geometry in Physics and Mathematics" for kind hospitality during this interesting and inspiring meeting. The work was supported by the National Academy of Sciences of Ukraine (project No.0112U000054) and the ICTP – SEENET-MTP grant PRJ-09 "Strings and Cosmology".

References

[1] H.B.G.Casimir, *Proc. Kon. Ned. Akad. Wetenschap B* **51**, 793 (1948).

[2] M.Bordag, G.L.Klimchitskaya, U.Mohideen and V.M.Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

[3] N.I.Akhiezer and I.M.Glazman, *Theory of Linear Operators in Hilbert Space* (Pitman, Boston, 1981).

[4] A.Chodos, R.L.Jaffe, K.Johnson, C.B.Thorn and V.Weisskopf, *Phys. Rev. D* **9**, 3471 (1974).

[5] A.Chodos, R.L.Jaffe, K.Johnson and C.B.Thorn, *Phys. Rev. D* **10**, 2599 (1974).

[6] K.Johnson, *Acta Phys. Polon. B* **6**, 865 (1975).

[7] M.H.Al-Hashimi and U.-J.Wiese, *Annals Phys.* **327**, 1 (2012).

[8] Yu.A.Sitenko, *Phys. Rev. D* **91**, 085012 (2015).

[9] W.Heisenberg and H.Euler, *Z. Phys.* **98**, 714 (1936).

[10] J.Schwinger, *Phys. Rev.* **82**, 662 (1951).

[11] A.K.Harding and D.Lai, *Rep. Progr. Phys.* **69**, 2631 (2006).

[12] B.D.Metzger, D.Giannios, T.A.Thompson, N.Bucciantini and E.Quataert, *Mon. Not. Roy. Astron. Soc.* **413**, 2031 (2011).

[13] J.A.A.J.Perenboom, J.C.Maan, M.R.van Breukelen, S.A.J.Wiegers, A.den Ouden, C.A.Wulfers, W.J.van der Zande, R.T.Jongma, A.F.G.van der Meer and B.Redlich, *J. Low Temp. Phys.* **170**, 520 (2013).

Тензорная теория гравитации с линейной связью метрики и потенциала

Ю.П. Выблый
Vyblyi@gmail.com; Институт физики им. Б.И. Степанова, Минск
И.Г. Дудко
igordudko89@gmail.com; Институт физики им. Б.И. Степанова, Минск
А.А. Леонович
kaffiz@bsuir.by; Белгосуниверситет информатики и радиоэлектроники, Минск

Аннотация

Рассмотрена тензорная теория гравитации в пространстве Минковского, приводящая к полевым уравнениям Эйнштейна для эффективной метрики, представляемой в виде суммы метрики Минковского и тензорного гравитационного потенциала. Дополнительные к эйнштейновскими уравнения, описывающие ограничение поля по спину, явно содержат метрику Минковского и позволяют в случак безмассовой теории, в отличие от релятивистской теории гравитации, однозначно задавать систему координат. Рассмотрены статическое сферически-симметричное и космологическое решения полевых уравнений.

1 Тензорная теория гравитации

Одним из подходов к получению полевых уравнений Эйнштейна общей теории относительности является их вывод в рамках нелинейной тензорной теории гравитации в пространстве Минковского. Этот подход рассматривался в ряде работ, начиная с [1], позднее в работах Тирринга [2], Р. Фейнмана [3], С. Дезера [4] . В наиболее полно этот подход был реализован в релятивистской теории гравитации [5], где было явно показано, что уравнения Эйнштейна имеют структуру нелинейных тензорных уравнений в плоском пространстве с источником в виде полного тензора энергии-импульса Гильберта материи и самого гравитационного поля. Возникающая в этом подходе эффективная риманова метрика пространства-времени имеет вид

$$\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}\left(\gamma^{\mu\nu} + k\Psi^{\mu\nu}\right), \tag{1}$$

где $\Psi^{\mu\nu}$ -тензорный потенциал, k - константа гравитационного взаимодействия. Важно отметить, что в ОТО и полевом подходе метрика задается на различных, вообще говоря, многообразиях. В ОТО структура многообразия определяется метрикой $g_{\mu\nu}$ и неодинакова для различных физических задач, в то время как в полевой теории она фиксирована и имеет простую топологию пространства Минковского. Таким образом, полевой подход в теории гравитации представляет самостоятельный интерес. Уравнения Эйнштейна подчиняются тождествам Бианки и должны быть дополнены в общем случае четырьмя условиями. В полевом подходе это уравнения $D_\mu\Psi^{\mu\nu} = 0$, (D_μ - ковариантная производная по метрике Минковского), исключающие из тензорного поля спиновое состояние единица и одно из состояний со спином ноль. С учетом (1) эти уравнения принимают вид $D_\mu\tilde{g}^{\mu\nu} = 0$ и являются

общековариантным обобщением условий гармоничности Фока в ОТО. Однако, эти упавнения не позволяют однозначно фиксировать систему координат, поскольку содежат в явном виде не метрику $g_{\mu\nu}$, а связность пространства Минковского. Для того, чтобы система координат могла бвть задана однозначно в РТГ вводится в рассмотрение массивное тензорное поле, при этом массивная часть лагранжиана содержит тензор Минковского

$$R^{\mu\nu} - \frac{1}{2}m^2\left(g^{\mu\nu} - g^{\mu\alpha}g^{\beta\gamma}\gamma_{\gamma\nu}\right) = \frac{8\pi}{\sqrt{-g}}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right).$$

Массивная тензорная теория гравитации позволяет получить новые, по сравнению с ОТО, результаты в области сильных полей. В частности, отсутствие сингулярности в решении о коллапсе пылевого шара и космологической сингулярности. Однако, существенным ненедостатком этого подхода является некорректность линейного приближения из-за присутствия в уравнениях состояний с отрицательной энергией ("духов").

В настоящей работе рассмотрена модификация РТГ, позволяющая в рмках полевого подхода описывать безмассовое гравитационное поле. Рассмотрим вместо (1) линейгую связь метрики Минковского и потенциала в виде $g_{\mu\nu} = \gamma_{\mu\nu} + k\Psi_{\mu\nu}$. [6]. Тогда уравнения ограничения поля по спину принимают вид

$$\gamma^{\mu\nu}D_\nu g_{\mu\alpha} = 0 \tag{2}$$

и яно содержат метрику Минковского. Из требования калмбровочной инвариантности теории относительно группы вариаций Ли динамических переменных вытекает, что лагранжиан нелинейного гравитационного поляможет быть получен из соответствующего линейнго лагранжиана путем замены метрики Минковского на метрику $g_{\mu\nu}$, приэтом мы получаем общековариантное обобщение лагранжиана ОТО, содержащего только первые производные от метрики и, следовательно, уравнения Эйнштейна. Тензор энергии-импульса гравитационного поля $t^g_{\mu\nu}$, стоящий в правой части уравнений Эйнштейна, записанных в пространстве Минковского,

$$\Box\Psi_{\mu\nu} + D_\mu D^\alpha\Psi_{\alpha\nu} + D_\nu D^\alpha\Psi_{\alpha\mu} - \gamma_{\mu\nu}D^\alpha D^\beta\Psi_{\alpha\beta} = k(t^g_{\mu\nu} + t^M_{\mu\nu}) \tag{3}$$

уже не будет тензором Гильберта для рассматриваемого лагранжиана, при этом он является тензорным продолжением псевдотензора энергии-импульса Вейнберга. Отметим, что раздичные аспекты РТГ с линейной связью, но сдополнительными условиями гармоничности Фока рассматривались в [7].

2 Статическое сферически-симметричное решение

Рассмотрим центрально-симметричное статическое вакуумное решение полевых уравнений, которые с учетом уравнения (2) имеют вид

$$1 - \frac{1}{V}\left(\frac{dW}{dr}\right)^2 - 2\frac{W}{V}\frac{d^2W}{dr^2} + \frac{W}{V^2}\frac{dW}{dr}\frac{dV}{dr} = 0, \tag{4}$$

$$1 - \frac{1}{V}\left(\frac{dW}{dr}\right)^2 - \frac{W}{UV}\frac{dW}{dr}\frac{dU}{dr} = 0, \tag{5}$$

$$\frac{dV}{dr} + \frac{2}{r}V - \frac{2}{r^3}W^2 = 0, \tag{6}$$

где $U = g_{00}$,$V = -g_{11}$,$W = -g_{22}$. Решение путем непосредственного интегрирования этой системы не может быть получено в аналитической форме, однако, можно использовать

точное решение эйнштейновских уравнений в какой-либо системе координат, напрмер со стандартной координатой ρ, а затем перейти к сферической координате rпространства Минковского так, чтобы удовлетворялоссь уравнение (6). Интервал в сферических координатах будет иметь вид

$$ds^2 = g_{00}\left(\rho\left(r\right)\right)dt^2 + g_{11}\left(\rho\left(r\right)\right)\left(\frac{d\rho}{dr}\right)^2 dr^2 - \rho\left(r\right)^2 d\Omega^2. \tag{7}$$

Здесь стандартная координата связана со сферической соотношением, вытекающим из (2)

$$\partial_1 g_{11}(r) + \frac{2}{r}g_{11}(r) - \frac{2}{r^3}g_{22}(r) = 0. \tag{8}$$

Выражая производную по сферической координате через производную по стандартной координате и используя метрику Шварцшильда

$$g_{00} = \left(1 - \frac{2m}{\rho}\right), g_{11} = -\left(1 - \frac{2m}{\rho}\right)^{-1}, g_{22} = -\rho^2, \tag{9}$$

получим уравнение, связывающее стандартную и сферичекую координаты

$$r^3 r'' + \rho(\rho - 2m)r'^4 - r^2 r'^2 + \frac{m}{\rho(\rho - 2m)}r^3 r' = 0, \tag{10}$$

где r' означает производную по стандартной координате.

Уравнение (10) путём двух подстановок $\frac{r'}{r} = z$ и $z = \sqrt[3]{U}$ сводится к уравнению Бернулли:

$$U' + 3\rho(\rho - 2m)U^2 + \frac{3m}{\rho(\rho - 2m)}U = 0. \tag{11}$$

Решение для z имеет вид:

$$z(\rho) = \frac{2^{1/3}\sqrt{\rho}}{\sqrt{\rho - 2m}\left(\sqrt{\rho}\sqrt{\rho - 2m}\left(2\rho^2 + 5m\rho + 15m^2\right) + 30m^2 \ln\left(C(\sqrt{\rho} + \sqrt{\rho - 2m})\right)\right)^{1/3}} \tag{12}$$

Окончательно выражение для r будет иметь вид:

$$r = A\exp\left(\int z(\rho)d\rho\right), \tag{13}$$

где C и A - константы интегрирования. Интеграл в (13) не может быть вычислен в элементарных функциях.Рассмотрение асимптотического решения при ρ стремящимся к бесконечности, проиводит к решению $r = \rho$ и $A = 1$. При $\rho = 2m$, $r = (2m)^{\frac{C}{2}}$ и всегда положительна. Таким образом, использованный метод позволяет найти решение не во всем пространстве Минковского, а только в области $r > (2m)^{\frac{C}{2}}$.

Покажем теперь, что в начале координат существует регулярное решение. Для этого вернемся к системе полевых уравнений (4) - (6) и предположим, что в окрестности начала координат величины W и W'конечны, V не равна нулю, а величина $\frac{U'}{U}$ ведет себя как r^m, где $m > 0$. Тогда U в начале коодинат конечно, и из (5) следует, что $V = W'^2$. Подставляя это выражение в уравнение(4), находим, что оно удовлетворяется тождественно и система уравнений для определения V и W принимает вид:

$$V = W'^2, \tag{14}$$

$$V' + \frac{2}{r}V - \frac{2}{r^3}W^2 = 0. \tag{15}$$

Дифференцируя (14) и подставляя в (15), получаем уравнение для W:

$$W'W'' + \frac{2}{r}W'^2 - \frac{2}{r^3}W^2 = 0 \tag{16}$$

Положим, что что в окрестности начала координат $W = Dr^n$,(D - константа), откуда следует уравнение для n:

$$n^3 + n^3 - 2 = 0, \tag{17}$$

которое имеет действиетльное решение $n = 1$. Таким образом, в окрестности начала координат:

$$W = Dr, \qquad V = D^2 \tag{18}$$

в согласии со сделанным выше предположение о их конечности. Отметим, что этот резудьтат формально не противоречит сингулярности решения при $r = 0$ в приближении слабого поля, поскольку само это приближение предполагает рассмотрение вне начала координат.

3 Космологическое решение

Для однородного и изотропного космологического решения интервал в сферических координатах пространства Минковского имеет вид:

$$ds^2 = U(t)\,dt^2 - V(t)\left[\frac{dr^2}{1-qr^2} = r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{19}$$

где параметр q, задающей тип решения (пространственно-плоский, с положительной

или отрицательной пространственной кривизной) может принимать значения 0, +1, −1. Система () в данном случае приводит к следующим соотношениям:

$$\partial_0 g_{00} = 0, \qquad g_{11} - \frac{1}{r^2}g_{22} = 0, \tag{20}$$

откуда следует, что $g_{00} = 1$ (с точностью до выбора единиц измерения времени) и $q = 0$. Последний результат, вытекающий также и из уравнений гармоничности в массивной РТГ , имеет чрезвычайно важное значение, поскольку позволяет теоретически обосновать пространственно плоский характер однородной и изотропной Вселенной в соответствии с установленными наблюдательными данными и, таким образом, решает проблему <плоскостности>, имеющую место в ОТО. Равенство $g_{00} = 1$ означает, что так называемое <космическое время> в метрике Фридмана - Робертсона -Уолкера является в данном подходе временем декартовой четырехмерной системы координат в пространстве Минковского.

Отметим, что космологический сценарий релятивистской теории гравитации как массивной, так и безмассовой не позволяет объяснить ускоренное расширение Вселенной в современную эпоху. Поскольку структура уравнений поля в пространстве Минковского однозначно прводит к уравнениям Эйнштейна, модификация этих уравнений в РТГ не имеет смысла. Наиболее последовательным подходом к описанию темной энергии является поэтому моделирование ее с помощью вещественного нелинейного скалярного поля. В последние годы этот подход широко рассматртвается в литературе [8], однако трудность его заключается в отсутствии критериев выбора потенциала скалярного поля. В рамках тензорной

теории эта трудность может быть преодолена, если предположить, что в пространстве Минковского уравнение скалярного поле имеет структуру, аналогичную структуре тензорного поля, при этом источником скалярного поля является след его тензора энергии-импульса. Это предположение позволяет однозначно определить вид скалярного потенциала и дать описание ускоренного космологического расщирения в современную эпоху в согласии с современными наблюдательными данными [9].

Список литературы

[1] G.D. Birkhoff. Flat space-time and gravitation //Proc. Nat. Acad. Sci. 1944. v. 30, p.324–334.

[2] W. Tirring. An alternative approach to the theory of gravitation // Ann. Phys. 1961, v.16. p. 96–117.

[3] R.Feunman. Quantum theory of gravitation // Acta Phys. Pol. v. XXIV. p. 697–722.

[4] S. Deser. Self-interaction and gauge invariance // Gen. Rel. Grav. v. 1. p. 9–18.

[5] А. А. Логунов, М.А. Мествиришвили. Релятивистская теория гравитации. М. Наука, 1989. 301 с.

[6] Ю. П. Выблый, А.А. Леонович // Релятивичтская теория гравитации с линейной связью метрики и потенциала // ДАН Беларуси 2013. № 4. с. 57–60.

[7] 5. А.В. Генк. Релятивистская теория гравитации с линейной связью. // Депонировано в ВИНИТИ, № 5982-1387. Л. 1987.

[8] Д.С. Горбунов, В.А. Рубаков. Введение в теорию ранней Вселенной. М. Изд-во ЛКИ, 2008. 552 с.

[9] Ю.П. Выблый, А.Н. Тарасенко // Скалярное поле с источником в виде следа тензора энергии-импульса как модель темной энергии. В сб. "Ковариатнные методы в теоретической физике. Физика элементарных частиц и теория относительности". Минск. 2011. с. 36–44.

Геодезическая структура сферически симметричного статического решения для центрального тела в рамках 5-мерной проективной теории поля

А.О.Шаплов
Yoichi@list.ru; Белорусский государственный университет
А.К.Горбацевич
Agorbats@cosmostv.by; Белорусский государственный университет

Аннотация

В данной работе была исследована геодезическая структура сферически симметричной статической метрики в рамках 5-мерной проективной теории поля. На основе детального анализа эффективного потенциала времениподобных и изотропных геодезических были получены все возможные виды геодезических в зависимости от значений параметров. Численно были были построены траектории для некоторых геодезических.

1 Введение

Как известно, идея 5-мерной проективной теории возвращает нас к работам Калузы и Клейна, которые впоследствии развивали многие авторы. Существенный прорыв в 5-мерных теориях был сделан Джорданом [1], который впервые ввёл в рассмотрение вспомогательное скалярное поле, естественным образом встречающееся в подобных теориях. Однако, используемые им уравнения поля оказались неприемлемыми.

Принципиально новый вклад в развитие 5-мерных теорий внёс Э. Шмутцер [2]. Им была построена 5-мерная проективная теория поля (5-dimension Projective Unified Field Theory или PUFT), работу над которой он вёл с 1959-ого года. В отличие от теории Джордана, в ней скалярное поле рассматривается как новое фундаментальное физическое взаимодействие наравне с гравитационным и электромагнитным полями. Для обозначения этого взаимодействия Шмуцер ввёл термин “скаляризм”, обозначив тем самым его отличие от остальных скалярных полей, используемых в физике.

Экспериментальные данные, полученные в последнее время, выявили ряд проблем в общей теории относительности, к которым можно отнести проблему тёмной энергии и тёмной материи. В связи с этим возрос интерес к альтернативным теориям гравитации, покрывающим всё пространство-время, а не какой-нибудь отдельный его участок.

В настоящее время в рамках 5-мерной проективной теории поля была построена космологическая модель без сингулярностей, а также был решён ряд проблем современной космологии [2, 3]. На фоне успехов PUFT очевиден интерес к точным решениям и их исследованиям. Эта работа посвящена исследованию геодезической структуры точного сферически симметричного статического решения в рамках PUFT.

2 Основы 5-мерной проективной теории поля

Всякая 5-мерная проективная теория основана на следующей теоретико-групповой теореме [1, 4]:

Полупрямое произведение абелевой группы калибровочных преобразований (электромагнетизм) и группы общекоординатных преобразований (гравитация) изоморфно группе однородных (степени 1) 5-мерных координатных преобразований:

$$X^{\mu'} = X^{\mu'}\left(X^{\nu}\right) = \frac{1}{\alpha} X^{\mu'}\left(\alpha X^{\nu}\right) \qquad (\alpha = const). \tag{1}$$

Эта математическая теорема позволяет нам предположить, что геометрия, построенная на такой группе, может быть использована как основа для геометризации электромагнитного, гравитационного, а также скалярного полей.

Подобно общей теории относительности в 4-мерном пространстве-времени, в 5-мерном пространстве постулируются уравнения типа Эйнштейна, которые затем с помощью специальной процедуры [5] проецируются на 4-мерное пространство-время. В результате такого проецирования получаются следующие уравнения поля (для простоты приведены уравнения поля в случае вакуума).

Обобщённые уравнения Эйнштейна:

$$R_{mn} - 2\sigma_{,m}\sigma_{,n} = 0, \tag{2}$$

где σ — так называемое скалярное поле.

Обобщённые уравнения Максвелла:

$$a)\, H^{mn}{}_{;n} = \frac{4\pi}{c} j^{m}, \qquad b)\, B_{[mn,k]} = 0, \qquad c)\, H_{mn} = e^{2\sigma} B_{mn}\,, \tag{3}$$

где B^{mn} — тензор напряжённости электромагнитного поля, H^{mn} — тензор индукции электромагнитного поля, j^{m} — плотность электрического тока.

Уравнения скалярного поля σ:

$$\sigma^{,m}{}_{;m} = -\frac{\kappa_0}{2}\vartheta, \tag{4}$$

где κ — аналог постоянной Эйнштейна, а ϑ — скалярная величина, являющаяся источником скалярного поля.

Уравнения движения произвольного объекта, как и в общей теории относительности, могут быть получены из следующего соотношения [6]:

$$T^{mn}{}_{;n} = 0, \tag{5}$$

где T^{mn} — тензор энергии-импульса объекта. Для примера возьмём пробную точечную частицу:

$$m\frac{\mathrm{D}U^{i}}{\mathrm{D}\tau} = -c^{2}\mu h^{i}{}_{k}\sigma^{,k}, \qquad \frac{\mathrm{d}m}{\mathrm{d}\tau} = \mu\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}, \tag{6}$$

где m — масса частицы, μ — так называемая скалярная масса частицы, $h^{i}{}_{k} = g^{i}_{k} + \frac{1}{c^{2}}\, u^{i}u_{k}$ — проекционный тензор, $u^{i} \equiv \frac{\mathrm{d}x^{i}}{\mathrm{d}\tau}$ — 4-скорость частицы. Стоит отметить, что в отличие от ОТО мировая линия пробной частицы не является времениподобной геодезической, а масса частицы не является постоянной и зависит от σ-поля.

3 Электровакуумное сферически симметричное решение уравнений PUFT

Общая форма сферически симметричной статической метрики в изотропных координатах имеет следующий вид [7]:

$$ds^2 = e^{2u(r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] - e^{2\upsilon(r)}c^2dt^2, \tag{7}$$

где $u(r)$ и $\upsilon(r)$ являются некоторыми пока неизвестными функциями. Подставляя метрику (7) в уравнения поля (2) – (4), мы получаем следующую систему уравнений:

$$\upsilon'' + 2u'' + \upsilon'(\upsilon' - u') + \frac{2u'}{r} - 2\left(\sigma'\right)^2 = 0, \tag{8}$$

$$u'' + \upsilon'\left(u' + \frac{1}{r}\right) + \left(u'\right)^2 + \frac{3u'}{r} + = 0, \tag{9}$$

$$\upsilon'' + \upsilon'\left(\upsilon' + u'\right) + \frac{2\upsilon'}{r} = 0, \tag{10}$$

$$\sigma'' + \sigma'\left(u' + \upsilon'\right) + \frac{2\sigma'}{r} = 0. \tag{11}$$

В результате в зависимости от соотношения констант интегрирования M и $\mathcal{M}$ мы получаем 3 вида сферически симметричной метрики. Физическая интерпретация этих постоянных будет дана ниже.

1) Случай для $\mathcal{M} < M$

$$ds^2 = \omega(r)\left(\frac{\mathrm{d}r^2}{\left(1 - \frac{4R_2}{r}\right)} + r^2\mathrm{d}\Omega^2\right) - \frac{\left(1 - \frac{4R_2}{r}\right)}{\omega(r)}c^2\mathrm{d}t^2, \tag{12}$$

$$\sigma(r) = -\frac{R_1^2 - 4R_2^2}{8R_1R_2}\ln\left|1 - \frac{4R_2}{r}\right|,$$

$$\omega(r) \equiv \frac{1}{\left|1 - \frac{4R_2}{r}\right|^{2k}},$$

где $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$, $R_1 \equiv M + \mathcal{M}$, $R_2 \equiv \frac{1}{2}\sqrt{M^2 - \mathcal{M}^2}$, $k \equiv \frac{(R_1 - 2R_2)^2}{8R_1R_2}$, $k \geq 0$. Как мы можем видеть, при $r = 4R_2$ в метрике наблюдается особенность. Также примечательно, что при обращении скалярной массы $\mathcal{M}$ в 0 метрика (12) переходит в метрику Шварцшильда. В этой работе мы будем рассматривать только метрику при $\mathcal{M} < M$.

2) Случай для $\mathcal{M} = M$

$$ds^2 = e^{2u(r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] - e^{2\upsilon(r)}c^2dt^2, \tag{13}$$

$$e^{2u} = exp\left\{\frac{R_0}{r}\right\}, \qquad e^{2\upsilon} = e^{-2u}$$

$$\sigma(r) = \frac{R_0}{2r}, \tag{14}$$

где $R_0 \equiv M + \mathcal{M}$.

3) Случай для $\mathcal{M} > M$

$$ds^2 = e^{2u(r)}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] - e^{2v(r)}c^2dt^2, \tag{15}$$

$$\begin{aligned} e^{2v} &= exp\left\{-\frac{1}{2R_1^2}\left[(R_1^2 - R_2^2)\Pi(r)\right]\right\}, \\ e^{2u} &= \left(1 + \frac{R_2^2}{4r^2}\right)^2 e^{-2v}, \\ \sigma(r) &= \frac{1}{4R_1^2}\left[(R_1^2 - R_2^2)\Pi(r)\right], \end{aligned} \tag{16}$$

где $R_1 \equiv M + \mathcal{M}$, $R_2 \equiv \sqrt{\mathcal{M}^2 - M^2}$, $\Pi(r) \equiv \frac{2R_1}{R_2}\left(\pi - 2\arctan\frac{2r}{R_2}\right)$.

Все три вида полученной метрики имеют одинаковую асимптотику на бесконечности [9]:

$$\begin{aligned} \sigma(r \to \infty) &= \frac{G\mathcal{M}}{c^2 r} + O(1/r^3) = \frac{\mathcal{M}}{r} + O(1/r^3), \\ g_{44}(r \to \infty) &= -\left(1 + \frac{2\varphi}{c^2}\right) = -1 + \frac{2GM}{rc^2}. \end{aligned}$$

Это позволяет нам интерпретировать константы M и $\mathcal{M}$ как массу и так называемую скалярную массу центрального тела.

Как отмечалось выше, времениподобные геодезические не являются мировыми линиями пробных частиц. Однако в большинстве случаев отклонения мировой линии от соответствующей ей времениподобной геодезической ничтожны. Также времениподобные геодезические важны для исследования структуры пространства-времени. Изотропные же геодезические, как и в ОТО, являются траекториями электромагнитного излучения, что следует из приближения геометрической оптики. Исходя из вышесказанного, очевиден интерес к исследованию геодезической структуры полученной метрики.

4 Уравнения времениподобных и изотропных геодезических

Уравнения геодезических могут быть найдены стандартным способом через эффективный лагранжиан [10]:

$$\mathcal{L}_{eff} = \frac{1}{2}g_{ij}u^i u^j, \tag{17}$$

где $u^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}\tau}$ — 4-скорость. Для сферически симметричной метрики вида (12) мы получаем следующие уравнения геодезических:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = E^2 - V_{eff}^2, \tag{18}$$

$$V_{eff}^2 = \left(1 - \frac{4R_2}{r}\right)^{2k+1}\left[h + \frac{L^2}{r^2}\left(1 - \frac{4R_2}{r}\right)^{2k}\right], \tag{19}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 0, \qquad \theta \equiv \pi/2, \tag{20}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{L}{r^2}\left(1 - \frac{4R_2}{r}\right)^{2k}, \tag{21}$$

$$c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E}{\left(1-\frac{4R_2}{r}\right)^{2k+1}}, \tag{22}$$

где E и L являются интегралами движения, $h = 0$ для изотропных геодезических и $h = 1$ для времениподобных геодезических. Как и в [10] для метрики Шварцшильда, была введена функция эффективного потенциала V^2_{eff}, определяющая поведение геодезической. Очевидно, что решение уравнений (18) – (22) существует только в тех областях, в которых эффективный потенциал $V^2_{eff}(r) \leq E^2$.

5 Времениподобные геодезические

Эффективный потенциал V^2_{eff} для времениподобных геодезических имеет три принципиально разных случая в зависимости от значения параметра L^2 (рис. 1).

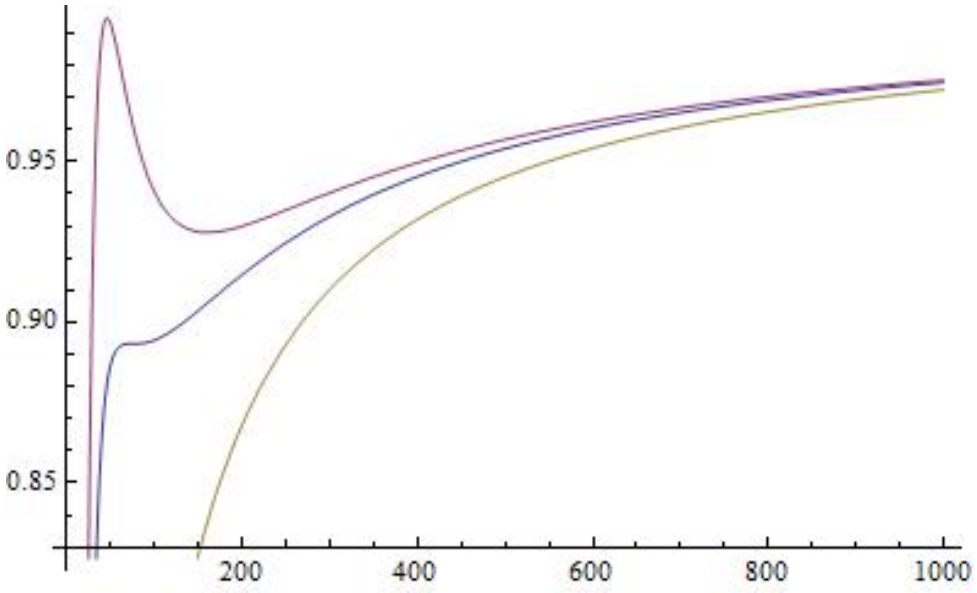


Рис. 1: Эффективный потенциал V^2_{eff} для различных значений параметра L^2.

При всех значениях параметра L^2 эффективный потенциал V^2_{eff} равен 0 на гиперповерхности $r = 4R_2$ и асимптотически стремится к 1 на бесконечности, а также имеет либо один максимум и один минимум, либо одну точку перегиба, либо не имеет экстремумов вовсе.

При $L^2 < L^2_{cr}$ функция эффективного потенциала V^2_{eff} монотонно возрастает, таким образом в этом случае возможен только один вид времениподобных геодезических.

При $L^2 = L^2_{cr}$ функция эффективного потенциала имеет один экстремум, при этом не убывает. Поэтому в дополнение к предыдущему случаю становится возможной единственная неустойчивая круговая времениподобная геодезическая.

При $L^2 > L^2_{cr}$ функция эффективного потенциала имеет максимум и минимум, поэтому в этом случае возможны два вида времениподобных геодезических.

Критическое значение параметра L^2_{cr}, при котором происходит принципиальный переход эффективного потенциала от одного вида к другому, может быть получен аналитически и принимает следующее значение:

$$r_{cr} = 4R_2\left[(3k+2) + \sqrt{5k^2+5k+1}\right] \tag{23}$$

$$L_{cr}^2 = \frac{4R_2(2k+1)r_{cr}^3\left(1-\frac{4R_2}{r_{cr}}\right)^{2k+1}}{\left[r_{cr}^2+4(2k-1)R_2r_{cr}-16k(4k+3)R_2^2\right]}. \tag{24}$$

Прежде всего рассмотрим простейший пример радиальной геодезической. В этом случае уравнения геодезической принимают простейший вид, сама геодезическая имеет форму прямой, а зависимость радиальной координаты r от параметра геодезической τ представлена на графике (рис. 2).

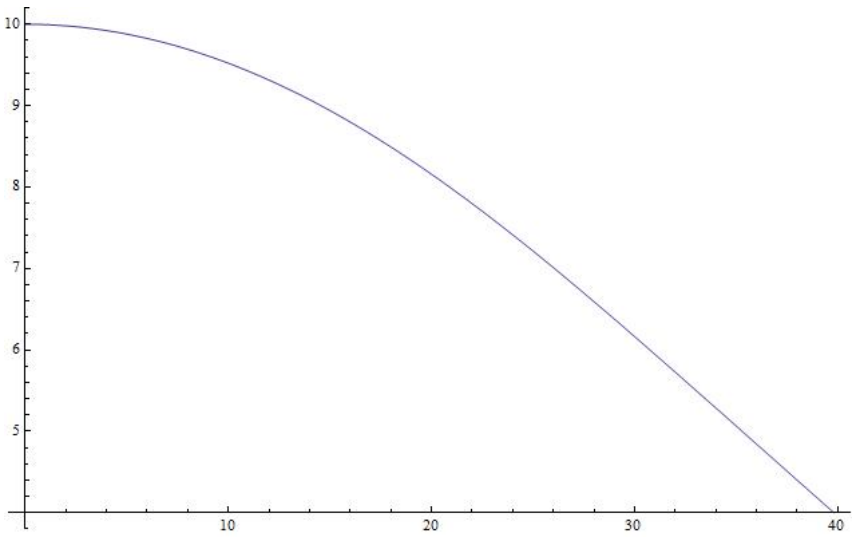


Рис. 2: Радиальная геодезическая ($L^2 = 0$). Зависимость радиальной координаты r от параметра геодезической τ.

Примечательно, что геодезическая достигает "особенной" гиперповерхности $r = 4R_2$ за конечное значение параметра τ. При этом, рассмотрев зависимость $r(t)$, для удалённого наблюдателя геодезическая будет стремиться к этой гиперповерхности бесконечно долго.

Если мы рассмотрим нерадиальные геодезические при значениях параметра $L^2 \leq L_{cr}^2$, результат будет тем же за исключением формы геодезической. Принципиально зависимость радиальной координаты r от параметра геодезической τ будет иметь такой же вид, как и для радиальной геодезической, к ней добавится ненулевая циклическая составляющая, поэтому геодезическая будет иметь форму спирали.

Наиболее интересен случай при значениях параметра $L^2 > L_{cr}^2$. Как отмечалось выше, тут появляются два экстремума: максимум при r_2 и минимум при r_3 (рис.3).

Времениподобная геодезическая существует в областях $r < r_1$ и $r_3 < r < r_5$. Первый случай принципиально не отличается от уже рассмотренных, поэтому мы не будем подробно на нём останавливаться. Отметим только, что вслед за Чандрасекаром [10] будем называть такие геодезические ограниченными в пространстве геодезическими второго рода. Во втором случае мы имеем ограниченные с двух сторон (r_3 и r_5) геодезические, которую вслед за Чандрасекаром будем называть ограниченными в пространстве геодезическими первого рода. В общем случае такая геодезическая, в отличие от метрики Шварцшильда, незамкнута. В качестве примера приведём некоторую геодезическую при произвольном выборе параметров для 2-х и 10-ти витков (рис. 4).

Как отмечалось выше, эффективный потенциал V_{eff}^2 в бесконечности монотонно возрастает к 1. Это означает, что геодезические для значений параметра $E^2 > 1$ могут быть неограниченными в пространстве. При этом если $V_{eff}^2(r_2) < E^2$, геодезическая "упадёт" на гиперповерхность $r = 4R_2$. Если же $V_{eff}^2(r_2) > E^2$, то возможны два вида геодезических,

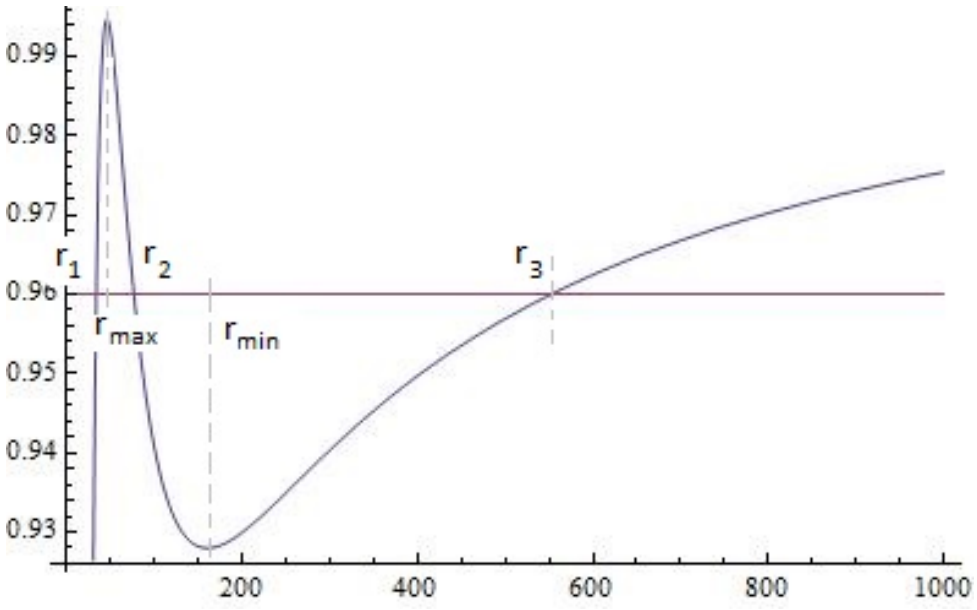


Рис. 3: Зависимость эффективного потенциала V^2_{eff} от координаты r для значений параметра $L^2 > L^2_{cr}$

a) b)

Рис. 4: Ограниченная в пространстве времениподобная геодезическая первого рода. a) $\Delta\varphi = 4\pi$, b) $\Delta\varphi = 20\pi$

один из которых аналогичен уже рассмотренным ограниченными в пространстве геодезическим второго рода, а второй будет ограничен в пространстве только с одной стороны. При значении параметра $E^2 = V^2_{eff}(r_2)$ возможна неустойчивая круговая геодезическая.

При значениях параметра $E^2 = V^2_{eff}(r_4)$ времениподобная геодезическая первого рода будет круговой. Стоит отметить, что $r_4 > r_{cr}$, т.е. ранее упомянутая неустойчивая круговая геодезическая в r_{cr} является пределом существования круговых времениподобных геодезических.

6 Изотропные геодезические

Уравнения для радиальной изотропной геодезической ($h = 0$ и $L^2 = 0$ в (18)–(22)) являются тривиальными с очевидным решением, поэтому не будем останавливаться на них более подробно. Для нерадиальных же изотропных геодезических удобней использовать в качестве параметра не параметр геодезической τ, а циклическую координату φ. Тогда уравнения

принимают вид:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = r^4\left(1-\frac{4R_2}{r}\right)^{-4k}\left[\frac{1}{D^2}-U_{eff}^2\right], \tag{25}$$

$$U_{eff}^2 \equiv \frac{1}{r^2}\left(1-\frac{4R_2}{r}\right)^{4k+1}. \tag{26}$$

где $D^2 \equiv \frac{L^2}{E^2}$ — прицельный параметр геодезической.

Мы ввели функцию эффективного потенциала U_{eff}^2, которая определяет поведение изотропных геодезических. Отметим, что потенциал U_{eff}^2 не зависит от параметров геодезической и определяется самой метрикой (рис.5).

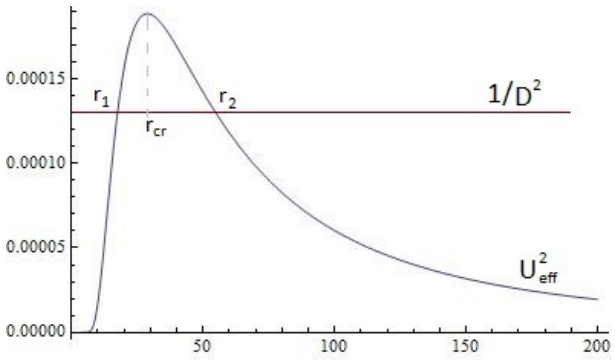


Рис. 5: Зависимость эффективного потенциала $U_{eff}^2(r)$

Очевидно, что изотропные геодезические при значении прицельного параметра $D^2 < D_{cr}^2 = U_{eff}^2(r_{cr})$ существуют в областях $r < r_1$ и $r > r_2$. В первой области геодезические ограничены со стороны $r = r_1$ и "падают" на гиперповерхность $r = 4R_2$. Во второй области мы имеем неограниченные в пространстве геодезические. При критическом значении прицельного параметра D_{cr} существует единственная неустойчивая круговая изотропная геодезическая. В качестве примера на (рис.6) показаны три принципиальных случая изотропных геодезических.

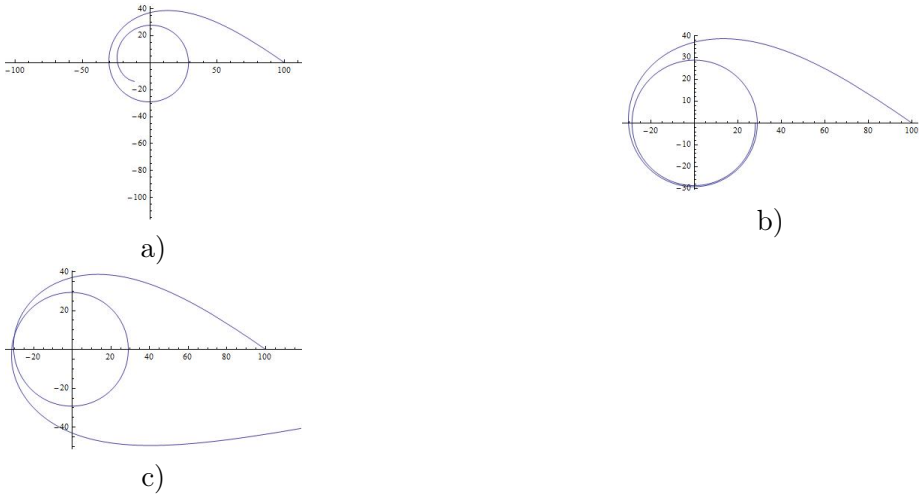


Рис. 6: Изотропные геодезические при разных значениях прицельного параметра (a) $D^2 < D_{cr}^2$, (b) $D^2 = D_{cr}^2$ and (c) $D^2 > D_{cr}^2$.

Список литературы

[1] Jordan P.Schwerkraft und Weltall: Grundlagen der theoretischen Kosmologie. — Braunschweig: Vieweg & Sohn GmbH, 1951. — 327 S.

[2] E.Schmutzer, F"unfdimensionale Physik — Projektive Einheitliche Feldtheorie mit Einbeziehung der Quantentheorie (Mechanik, Astrophysik, Kosmologie ohne Urknall, Spinoren) Wissenschaftsverlag Thuringen, Langewiesen, 2009

[3] Schmutzer E. Projektive Einheitliche Feldtheorie mit Anwendungen in Kosmologie und Astrophysik. Neues Weltbild ohne Urknall? (Mit einem Anhang von A. K. Gorbatsievich). Berlin, 2004.

[4] Schmutzer E. Relativistische Physik. Leipzig: Teubner Verlagsgesellschaft, 1968. — 974 S.

[5] A. K. Gorbatsievich. On the Axiomatics of the 5-Dimensional Projective Unufied Field Theory of Schmutzer. General Relativity and Gravitation, Vol. **33**, No. 6, 2001.

[6] A. K. Gorbatsievich, E. Schmutzer, Ho Si Mau Thuc. "Equation of Motion of a Point-Like Test Body and Post-Newtonian Approximation in the Projective Unified Field Theory". Acta Physica Polonica. V. **B27**(1996) 1991.

[7] A. Gorbatsievich. Investigation of Schmutzer's Exact External Spherically Symmetric Static Solution for a Central Body within the Framework of the 5-Dimensional Projective Unified Field Theory.

[8] A. Gorbatsievich, Fatma Desokey. "On the Exact Spherically Symmetric Static Vacuum Solution of the 5-dimensional Projective Unified Field Theory". Nonlinear Phenomena in Complex Systems. 2005. No 1. P. 35–41.

[9] A. Gorbatsievich, Fatma Desokey. "On the Post-Newtonian approximation of the 5-dimensional Projective Unified Field Theory ". In: Proceedings of the XXII Intern. School-Conference "Foundations & Advances in Nonlinear Science" / Ed. V.I.Kuvshinov, G.G.Krylov. — Belorussian State University, Minsk, 2005, P. 8–12.

[10] S. Chandrasekhar, The Mathematical Theory of Black Holes, Clarendon Press Oxford Oxford University Press New York 1983

Elementary particles theory in the early Universe

N. A. Gromov
gromov@dm.komisc.ru; Komi Science Center UrD RAS, Syktyvkar, Russia

Abstract

We discuss the high-energy (high-temperature) limit of Standard Model generated by the contractions of gauge groups. Contraction parameters of gauge group $SU(2)$ of Electroweak Model and gauge group $SU(3)$ of Quantum Chromodynamics are taken identical and tending to zero when energy increase. At the infinite energy limit all particles lose masses, all quarks are monochromatic. Electroweak interactions become long-range and are mediated by the neutral currents. Particles of different kind do not interact. It looks like some stratification with only one sort of particles in each stratum. The Standard Model passes in this limit through several stages, which are distinguished by the powers of contraction parameter. For any stage the intermediate models are constructed and the exact expressions for the respective Lagrangians are presented. The developed approach describes the evolution of Standard Model in the early Universe from the Big Bang up to the end of several milliseconds.

1 Introduction

Modern theory of elementary particles known as Standard Model (SM) consist of Electroweak Model (EWM), which unified electromagnetic and weak interactions, as well as Quantum Chromodynamics (QCD), describing their strong interactions. Standard Model gives a good description of the experimental data and was recently confirmed by discovering of Higgs boson at LHC. Standard Model is a gauge theory with $SU(3) \times SU(2) \times U(1)$ gauge group, which is direct product of a simple groups. The operation of group contraction (or limit transition) [1] well known in physics transforms a simple group to a non-semisimple one. For a symmetric physical system the contraction of its symmetry group means a transition to some limit state. In the case of a complicated physical system the investigation of its limit states under the limit values of some of its parameters enables to better understand the system behavior. When contraction of some mathematical or physical structure is performed one can reconstruct the initial structure moving back along the contraction way.

We use this method in order to re-establish the evolution of elementary particles and their interactions in the early Universe. We are based on the modern knowledge of the particle world which is concentrated in Standard Model. The low energy limit of Electroweak Model was discussed early [2–4]. In this paper we investigate the high-energy limit of Standard Model generated by contraction of the gauge groups $SU(2)$ and $SU(3)$. Similar very high energies can exist in the early Universe after inflation and reheating on the first stages of the Hot Big Bang [5]. It appears that the SM Lagrangian falls to a number of terms which are distinguished by the powers of the contraction parameter $\epsilon \to 0$. As far as the average energy (or temperature) in the hot Universe is connected with its age, then moving forward in time, i.e. back to high energy contraction, we conclude that after the Universe creation elementary particles and their interactions pass a number of stages in their evolution from infinite energy state up to SM state. These stages of quark-gluon plasma formation and color symmetries restoration are distinguished by the powers of contraction parameter and consequently by times of its creations. From the

contraction of Standard Model we can classify the stages in time as earlier-later, but we can not determine their absolute date. To estimate the absolute date we use additional assumptions.

2 High-Energy Lagrangian of EWM

It was demonstrated [2–4] that in zero energy limit the first components of the lepton and quark doublets become infinitely small in comparison to their second components. On the contrary, when energy increases the first components of the doublets become greater than their second ones. In the infinite energy limit the second components of the lepton and quark doublets will be infinitely small as compared to their first components. To describe this limit we introduce [6, 7] new contraction parameter ϵ and **consistent rescaling** of the group $SU(2)$ and the space $\mathbf{C}_2$ as follows

$$z'(\epsilon) = \begin{pmatrix} z'_1 \\ \epsilon z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & \epsilon\beta \\ -\epsilon\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ \epsilon z_2 \end{pmatrix} = u(\epsilon)z(\epsilon),$$

$$\det u(\epsilon) = |\alpha|^2 + \epsilon^2|\beta|^2 = 1, \quad u(\epsilon)u^\dagger(\epsilon) = 1. \tag{1}$$

Hermite form

$$z^\dagger(\epsilon)z(\epsilon) = |z_1|^2 + \epsilon^2|z_2|^2 \tag{2}$$

remain invariant under contraction $\epsilon \to 0$.

In the contraction scheme (1) the standard boson fields and left lepton and quark fields are transformed as follows

$$W^\pm_\mu \to \epsilon W^\pm_\mu, \;\; Z_\mu \to Z_\mu, \;\; A_\mu \to A_\mu.$$

$$e_l \to \epsilon e_l, \quad d_l \to \epsilon d_l, \quad \nu_l \to \nu_l, \quad u_l \to u_l. \tag{3}$$

The next reason for inequality of the first and second doublet components is the special mechanism of spontaneous symmetry breaking, which is used to generate mass of vector bosons and other elementary particles of the model. In this mechanism one of Lagrangian ground states $\phi^{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}$ is taken as vacuum of the model and then small field excitations $v + \chi(x)$ with respect to this vacuum are regarded. So Higgs boson field χ and constant v are multiplied by ϵ. As far as masses of all particles are proportionate to v we obtain the following transformation rule

$$\chi \to \epsilon\chi, \quad v \to \epsilon v, \quad m_p \to \epsilon m_p, \quad p = \chi, W, Z, e, u, d. \tag{4}$$

After transformations (3), (4) the EWM boson Lagrangian [8, 9] can be represented in the form

$$L_B(\epsilon) = -\frac{1}{4}\mathcal{Z}^2_{\mu\nu} - \frac{1}{4}\mathcal{F}^2_{\mu\nu} + \epsilon^2 L_{B,2} + \epsilon^3 g W^+_\mu W^-_\mu \chi + \epsilon^4 L_{B,4}, \tag{5}$$

where

$$L_{B,4} = m^2_W W^+_\mu W^-_\mu - \frac{1}{2}m^2_\chi\chi^2 - \lambda v\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{g^2}{4}\left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\right)^2 + \frac{g^2}{4}W^+_\mu W^-_\nu\chi^2, \tag{6}$$

$$L_{B,2} = \frac{1}{2}\left(\partial_\mu\chi\right)^2 + \frac{1}{2}m^2_Z\left(Z_\mu\right)^2 - \frac{1}{2}\mathcal{W}^+_{\mu\nu}\mathcal{W}^-_{\mu\nu} + \frac{gm_z}{2\cos\theta_W}\left(Z_\mu\right)^2\chi + \frac{g^2}{8\cos^2\theta_W}\left(Z_\mu\right)^2\chi^2 -$$

$$-2ig\left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\right)\left(\mathcal{F}_{\mu\nu}\sin\theta_W + \mathcal{Z}_{\mu\nu}\cos\theta_W\right) -$$

$$-\frac{i}{2}e\left[A_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu - \mathcal{W}^-_{\mu\nu}W^+_\nu\right) + \frac{i}{2}eA_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu - \mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right] -$$

$$-\frac{i}{2}g\cos\theta_W\left[Z_\mu\left(\mathcal{W}^+_{\mu\nu}W^-_\nu-\mathcal{W}^-_{\mu\nu}W^+_\nu\right)-\right.$$
$$\left.-Z_\nu\left(\mathcal{W}^+_{\mu\nu}W^-_\mu-\mathcal{W}^-_{\mu\nu}W^+_\mu\right)\right]-\frac{e^2}{4}\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](A_\nu)^2-\right.$$
$$\left.-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)A_\mu A_\nu+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](A_\mu)^2\right\}-$$
$$-\frac{g^2}{4}\cos\theta_W\left\{\left[\left(W^+_\mu\right)^2+\left(W^-_\mu\right)^2\right](Z_\nu)^2-\right.$$
$$\left.-2\left(W^+_\mu W^+_\nu+W^-_\mu W^-_\nu\right)Z_\mu Z_\nu+\left[\left(W^+_\nu\right)^2+\left(W^-_\nu\right)^2\right](Z_\mu)^2\right\}-$$
$$-eg\cos\theta_W\left[W^+_\mu W^-_\mu A_\nu Z_\nu+W^+_\nu W^-_\nu A_\mu Z_\mu-\frac{1}{2}\left(W^+_\mu W^-_\nu+W^+_\nu W^-_\mu\right)(A_\mu Z_\nu+A_\nu Z_\mu)\right]. \quad (7)$$

The lepton Lagrangian in terms of electron and neutrino fields takes the form

$$L_L(\epsilon)=L_{L,0}+\epsilon^2 L_{L,2}=$$
$$=\nu^\dagger_l i\tilde\tau_\mu\partial_\mu\nu_l+e^\dagger_r i\tau_\mu\partial_\mu e_r+g'\sin\theta_w e^\dagger_r\tau_\mu Z_\mu e_r-g'\cos\theta_w e^\dagger_r\tau_\mu A_\mu e_r+\frac{g}{2\cos\theta_w}\nu^\dagger_l\tilde\tau_\mu Z_\mu\nu_l+$$
$$+\epsilon^2\left\{e^\dagger_l i\tilde\tau_\mu\partial_\mu e_l-m_e(e^\dagger_r e_l+e^\dagger_l e_r)+\frac{g\cos 2\theta_w}{2\cos\theta_w}e^\dagger_l\tilde\tau_\mu Z_\mu e_l-\right.$$
$$\left.-ee^\dagger_l\tilde\tau_\mu A_\mu e_l+\frac{g}{\sqrt{2}}\left(\nu^\dagger_l\tilde\tau_\mu W^+_\mu e_l+e^\dagger_l\tilde\tau_\mu W^-_\mu\nu_l\right)\right\}. \quad (8)$$

The quark Lagrangian in terms of u- and d-quarks fields can be written as

$$L_Q(\epsilon)=L_{Q,0}-\epsilon\, m_u(u^\dagger_r u_l+u^\dagger_l u_r)+\epsilon^2 L_{Q,2}, \quad (9)$$

where

$$L_{Q,0}=d^\dagger_r i\tau_\mu\partial_\mu d_r+u^\dagger_l i\tilde\tau_\mu\partial_\mu u_l+u^\dagger_r i\tau_\mu\partial_\mu u_r-$$
$$-\frac{1}{3}g'\cos\theta_w d^\dagger_r\tau_\mu A_\mu d_r+\frac{1}{3}g'\sin\theta_w d^\dagger_r\tau_\mu Z_\mu d_r+\frac{2e}{3}u^\dagger_l\tilde\tau_\mu A_\mu u_l+$$
$$+\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u^\dagger_l\tilde\tau_\mu Z_\mu u_l+\frac{2}{3}g'\cos\theta_w u^\dagger_r\tau_\mu A_\mu u_r-\frac{2}{3}g'\sin\theta_w u^\dagger_r\tau_\mu Z_\mu u_r, \quad (10)$$

$$L_{Q,2}=d^\dagger_l i\tilde\tau_\mu\partial_\mu d_l-m_d(d^\dagger_r d_l+d^\dagger_l d_r)-\frac{e}{3}d^\dagger_l\tilde\tau_\mu A_\mu d_l-$$
$$-\frac{g}{\cos\theta_w}\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)d^\dagger_l\tilde\tau_\mu Z_\mu d_l+\frac{g}{\sqrt{2}}\left[u^\dagger_l\tilde\tau_\mu W^+_\mu d_l+d^\dagger_l\tilde\tau_\mu W^-_\mu u_l\right]. \quad (11)$$

The complete Lagrangian of the modified model is given by the sum $L(\epsilon)=L_B(\epsilon)+L_L(\epsilon)+L_Q(\epsilon)$ and can be written in the form

$$L(\epsilon)=L_\infty+\epsilon L_1+\epsilon^2 L_2+\epsilon^3 L_3+\epsilon^4 L_4. \quad (12)$$

The contraction parameter is monotonous function $\epsilon(E)$ of the representative energy (or temperature) of a system under consideration with the property $\epsilon(E)\to 0$ for $E\to\infty$. Very high energies can exist in the early Universe just after its creation.

It is well known that to gain a better understanding of a physical system it is useful to investigate its properties for limiting values of physical parameters. It follows from (12) that there are five stages in evolution of the Electroweak Model after the creation of the Universe which are distinguished by the powers of the contraction parameter ϵ. This offers an opportunity

for construction of intermediate limit models. One can take the Lagrangian L_∞ for the initial limit system, then add L_1 and obtain the second limit model with the Lagrangian $\mathcal{L}_1 = L_\infty + L_1$. After that one can add L_2 and obtain the third limit model $\mathcal{L}_2 = L_\infty + L_1 + L_2$. The last limit model has the Lagrangian $\mathcal{L}_3 = L_\infty + L_1 + L_2 + L_3$.

In the infinite energy limit ($\epsilon = 0$) Lagrangian (12) is equal to

$$L_\infty = -\frac{1}{4}\mathcal{Z}_{\mu\nu}^2 - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 + \nu_l^\dagger i\tilde{\tau}_\mu \partial_\mu \nu_l + u_l^\dagger i\tilde{\tau}_\mu \partial_\mu u_l +$$

$$+ e_r^\dagger i\tau_\mu \partial_\mu e_r + d_r^\dagger i\tau_\mu \partial_\mu d_r + u_r^\dagger i\tau_\mu \partial_\mu u_r + L_\infty^{int}(A_\mu, Z_\mu), \tag{13}$$

where

$$L_\infty^{int}(A_\mu, Z_\mu) = \frac{g}{2\cos\theta_w}\nu_l^\dagger \tilde{\tau}_\mu Z_\mu \nu_l + \frac{2e}{3}u_l^\dagger \tilde{\tau}_\mu A_\mu u_l +$$

$$+ g'\sin\theta_w e_r^\dagger \tau_\mu Z_\mu e_r + \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) u_l^\dagger \tilde{\tau}_\mu Z_\mu u_l - g'\cos\theta_w e_r^\dagger \tau_\mu A_\mu e_r -$$

$$- \frac{1}{3}g'\cos\theta_w d_r^\dagger \tau_\mu A_\mu d_r + \frac{1}{3}g'\sin\theta_w d_r^\dagger \tau_\mu Z_\mu d_r + \frac{2}{3}g'\cos\theta_w u_r^\dagger \tau_\mu A_\mu u_r - \frac{2}{3}g'\sin\theta_w u_r^\dagger \tau_\mu Z_\mu u_r. \tag{14}$$

The limit model includes only **massless particles**: photons A_μ and neutral bosons Z_μ, left quarks u_l and neutrinos ν_l, right electrons e_r and quarks u_r, d_r. This phenomenon has simple physical explanation: the average energy (or temperature) is so high ($> 10^{18} GeV$), that particle mass becomes negligible quantity as compared to its kinetic energy. The electroweak interactions become long-range because they are mediated by the massless neutral Z-bosons and photons. Let us note that $W_\mu^\pm$-boson fields, which correspond to the translation subgroup of Euclid group $E(2)$, are absent in the limit Lagrangian L_∞ (13).

Similar high energies can exist in early Universe after inflation and reheating on the first stages of the Hot Big Bang [5, 10] in pre-electroweak epoch. However more interesting is the Universe evolution and the limit Lagrangian L_∞ can be considered as a good approximation near the Big Bang just as the nonrelativistic mechanics is a good approximation of the relativistic one at low velocities.

From the explicit form of the interaction part $L_\infty^{int}(A_\mu, Z_\mu)$ it follows that there are no interactions between particles of different kind, for example neutrinos interact only with each other by neutral currents. All other particles are charged and interact with particles of the same sort by massless Z_μ-bosons and photons. Particles of different kind do not interact. It looks like some stratification of the Electroweak Model with only one sort of particles in each stratum.

From contraction of the Electroweak Model we can classify events in time as earlier-later, but we can not determine their absolute time without additional assumptions. Already at the level of classical gauge fields we can conclude that the u-quark first restores its mass in the evolution of the Universe. Indeed the mass term of u-quark in the Lagrangian (12) $L_1 = -m_u(u_r^\dagger u_l + u_l^\dagger u_r)$ is proportional to the first power ϵ, whereas the mass terms of Z-boson, electron and d-quark are multiplied by the second power of the contraction parameter

$$\epsilon^2 \left[\frac{1}{2}m_Z^2 \left(Z_\mu\right)^2 + m_e(e_r^\dagger e_l + e_l^\dagger e_r) + m_d(d_r^\dagger d_l + d_l^\dagger d_r)\right]. \tag{15}$$

At the same time massless Higgs boson χ and charged W-boson are appeared. They restore their masses after all other particles of the Electroweak Model because their mass terms are multiplied by ϵ^4.

The electroweak interactions between elementary particles are restored mainly in the epoch which corresponds to the second order of the contraction parameter. There is one term in

Lagrangian (5) $L_3 = gW_\mu^+ W_\mu^- \chi$ proportionate to ϵ^3. The final reconstruction of the electroweak interactions takes place at the last stage ($\approx \epsilon^4$) together with restoration of mass of all particles.

Two other generations of leptons and quarks are developed in a similar way: for infinite energy there are only massless right μ- and τ-muons, left μ- and τ-neutrinos, as well as massless left and right quarks $c_l, c_r, s_r, t_l, t_r, b_r$. c- and t-quarks first acquire their mass and after that μ-, τ-muons, s-, b-quarks become massive.

3 QCD with contracted gauge group

Strong interactions of quarks are described by the QCD. Like the Electroweak Model QCD is a gauge theory based on the local color degrees of freedom [11]. The QCD gauge group is $SU(3)$, acting in three dimensional complex space $\mathbf{C}_3$ of color quark states. The $SU(3)$ gauge bosons are called gluons. There are eight gluons in total, which are the force carrier of the theory between quarks. The QCD Lagrangian is taken in the form

$$\mathcal{L} = \sum_q \bar{q}^i (i\gamma^\mu)(D_\mu)_{ij} q^j - \frac{1}{4} \sum_{\alpha=1}^{8} F_{\mu\nu}^\alpha F^{\mu\nu\,\alpha}, \tag{16}$$

where $D_\mu q$ are covariant derivatives of quark fields

$$D_\mu q = \left(\partial_\mu - i g_s \left(\frac{\lambda^\alpha}{2} \right) A_\mu^\alpha \right) q, \tag{17}$$

g_S is the strong coupling constant, $t^a = \lambda^a/2$ are generators of $SU(3)$, λ^a are Gell-Mann matrices, $F_{\mu\nu}^\alpha$ is gluon stress tensor. Mass terms $-m_q \bar{q}^i q_i$ are not included as far as they are present in the electroweak Lagrangian. Let us note, that in QCD the special mechanism of spontaneous symmetry breaking is absent, which provides masses for vector bosons, therefore gluons are massless particles.

The contracted special unitary group $SU(3;\kappa)$ is defined by the action

$$q'(\kappa) = \begin{pmatrix} q_1' \\ \kappa_1 q_2' \\ \kappa_1\kappa_2 q_3' \end{pmatrix} = \begin{pmatrix} u_{11} & \kappa_1 u_{12} & \kappa_1\kappa_2 u_{13} \\ \kappa_1 u_{21} & u_{22} & \kappa_2 u_{23} \\ \kappa_1\kappa_2 u_{31} & \kappa_2 u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} q_1 \\ \kappa_1 q_2 \\ \kappa_1\kappa_2 q_3 \end{pmatrix} =$$

$$= U(\kappa) q(\kappa), \quad \det U(\kappa) = 1, \quad U(\kappa) U^\dagger(\kappa) = 1 \tag{18}$$

on the complex space $\mathbf{C}_3(\kappa)$ in such a way that the hermitian form

$$q^\dagger(\kappa) q(\kappa) = |q_1|^2 + \kappa_1^2 \left(|q_2|^2 + \kappa_2^2 |q_3|^2 \right) \tag{19}$$

remains invariant, when the contraction parameters tend to zero: $\kappa_1, \kappa_2 \to 0$. Transition from the classical group $SU(3)$ and space $\mathbf{C}_3$ to the group $SU(3;\kappa)$ and space $\mathbf{C}_3(\kappa)$ is given by the substitution

$$q_1 \to q_1, \quad q_2 \to \kappa_1 q_2, \quad q_3 \to \kappa_1\kappa_2 q_3,$$
$$A_\mu^{GR} \to \kappa_1 A_\mu^{GR}, \quad A_\mu^{BG} \to \kappa_2 A_\mu^{BG}, \quad A_\mu^{BR} \to \kappa_1\kappa_2 A_\mu^{BR}, \tag{20}$$

and diagonal gauge fields $A_\mu^{RR}, A_\mu^{GG}, A_\mu^{BB}$ remain unchanged.

Substituting (20) in (16), we obtain the quark part of Lagrangian in the form

$$\mathcal{L}_q(\kappa) = \sum_q \Bigg\{ i\bar{q}_1 \gamma^\mu \partial_\mu q_1 + \frac{g_s}{2} |q_1|^2 \gamma^\mu A_\mu^{RR} +$$

$$+\kappa_1^2\Big[i\bar{q}_2\gamma^\mu\partial_\mu q_2+\frac{g_s}{2}\Big(|q_2|^2\,\gamma^\mu A_\mu^{GG}+q_1\bar{q}_2\gamma^\mu A_\mu^{GR}+\bar{q}_1 q_2\gamma^\mu\bar{A}_\mu^{GR}\Big)\Big]+$$

$$+\kappa_1^2\kappa_2^2\Big[i\bar{q}_3\gamma^\mu\partial_\mu q_3+\frac{g_s}{2}\Big(|q_3|^2\,\gamma^\mu A_\mu^{BB}+q_1\bar{q}_3\gamma^\mu A_\mu^{BR}+\bar{q}_1 q_3\gamma^\mu\bar{A}_\mu^{BR}+$$

$$+q_2\bar{q}_3\gamma^\mu A_\mu^{BG}+\bar{q}_2 q_3\gamma^\mu\bar{A}_\mu^{BG}\Big)\Big]\Big\}=L_q^\infty+\kappa_1^2L_q^{(2)}+\kappa_1^2\kappa_2^2L_q^{(4)}. \tag{21}$$

Let us introduce the notations

$$\partial A^k\equiv\partial_\mu A_\nu^k-\partial_\nu A_\mu^k,\quad [k,m]\equiv A_\mu^k A_\nu^m-A_\mu^m A_\nu^k, \tag{22}$$

then the gluon tensor has the following components

$$F_{\mu\nu}^1=\kappa_1\Big\{\partial A^1+\frac{g_s}{2}\Big(2[2,3]+\kappa_2^2\left([4,7]-[5,6]\right)\Big)\Big\},$$

$$F_{\mu\nu}^2=\kappa_1\Big\{\partial A^2+\frac{g_s}{2}\Big(-2[1,3]+\kappa_2^2\left([4,6]+[5,7]\right)\Big)\Big\},$$

$$F_{\mu\nu}^3=\partial A^3+\frac{g_s}{2}\Big(\kappa_1^2 2[1,2]-\kappa_2^2[6,7]+\kappa_1^2\kappa_2^2[4,5]\Big),$$

$$F_{\mu\nu}^4=\kappa_1\kappa_2\Big\{\partial A^4-\frac{g_s}{2}\Big([1,7]+[2,6]+[3,5]-\sqrt{3}[5,8]\Big)\Big\},$$

$$F_{\mu\nu}^5=\kappa_1\kappa_2\Big\{\partial A^5+\frac{g_s}{2}\Big([1,6]-[2,7]+[3,4]-\sqrt{3}[4,8]\Big)\Big\},$$

$$F_{\mu\nu}^6=\kappa_2\Big\{\partial A^6+\frac{g_s}{2}\Big(\kappa_1^2\left([2,4]-[1,5]\right)+[3,7]+\sqrt{3}[7,8]\Big)\Big\},$$

$$F_{\mu\nu}^7=\kappa_2\Big\{\partial A^7+\frac{g_s}{2}\Big(\kappa_1^2\left([1,4]+[2,5]\right)-[3,6]-\sqrt{3}[6,8]\Big)\Big\},$$

$$F_{\mu\nu}^8=\partial A^8+\frac{g_s\sqrt{3}}{2}\kappa_2^2\Big(\kappa_1^2[4,5]+[6,7]\Big). \tag{23}$$

In the framework of Cayley-Klein scheme [2] the gauge group $SU(3;\kappa)$ has two one-parameter contractions $\kappa_1\to 0,\kappa_2=1$ and $\kappa_2\to 0,\kappa_1=1$, as well as one two-parameter contraction $\kappa_1,\kappa_2\to 0$. We consider the following contraction: $\kappa_1=\kappa_2=\kappa=\epsilon\to 0$, which corresponds to the infinite energy limit of QCD. The quark part of Lagrangian (21) is represented as a sum of terms proportional to zero, the second and forth powers of ϵ

$$\mathcal{L}_q(\epsilon)=L_q^\infty+\epsilon^2L_q^{(2)}+\epsilon^4L_q^{(4)} \tag{24}$$

and gluon part is represented as a sum

$$\mathcal{L}_{gl}(\epsilon)=L_{gl}^\infty+\epsilon^2L_{gl}^{(2)}+\epsilon^4L_{gl}^{(4)}+\epsilon^6L_{gl}^{(6)}+\epsilon^8L_{gl}^{(8)}. \tag{25}$$

In the infinite energy (temperature) limit $\kappa=\epsilon\to 0$ the most parts of gluon tensor components are equal to zero and the expressions for two nonzero components are simplified

$$F_{\mu\nu}^3=\partial_\mu A_\nu^3-\partial_\nu A_\mu^3=\frac{1}{2}\left(F_{\mu\nu}^{RR}-F_{\mu\nu}^{GG}\right),\quad F_{\mu\nu}^8=\partial_\mu A_\nu^8-\partial_\nu A_\mu^8=\frac{\sqrt{3}}{2}\left(F_{\mu\nu}^{RR}+F_{\mu\nu}^{GG}\right), \tag{26}$$

so we can write out the QCD Lagrangian $\mathcal{L}_\infty$ in this limit explicitly

$$\mathcal{L}_\infty=L_q^\infty+L_{gl}^\infty=$$

$$= \sum_q \left\{ i\bar{q}_R \gamma^\mu \partial_\mu q_R + \frac{g_s}{2} |q_R|^2 \gamma^\mu A_\mu^{RR} \right\} - \frac{1}{4} \left(F_{\mu\nu}^{RR} \right)^2 - \frac{1}{4} \left(F_{\mu\nu}^{GG} \right)^2 - \frac{1}{4} F_{\mu\nu}^{RR} F_{\mu\nu}^{GG}. \qquad (27)$$

From $\mathcal{L}_\infty$ we conclude that only the dynamic terms for the first color component of massless quarks survive under infinite energy, which means that the quarks are monochromatic, and the terms also survive, which describe the interactions of these components with R-gluons. Besides R-gluons there are also G-gluons, which do not interact with the quarks.

Similarly to the Electroweak Model starting with $\mathcal{L}_q(\epsilon)$ (24) and $\mathcal{L}_{gl}(\epsilon)$ (25) one can construct a number of intermediate models for QCD, which describe the restoration of color degrees of freedom for the quarks and the gluon interactions in the Universe evolution.

It follows from Lagrangian $\mathcal{L}(\epsilon) = \mathcal{L}_q(\epsilon) + \mathcal{L}_{gl}(\epsilon)$, that the total reconstruction of the quark color degrees of freedom will take place after the restoration of all quark mass ($\approx \epsilon^2$) at the same time with the reestablishment of all electroweak interactions ($\approx \epsilon^4$). All color interactions start to work later because some of them are proportionate to the eighth power ϵ^8.

4 Estimation of boundary values

As it was mentioned the contraction of gauge group of QCD gives an opportunity to order in time different stages of its development, but does not make it possible to bear their absolute date. Let us try to estimate this date with the help of additional assumptions. The equality of the contraction parameters for QCD and the EWM is one of these assumptions.

Then we use the fact that the electroweak epoch starts at the energy $E_4 = 100\,GeV$ and the QCD epoch begins at $E_8 = 0,2\,GeV$. In other words we assume that complete reconstruction of the EWM, whose Lagrangian has minimal terms proportionate to ϵ^4, and QCD, whose Lagrangian has minimal terms proportionate to ϵ^8, take place at these energies. Let us denote by Δ cutoff level for ϵ^k, $k = 1, 2, 4, 6, 8$, i.e. for $\epsilon^k < \Delta$ all the terms proportionate to ϵ^k are negligible quantities in Lagrangian. At last we suppose that the contraction parameter inversely depends on energy

$$\epsilon(E) = \frac{A}{E}, \qquad (28)$$

where A is constant.

As far as the minimal terms in the QCD Lagrangian are proportional to ϵ^8 and QCD is completely reconstructed at $E_8 = 0,2\,GeV$, we have the equation $\epsilon^8(E_8) = A^8 E_8^{-8} = \Delta$ and obtain $A = E_8 \Delta^{1/8} = 0,2\Delta^{1/8}\,GeV$. The minimal terms in the EWM Lagrangian are proportional to ϵ^4 and it is reconstructed at $E_4 = 100\,GeV$, so we have $\epsilon^4(E_4) = A^4 E_4^{-4} = \Delta$, i.e. $E_4 = A\Delta^{-1/4} = E_8 \Delta^{1/8} \Delta^{-1/4} = E_8 \Delta^{-1/8}$ and we obtain the cutoff level $\Delta = (E_8 E_4^{-1})^8 = (0,2 \cdot 10^{-2})^8 \approx 10^{-22}$, which is consistent with the typical energies of the Standard Model. From the equation $\epsilon^k(E_k) = A^k E_k^{-k} = \Delta$ we obtain

$$E_k = \frac{A}{\Delta^{1/k}} = \frac{E_8 \Delta^{1/8}}{\Delta^{1/k}} = E_8 \Delta^{\frac{k-8}{8k}} \approx 10^{\frac{88-15k}{4k}}\,GeV. \qquad (29)$$

Simple calculations give the following estimations for the boundary values of the average energy (or temperature) in the early Universe (GeV): $E_1 = 10^{18}$, $E_2 = 10^7$, $E_3 = 10^3$, $E_4 = 10^2$, $E_6 = 1$, $E_8 = 2 \cdot 10^{-1}$. The obtained estimation for "infinity" energy $E_1 \approx 10^{18}\,GeV$ is comparable with Planck energy $\approx 10^{19}\,GeV$, where the gravitation effects are important. So the developed evolution of the elementary particles does not exceed the range of the problems described by electroweak and strong interactions.

It should be noted that for the power function class $\epsilon(E) = BE^{-p}$ the estimations for energy boundary values are very weakly dependent on power p. So for $p = 10$ we obtain practically the same E_k as for the simplest function (28) with $p = 1$.

Conclusion

We have investigated the high-energy limit of the SMwhich was obtained from the first principles of the gauge theory as contraction of its gauge group. It was shown that the mathematical contraction parameter is proportional to the average energy and its zero limit corresponds to the infinite energy limit of the Model. The SMpasses in this limit through several stages, which are distinguished by the powers of contraction parameter, what gives the opportunity to classify them in time as earlier-later. To determine the absolute date of these stages the additional assumptions was used, namely: the inversely dependence ϵ on the energy (28) and the cutoff level Δ for ϵ^k. Unknown parameters are determined with the help of the QCD and EWM typical energies.

The exact expressions for the respective Lagrangians for any stage in the SMevolution are presented. On the base of decompositions (12),(24),(25) for any energy scale the intermediate models $\mathcal{L}_k$ are constructed. It gives an opportunity to draw a conclusions on interactions and properties of elementary particles in each of considered epoch. The presence of the several intermediate models in the interval from Plank energy $10^{19}\,GeV$ up to the EWM typical energy $10^2\,GeV$ instead of one model solves so-called the hierarchy problem of the SM[11].

At the infinite energy limit ($E > 10^{18}\,GeV$) all particles including vector bosons lose their masses and electroweak interactions become long-range. Monochromatic massless quarks exchange by only one sort of R-gluons. Besides R-gluons there are also G-gluons, which do not interact with quarks. It follows from the explicit form of Lagrangians $L_\infty^{int}(A_\mu, Z_\mu)$ (14) and $\mathcal{L}_\infty$ (27) that only the particles of the same sort interact with each other. Particles of different sorts do not interact. It looks like some stratification of leptons and quark-gluon plasma with only one sort of particles in each stratum.

At the level of classical gauge fields it is already possible to give some conclusions on the appearance of elementary particles mass on the different stages of the Universe evolution. In particular we can conclude that half of quarks ($\approx \epsilon$, $10^{18}\,GeV > E > 10^7\,GeV$) first restore they mass. Then Z-bosons, electrons and other quarks become massive ($\approx \epsilon^2$, $10^7\,GeV > E > 10^3\,GeV$). Finally Higgs boson χ and charged $W^\pm$-bosons restore their masses because their mass terms are multiplied by ϵ^4 $(E < 10^2\,GeV)$.

In a similar way it is possible to describe the evolution of particle interactions. Self-action of Higgs boson appears with its mass restoration. At the same epoch start interactions of four $W^\pm$-bosons, as well as of two Higgs and two W-bosons (6). The only one term in Lagrangian, which is proportional to the third power of ϵ, describes interaction of Higgs boson with charged $W^\pm$-bosons ($E < 10^3\,GeV$). The rest of the electroweak particle interactions are appeared in the second order of the contraction parameter ($10^7\,GeV > E > 10^3\,GeV$). Some part of color interactions between quarks in Lagrangian (21) is proportional to ϵ^2 $(E < 10^7\,GeV)$ and the rest part is proportional to ϵ^4 $(E < 10^2\,GeV)$. Therefore the complete restoration of quark color degrees of freedom takes place after the appearance of quark masses ($\approx \epsilon^2$, $E < 10^7\,GeV$) (15) together with the restoration of all electroweak interactions ($\approx \epsilon^4$, $E < 10^2\,GeV$). Complete color interactions start later because they are proportional to ϵ^8 $(E < 10^{-1}\,GeV)$.

The evolution of elementary particles and their interactions in the early Universe obtained with the help of contractions of gauge groups of the SMdoes not contradict the canonical one [5], according to which the QCD phase transitions take place later then the electroweak phase transitions. The developed evolution of the SMpresent the basis for a more detailed analysis of different phases in the formation of leptons and quark-gluon plasma.

The author is thankful to V. V. Kuratov and V. I. Kostyakov for helpful discussions.

The study is supported by Program of UrD RAS project N 15-16-1-3.

References

[1] E. Inönü, E.P. Wigner. On the contraction of groups and their representations. Proc. Nat. Acad. Sci. USA, 1953. Vol. 39. P. 510–524.

[2] N.A. Gromov. Contractions of Classical and Quantum Groups. Moscow: Fizmatlit, 2012. – 318 p. [in Russian].

[3] N.A. Gromov. Contraction of Electroweak Model and neutrino. Physics of Atomic Nuclei, 2012. Vol. 75. N 10. P. 1203–1209.

[4] N.A. Gromov. Interpretation of neutrino-matter interactions at low energies as contraction of gauge group of Electroweak Model. Physics of Atomic Nuclei, 2013. Vol. 76. N 9. P. 1144–1148.

[5] D.S. Gorbunov, V.A. Rubakov. Introduction to the Theory of the Early Universe: Hot Big Bang Theory. Singapure: World Scientific, 2011. – 488 p.

[6] N.A. Gromov. High and low energy limits of Electroweak Model. Proceedings of Komi SC UrD RAS, 2014. Issue 1(17). P. 5–9 [in Russian].

[7] N.A. Gromov. Natural Limits of Electroweak Model as Contraction of its Gauge Group. J. Phys.: Conf. Ser., 2015. Vol. 597. 012040.

[8] V.A. Rubakov. Classical Theory of Gauge Fields. Princeton, USA: University Press, 2002. – 444 p.

[9] M.E. Peskin, D.V. Schroeder. An Introduction to Quantum Field Theory. Addison-Wesley, 1995.

[10] L.D. Linde. Particle Physics and Inflationary Cosmology. Moscow: Nauka, 1990. – 280 p. [in Russian].

[11] V.M. Emel'yanov. Standard Model and its expansion. Moscow: Fizmatlit, 1990. – 584 p. [in Russian].

Spherical gravitational waves in the week gravitational field

A. Leonovich
kaffiz@bsuir.by; Belarusian State University of Informatics and Radioelectronics, Minsk
Yu. Vyblyi
Vyblyi@gmail.com; Stepanov Institute of Physics, Minsk

Abstract

Within the framework of relativistic theory of gravitation the exact spherically-symmetric wave solution is received. It is shown that this solution possesses the positive-definite energy and momentum deriving with the Fock energy-momentum density tensor of gravitational field. Also the exact solution of linearized Einstein equations with zero trace of tensor gravitational potential are presented. In this connection the sense of Birkhoff theorem in General Relativity is discussed.

1 Spherically-symmetric wave solution

In General Relativity the energy of gravitational field is described with help the energy-momentum pseudotensor, but the expression for corresponding tensor is absent. It is one of reasons for regarding the gravitation interaction as a tensor interaction in Minkowski space-time (see, for example [1]). The most consistently such approach was realized in the so-called relativistic theory of gravitation (RTG) [2]. This theory can be considered as a gauge theory of the group of Lie variations for dynamical variables. The related transformations are variations of the form of the function for generally covariant transformations. In order to the action be invariant for this group under the transformations of the dynamic variables alone one needs replacing the "nondynamic" Minkowski metric γ^{ik} with expression g^{ik}: $\tilde{g}^{ik} = \sqrt{-g}g^{ik} = \sqrt{-\gamma}(\gamma^{ik} + k\psi^{ik})$, where $\gamma = det\gamma_{ik}, g = detg_{ik}$, k^2 - is the Einstein constant, and thus introducing the gauge gravitational potential ψ^{ik}. The quantity g^{ik} is interpreted here as a metric of an effective space-time allowing an unique construction of the connection (the Cristoffel brackets). The RTG field equations at its massless variant are the Einstein ones for this effective metric, added the conditions, restricting the spin states of the field ψ^{ik}

$$D_i\tilde{g}^{ik} = 0 \ , \tag{1}$$

where D_i - the covariant derivative in the Minkowski space. This condition plays the significant role in RTG. It removes the gauge arbitrariness of Einstein equations and coincides with the Fock harmonical condition in Galilean coordinates [3].Note that this conditions fix the Minkowski coordinates with the exactness up to a linear transformations. This uncertain may be removes with help any additional conditionals, containing the Minkowski metric tensor obviously.

Although the massless RTG field equations locally coincide with General Relativity ones, their global solutions, generally speaking, will be different, since this solutions are defined on the various manifolds. RTG, being founded on the simple space-time topology, allows to introduce the global Galilean coordinate system, that distinguishes RTG from the bimetric theories, in which a flat space plays the auxiliary role and its topology does not define the character of

the physical processes. This distinction may take place at interpretation of the field solutions, since the coordinate system in RTG is defined by Minkowski metric, but it is fixed by noncovariant coordinate conditions in GR. Just this situations takes place for spherical-symmetric gravitational fields. In GR according the Birkhoff theorem [4] any spherical gravitational field in vacuum is static. The proof of this theorem is grounded on the transformation of certain spherically-symmetric metric to the coordinates in which it has a static form. But in RTG such transformation is the transfer from the spherical coordinates in the Minkowski space with the metric $\gamma_{ik} = diag(1, -1, -r^2, -r^2 \sin^2\theta)$ to some "nonstatic" coordinates. The Birkhoff theorem means that in the case of spherical symmetry the coordinate system in which the vacuum metric depends from one coordinate always exists, but it not means that the field was static in the starting coordinates.

Hence the task of the investigation of nonstatic spherical-symmetric solutions, which was in general view investigated in [5], arises. In this paper one of the possible nonstatic spherically-symmetric wave solution is found in implicit form. Such solutions may play very important role in astrophysics and cosmology.

To find the spherical wave solutions we use the Birkhoff theorem and present a nonstatic spherical vacuum solution in the certain coordinate system (τ, R, θ, ϕ) in the form of Schwarzschild metric

$$ds^2 = (1 - \frac{2m}{R})d\tau^2 - (1 - \frac{2m}{R})^{-1}dR^2 - R^2 d\Omega^2 \ . \tag{2}$$

To find the solution in spherical coordinates of (t, r, θ, ϕ) we make the coordinate transformation

$$t = t(\tau, R), r = r(\tau, R) \ , \tag{3}$$

The transformation coefficients will be found from the condition (1). The corresponding equations connecting the variables (t, r) and (τ, R), take the form

$$\frac{R}{R-2m}\frac{\partial^2 t}{\partial \tau^2} - R^{-2}\partial_R[(R^2 - 2mR)\frac{\partial t}{\partial R}] = 0 \ , \tag{4}$$

$$\frac{R}{R-2m}\frac{\partial^2 r}{\partial \tau^2} - R^{-2}\partial_R[(R^2 - 2mR)\frac{\partial r}{\partial R}] + \frac{2r}{R^2} = 0 \ . \tag{5}$$

We search for the partial solution of the equations (4), (5) in the next form

$$\tau = t + T(u), R = r + m, u = t + f(r) \ , \tag{6}$$

where u - the retarded argument, which is finite at any values of r; the light velocity and gravitational constant are believed equal to unit. Another form of solution was considered in [6]. Finding with help of (6) the transformations coefficients and substituting its to equations (4),(5), we find that the equation (5) is satisfied identically, and the equation (4) after variables separation is reduced to ordinary differential equations with separation constant C for the functions $T(u)$ and $f(r)$

$$T_{uu} = CT_u(1 + T_u)^2 \ , \tag{7}$$

$$f_{rr} + C{f_r}^2 + \frac{2r}{r^2 - m^2}f_r - C\frac{(r+m)^2}{(r-m)^2} = 0 \ , \tag{8}$$

The first integral of equation (7) has a form

$$Cu = \frac{1}{1+z} - \ln|\frac{z+1}{z}| + A \ , \tag{9}$$

where $z = T_u$, A - integration constant. From (9) we find that function $z(u)$ is positive for positive A and large values of u ($u > 0$).

The equation (8) may be integrated with help the power series for $f(r)$ relative variable $1/r$. The analysis of this solution shows that for $C > 0$, $f_r < 0$ for all values of r. The metric components now have the next form

$$g_{00} = \frac{r-m}{r+m}(1+T_u)^2 \ , g_{01} = \frac{r-m}{r+m}T_u(1+T_u)f_r \ , \tag{10}$$

$$g_{11} = \frac{r-m}{r+m}{T_u}^2{f_r}^2 - \frac{r+m}{r-m} \ , g_{22} = -(r+m)^2 \ , g_{33} = -(r+m)^2\sin^2\theta \ . \tag{11}$$

Note that we did not use any additional conditions which fix the Minkowski coordinates up to the Lorents transformations. The linear approximation of the solution (10),(11) show that it does not satisfy to the linear approximation Einstein equations. This means that the finding solution relates to some coordinate system from the class of affine coordinates. Nevertheless, it illustrates the possibility the existence of spherical gravitational waves.

2 Energy-momentum tensor

The finding metric may be used for the investigation of a sign of gravitational radiate density t^{00}, which according to RTG has a form [2]

$$t^{00} = \frac{1}{\sqrt{-\gamma}}\gamma^{ik}D_iD_k\tilde{g}^{00} \ . \tag{12}$$

where Hilbert energy momentum tensor density is used. The problem concerning positive definite of gravitational energy density is not trivial, because in RTG the expression t^{00} don't possess square-law structure relative the field functions and its first derivatives and it contains second derivatives too. This situation is the consequence of a difference between spin one and spin two description. In the case of a vector field Lagrangian does not contain the Christoffel symbols (in arbitrary coordinate system) and the energy-momentum tensor can be easily found when this Lagrangian is varied in respect to the metric. The same is true for the Maxwell and Yang-Mills fields too. Quiet different situation arises in the case of a tensor field. Now the Christoffel symbols can not be shorten and thus the energy-momentum tensor contains the second derivatives. This fact leads, generally speaking to uncertainty of energy density sign. The possible approaches to the solving of this problem are regarded in [7],[8].

Let us consider Fock representation for Einstein tensor G^{ik} [3]

$$2gG^{ik} = \partial_m\partial_n(\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}) + L^{ik} \ , \tag{13}$$

where the term L^{ik} contain the first derivatives only. The expression

$$U^{ik} = \partial_m\partial_n(\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}) \ , \tag{14}$$

Fock interpreted as gravitational density energy-momentum pseudo-tensor of weight equal to +2. Due to field equations it equal to L^{ik} and coincides with well-known Landau-Lifshitz pseudo-tensor.

In RTG we may to receive the corresponding tensor density, replacing ordinary derivations to covariant derivations relative Minkowski metric γ^{ik}

$$T_g^{ik} = \frac{1}{\sqrt{-\gamma}}D_mD_n(\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}) \ , \tag{15}$$

This tensor may be received by means of variational procedure from the next Lagrangian

$$L = \tilde{R}(g^{ik}) + \frac{1}{\sqrt{-f}} R_{ijkl}(f_{mn})\tilde{g}^{ik}\tilde{g}^{jl} \ . \tag{16}$$

where $\tilde{g}^{ik} = \tilde{f}^{ik} + k\tilde{\psi}^{ik}$, f_{ik} is background metric with nonzero curvature tensor, which is believed equal to zero after variation. Analogously approach was used in [9]to receive the energy-momentum tensor of conformal-invariant scalar field.

Using Fock energy-momentum tensor, let us to calculate the energy and momentum density for finding solution. For energy density we have

$$T_g^{00} = -\frac{1}{r^2}[(\frac{R^4}{r^2})_r + 2\frac{R^2}{r}(\tau_r{}^2 g_{\tau\tau} + R_r{}^2 g_{RR})]_r \ , \tag{17}$$

This expression is the sum of static and wave parts. For later we have

$$T_w^{00} = -\frac{2}{r^2}[(\frac{R^2}{r}g_{\tau\tau})_r T_u{}^2 f_r{}^2 + 2\frac{R^2}{r}g_{\tau\tau}(T_u T_{uu} f_r{}^3 + T_u{}^2 f_r f_{rr})] \ . \tag{18}$$

The expression (17) may be present as

$$T_w^{00} = \frac{2T_u{}^2}{r^2}[(3 - \frac{m^2}{r^2})f_r{}^2 - 2Cf_r r(1 - \frac{m^2}{r^2})(\frac{(r+m)^2}{(r-m)^2} + f_r{}^2(T_u{}^2 + 2T_u))] \ . \tag{19}$$

The analysis of last expression shows that it is a positive-definite, if C and T_u are positive, but f_r is negative. As it was finding above this conditions follow from solutions of equations (7),(8).

Let us to find the expression for momentum density.In Cartesian coordinates x^i we have

$$T_w^{0i} = \frac{4Cx^i}{r^4}(r^2 - m^2)f_r{}^2 T_u{}^2(1 + T_u)^2 \ , \tag{20}$$

This expression is positive-definite for $C > 0$ that testify about energy transport and, consequently, about a wave character of finding solution.

For large values of r we have

$$f_r = -1 - \frac{1}{Cr} \ , \tag{21}$$

$$T_w^{00} = \frac{4C}{r}T_u{}^2(1 + T_u)^2 \ , \tag{22}$$

$$T_w^{0i} = \frac{4Cx^i}{r^2}T_u{}^2(1 + T_u)^2 \ . \tag{23}$$

The expressions (22),(23) show that energy and momentum are not transporting at infinity as it had to be, because wave solution was finding with help the coordinate transformation from asymptotically-flat metric.

Note that the asymptotical expression for retarded argument coincides with corresponding expression, using by Fock [4] in wave zone

$$u = t - (r + 2m\ln\frac{r}{r_0}) \ , \tag{24}$$

whence follow that $C = 1/2m$.

So, we may to conclude that Hilbert energy-momentum tensor play the role of gravitational current in the field equations although Fock tensor describes the energy characteristics of gravitational field.

3 Linear approximation

It may be shown that the received solution describes as a transversal as a longitudinal waves, because the spur of gravitational potential depends on retarding argument. Now we will show that in linear approximation the wave solution with condition $Sp\psi^{ik} = 0$ exist and consequently it describes the transversal waves with spin two only.

In linear approximation the field equation system has the next form

$$\gamma^{ik} D_i D_j \tilde{g}^{mn} = 0 \ , D_i \tilde{g}^{ik} = 0 \ , \tilde{\gamma}_{ik} \tilde{g}^{ik} = 4 \ . \tag{25}$$

In Cartesian coordinates we introduce the next definitions

$$\tilde{g}^{00} = \frac{u}{r^2} \ , \tilde{g}^{0k} = \frac{a}{r^3} x^k \ , \tilde{g}^{ik} = -w\eta^{ik} + (-\frac{w}{r^2} + \frac{v}{r^4}) x^i x^k \ . \tag{26}$$

Then the system (24) takes the form

$$\begin{gathered} \ddot{u} - u^{''} + \frac{2}{r} u^{'} - \frac{2}{r^2} u = 0 \ , \ddot{a} - a^{''} + \frac{2}{r} a^{'} = 0 \ , \\ \ddot{v} - v^{''} + \frac{2}{r} v^{'} + \frac{2}{r^2} v - 4w = 0 \ , \ddot{w} - w^{''} - \frac{2}{r} w^{'} + \frac{2}{r^2} w - \frac{2}{r^4} v = 0 \ , \\ \dot{u} + a^{'} = 0 \ , \dot{a} + v^{'} - 2rw = 0 \ , u - v - 2r^2 w = 4r^2 \ . \end{gathered} \tag{27}$$

The solution of this system, depending from retarding time $(\omega t - r)$ may be represented in the next form

$$\begin{gathered} u = r^2 - r\ddot{T} \ , a = -r\ddot{T} - \dot{T} \ , \\ v = -r^2 - r\ddot{T} - 2\dot{T} - 2\frac{T}{r} \ , w = -1 + \frac{\dot{T}}{r^2} + \frac{T}{r^3} \ . \end{gathered} \tag{28}$$

where $T = T(\phi)$ - arbitrary function, $\phi = \omega t - r$, where $\omega = 1$. Let us calculate the density of energy and momentum for this solution, using the expression (14)

$$\begin{gathered} T_w^{00} = \frac{2}{r^5}[\dot{X} + r\ddot{X}] + \frac{4}{r^5}\dot{X} + \frac{6}{r^6} X \ , \\ T_w^{0i} = \frac{2}{r^6} x^i [\dot{X} + r\ddot{X}] \ , \end{gathered} \tag{29}$$

where $X = T^2 \dot{x} \ , x = \dot{T}/T$. This expressions are positive-definite if the next conditions take place

$$X > 0 \ , \dot{X} > 0 \ , \ddot{X} > 0 \ . \tag{30}$$

Note that though in linear approximation the field equations are as is well known gauge invariant, the additional conditions for spin states and spur broke this invariance in full and make the solution and its energy characteristics strictly definite.

The existence of wave solution in weak spherically field is in a contradiction with ordinary physical interpretation of Birkhoff theorem not only in RTG but in General Relativity too, because in this approximation we must to consider a gravitational field as tensor field in Minkowsky space-time. Due to this result we must admit that weak spherical gravitational waves in General Relativity exist if they are considered far from compact sources.

4 Conclusion

The existence of spherically-symmetrical wave solutions and as a whole nonstatic spherically-symmetrical solutions is in a contradiction with ordinary physical interpretation of Birkhoff theorem in GR. In RTG such solutions have a physical sense as far as the temporal coordinate of the Minkowski space-time has it. Essentially that this wave solution possesses positive-defined energy and momentum densities.

Although the receiving solution has enough formal character, it illustrates the possibility of existence of spherical gravitational waves. A total solution must include in the interior one and the matching of these solutions.

References

[1] R.P. Feynman and all. Feynman Lectures on Gravitation. Addison-Wesley Publishing Company, 1965.

[2] A.A. Logunov. The Theory of Gravity. Nova Science Publ. N.Y., 1998.

[3] V.A. Fock. The Theory of Space Time and Gravity. Pergamon Press, 1964.

[4] G.Birkhoff, Relativity and Modern Physics. Harvard Un. Press, 1923.

[5] A.A. Vlasov, Nongauge Approach to Relativistic Theory of Gravity, (MSU, Moscow, 1992), (in Russian).

[6] A. Leonovich and Yu.P. Vyblyi. In: Nonlinear Phenomena in Complex Systems. Institute of Physics, Minsk, 2003, p. 330 - 333.

[7] A.A. leonovich and D.M. Mladenov. Modern Physics Letters. A, V.8, No 34, 1993, p.3251.

[8] A.A. Leonovich. In: Fundamental Problems on High Energy Physics and Field Theory, (Proc. of the 18 Workshop on High Energy Physics and Field Theory), Protvino, p. 35-39, 1995.

[9] N.A. Chernikov and E.A. Tagirov. Ann.Inst. H. Poincare, A9, 109, 1968, p.46.

Реконструкция движения компактного объекта вблизи черной дыры

А. Тарасенко
Белорусский государственный университет, Минск
С. Комаров
Белорусский государственный университет, Минск
А. Горбацевич
Белорусский государственный университет, Минск

Аннотация

Рассмотрена задача восстановления траектории движения компактного объекта вблизи черной дыры на основе анализа его электромагнитного излучения. Для ее решения использованы два подхода. Первый основан на наблюдениях сразу за несколькими изображениями одного и того же источника. Он требует анализа временно́й зависимостей интенсивности и красного смещения излучения от этих изображений. Такой метод позволяет определить траекторию движения любого объекта, не предполагая известными уравнения его движения. Второй метод применим только для двойных систем, движущихся во внешнем гравитационном поле, например, в поле сверхмассивной черной дыры. В этом случае для решения задачи используются данные только о красном смещении основного изображения, но при этом предполагаются известными уравнения движения данной двойной системы.

1 Красное смещение

Наблюдения окрестности Центра Галактики предоставляют нам новые возможности изучения гравитации [1–3]. Смещение частоты принимаемого из этой области излучения, относительно испущенной, является параметром, измеряемым с очень большой точностью. Так как при этом значительную роль играет гравитационное красное смещение (часто, определяющую), то его называют красным смещением. На величину красного смещения влияют одновременно как гравитационное поле, в котором находятся источник и приемник, так и их скорости. Поэтому, измерения рассматриваемой величины могут дать нам информацию о движении источника.

Здесь под источником электромагнитного излучения мы будем понимать одиночный или двойной пульсар, движущийся в окрестности сверхмассивной черной дыры. Интересным приложением нашего метода будет также и изучение движения звезд из S-кластера (см., напр., [4]).

Как известно (см., напр., [5]) в приближении геометрической оптики мировая линия луча света представляет собой изотропную геодезическою $x^i = x^i(\gamma)$[1]:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\gamma^2} + \Gamma^i_{mn}\frac{\mathrm{d}x^m}{\mathrm{d}\gamma}\frac{\mathrm{d}x^n}{\mathrm{d}\gamma} = 0 \quad \text{или} \quad k^i{}_{;j}k^j = 0\,, \quad k^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}\gamma}\,, \tag{1}$$

[1]Латинские индексы пробегают значения от 1 до 4. Сигнатура метрики пространства-времени — +2.

где γ — канонический параметр, а $k^i \equiv \dfrac{\mathrm{d}x^i}{\mathrm{d}\gamma}$ — касательный к геодезической изотропный вектор ($k_i k^i = 0$), который в дальнейшем будем называть волновым вектором.

Для вычисления красного смещения удобнее всего воспользоваться следующим выражением (см., напр., [9]):

$$z = \frac{\delta\lambda}{\lambda} = \frac{(k_i u^i)_0}{(k_i u^i)_1} - 1, \tag{2}$$

здесь z — красное смещение, λ — длина волны испущенного света, $(u^i)_{0,1}$ — вектор 4-скорости излучателя (индекс 0) и приемника (индекс 1), соответственно. При этом волновые векторы $(k_i)_0$ и $(k_i)_1$ связаны между собой посредством параллельного переноса вдоль изотропной геодезической $x^i(\gamma)$, соединяющей событие 0 (испускание света) с событием 1 (прием света). Выражение (3) учитывает полное красное смещение, обусловленное как гравитационным полем, так и эффектом Доплера. (Заметим, что эти эффекты не всегда можно однозначно разделить.) Для упрощения последующих вычислений будем пренебрегать как орбитальным движением Земли ($v/c \approx 10^{-4}$), так и кривизной пространства-времени в окрестности Земли. Заметим, что их учет не представляет никаких трудностей. Таким образом соотношение (2) принимает следующую форму:

$$z = \frac{(k_i u^i)_0}{k_4 c} - 1, \ , \quad \text{где} \quad k_4 = \frac{1}{c}\left(k_i\, u^i\right)_1 , \quad (u^i)_1 \approx \{0\,, 0\,, 0\,, c\}\,. \tag{3}$$

Нетрудно показать, что $k_4 = g_{44} k^4$ в случае статической метрики и при надлежащем выборе системы координат является интегралом движения.

Даже в случае нейтронной звезды вкладом собственного гравитационного поля звезды в красное смещение можно пренебречь.. Поэтому в соотношении (3) будем учитывать только внешнее гравитационное поле, создаваемое сверхмассивной черной дырой.

2 Уравнения движения двойной звезды во внешнем гравитационном поле

Рассмотрим связанную посредством гравитационного взаимодействия систему двух нейтронных звезд, свободно падающих во внешнем гравитационном поле, например, в поле сверхмассивной черной дыры. Уравнения движения такой системы были получены в работах [6–8] при следующих предположениях:

a) Среднее расстояние между заездами (ρ) существенно больше их собственных размеров (R_0); например, в случае нейтронных звезд $\rho \sim 10^4$км, а $R_0 \sim 10 \div 20$км. Данное предположение позволяет при выводе уравнений движения сами звезды рассматривать как материальные точки.

b) Размеры двойной системы существенно меньше характерных размеров неоднородности внешнего гравитационного поля. Так, например, в случае движения двойной нейтронной звезды в поле сверхмассивной черной дыры с массой порядка $3,6 \cdot 10^6\, M_\odot$ (именно такая черная дыра находится в центре нашей галактики) условие

$$\rho \ll r_g = 2M \quad (2M \sim 10^7\text{км})\,. \tag{4}$$

очевидно выполняется.

c) Относительно движение звезд в двойной системе нерелятивистское:

$$\frac{v}{c} \ll 1 \quad (\text{для нейтронных звезд } v/c \sim 10^{-2})\,. \tag{5}$$

В частности в отмеченных выше работах было показано, что движение центра масс двойной системы удовлетворяет уравнениям

$$(m_1+m_2)\frac{\mathrm{D}u^i}{\mathrm{D}\tau} = -\frac{1}{2c}\,R^i_{skm}u^s\varepsilon^{mkbn}M_b u_n - \frac{1}{3}\,h^i_s\frac{\mathrm{D}}{\mathrm{D}\tau}\left(R^s_{klm}Q^{kl}u^m\right)\,, \tag{6}$$

где ε^{mkbn} — псевдотензор Леви-Чивита, ($\varepsilon^{1234} = \dfrac{1}{\sqrt{-g}}$; $g = \det(g_{ij})$); $h^i_s = g^i_s + \frac{1}{c^2}u^i u_s$ — проекционный тензор, M^i и Q^{kl} — собственный момент импульса двойной системы и ее квадрупольный момент, вычисленные относительно ее центра инерции.

Переходя в сопутствующую систему отсчета и вводя в ней координаты Ферми, систему уравнений, описывающую двойную систему, можно представить в следующем виде:

$$\begin{aligned}(m_1+m_2)\,\frac{\mathrm{d}V^{(\kappa)}}{\mathrm{d}T} \;=&(m_1+m_2)(2\varepsilon^{(\kappa)}{}_{(\mu)(\nu)}V^{(\mu)}\omega^{(\nu)} - 2cR^{(\kappa)}{}_{(\nu)(\mu)(4)}X^{(\mu)}V^{(\nu)}+\\ &2D^{(\kappa)}{}_{(\nu)}X^{(\nu)} - W^{(\kappa)}) - 2mcR^{(\kappa)}{}_{(\nu)(\mu)(4)}x^{(\mu)}v^{(\nu)}.\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}v^{(\kappa)}}{\mathrm{d}T} \;=&\left(\frac{G(m_1+m_2)}{r}\right)_{,(\kappa)} - 2\varepsilon^{(\kappa)}{}_{(\alpha)(\tau)}\omega^{(\alpha)}v^{(\tau)} - \frac{2c(m_2-m_1)}{(m_1+m_2)}R^{(\kappa)}{}_{(\nu)(\mu)(4)}x^{(\mu)}v^{(\nu)}\\ &+2D^{(\kappa)}{}_{(\nu)}x^{(\nu)} + \frac{2}{3}c(R_{(\alpha)(\mu)(\kappa)(4)} + R_{(\alpha)(\kappa)(\mu)(4)})(X^{(\mu)}v^{(\alpha)} + x^{(\mu)}V^{(\alpha)}).\end{aligned} \tag{8}$$

Здесь $X^{(\alpha)}$ и $V^{(\alpha)}$ соответственно координаты и скорость центра инерции в координатах Ферми. $x^{(\alpha)}$ и $v^{(\alpha)}$ — параметры внутреннего движения двойной системы. m — ее приведенная масса. $\omega^{(\mu)}$ — скорость вращения тетрады;

$$D_{(\mu)(\nu)} = -\frac{c^2}{2}R_{(4)(\mu)(4)(\nu)} + \frac{1}{2}(\delta_{(\mu)(\nu)}\omega^2 - \omega_{(\mu)}\omega_{(\nu)}).$$

3 Решение обратной задачи

3.1 Реконструкция движения на основе яркости изображений

Состояние точечного объекта определяется в каждый момент времени шестью параметрами — тремя координатами и компонентами скорости. Чтобы восстановить произвольное, не обязательно свободное, движение объекта, необходимо как минимум 6 наблюдаемых параметров в каждый момент времени. Излучение точечного источника из-за гравитационного линзирования воспринимается наблюдателем как ряд изображений, яркость которых однако быстро убывает с увеличением расстояния до черной дыры [11]. Для восстановления движения в качестве наблюдаемых параметров можно использовать красное смещение и яркость первых трех изображений источника. Метод восстановления движения по двум и трем изображениям рассмотрен в [12], ниже приведены основные результаты.

В качестве характеристики яркости изображения источника будем использовать силу света I — энергию, излученную в единицу времени в единицу телесного угла

$$I = \frac{\mathrm{d}E}{\mathrm{d}\tau\mathrm{d}\Omega}. \tag{9}$$

Назовем коэффициентом усиление отношение принятой силы света к испущенной

$$K = \frac{I_P}{I_0}. \tag{10}$$

Коэффициент усиления можно разбить на 2 части — геометрическую K^{geom}, описывающую гравитационное линзирование, и множитель, зависящий от красного смещения:

$$K = \frac{K^{\text{geom}}(\Delta\Phi, r)}{(1+z)^4}. \tag{11}$$

Геометрический коэффициент усиления K^{geom} зависит только от радиальной координаты r и углового расстояния между источником излучения и наблюдателем $\Delta\Phi$, но не от скорости.

Составим систему алгебраических уравнений

$$\begin{aligned} \frac{K_{-1}^{\text{geom}}(\Delta\Phi, r)}{K_0^{\text{geom}}(\Delta\Phi, r)} &= \frac{I_{-1}^{\text{exp}}(s)(1+z_{-1}^{\text{exp}}(s))^4}{I_0^{\text{exp}}(s)(1+z_0^{\text{exp}}(s))^4}, \\ \frac{K_1^{\text{geom}}(\Delta\Phi, r)}{K_0^{\text{geom}}(\Delta\Phi, r)} &= \frac{I_1^{\text{exp}}(s)(1+z_1^{\text{exp}}(s))^4}{I_0^{\text{exp}}(s)(1+z_0^{\text{exp}}(s))^4}. \end{aligned} \tag{12}$$

Правая часть обоих уравнений может быть найдена из наблюдательных данных. Решение системы определяет r и $\Delta\Phi$, что позволяет найти закон движения источника.

В [12] исследуется существование, единственность и устойчивость решения системы (12). Показано, что система имеет единственное решение, которое будет устойчивым при $\Delta\Phi$ не очень близком к 0 и π.

3.2 Реконструкция движения по красному смещению

Описанный выше метод реконструкции движения имеет ряд ограничений. Он предполагает, что источник излучения находится достаточно близко к черной дыре, чтобы можно было наблюдать дополнительные изображения. Предполагается, что излучение изотропное, в противном случае количество степеней свободы возрастает, и недостаточно наблюдать 3 изображения. Кроме того, представляет интерес восстановление движения не только точечного объекта, но и компактной двойной системы. Ниже описан другой метод реконструкции движения, который решает задачу: найти параметры свободного движения компактной двойной системы в поле черной дыры по красному смещению основного изображения обеих компонент системы.

Решение обратной задачи заключается в том, чтобы определить все параметры траекторий движения рассматриваемой системы (см.Табл. 1). Для проверки результатов решения вначале была решена прямая задача вычисления красного смещения, исходя из закона движения излучающей компоненты, найденного в результате решения системы уравнений (7) и (8). Результаты численного решения данной системы представлены на графике (см. рис. 1).

Для восстановления параметров орбит, попытаемся разбить задачу на две: соответствующую движению центра масс системы, и соответствующую движению компонент относительно друг друга. Учитывая (5), в координатах Ферми красное смещение можно представить в виде следующих двух частей:

$$z + 1 = \frac{1}{k_4 c}\left(k_{(4)}c + k_{(\alpha)}v^{(\alpha)}\right) = z_0 + \tilde{z} + 1, \tag{13}$$

где

$$z_0 = \frac{1}{k_4 c}k_{(4)}c - 1 \tag{14}$$

Величина	Обозначение
Механическая энергия	ϵ
Момент импульса	l
Долгота восходящего узла	Ω
Долгота апоцентра	δ
Начальные положения внутреннего движения в координатах Ферми	x_0, y_0, z_0
Начальные скорости внутреннего движения в координатах Ферми	$x_0^{'}, y_0^{'}, z_0^{'}$

Таблица 1: Параметры движения, подлежащие определению в ходе решения задачи

– вклад в красное смещение, связанный с движением центра инерции системы во внешнем гравитационном поле,

$$\tilde{z} = \frac{1}{k_4 c} k_{(\alpha)} v^{(\alpha)} \tag{15}$$

— вклад в красное смещение, обусловленный "внутренним" движением системы. Для определения величин обеих частей красного смещения из графика, воспользуемся следующей оценкой:

$$k^{(\alpha)}(x) \approx k^{(\alpha)}(0) + k^{(\alpha)}{}_{,(\beta)} X^{(\beta)}. \tag{16}$$

$$\int k_{(\alpha)} v^{(\alpha)} \, \mathrm{d}\tau = k_{(\alpha)}(0) \int \mathrm{d}x^{(\alpha)} + k^{(\alpha)}{}_{,(\beta)}(0) \int X^{(\beta)} \, \mathrm{d}x^{(\alpha)} \Rightarrow$$

$$\int k_{(\alpha)} v^{(\alpha)} \, \mathrm{d}\tau < k \, \omega \, T \, x_a + k_{,(\alpha)} \, X^2.$$

В нашем случае (R — скалярная кривизна пространства-времени, $k_{,(\alpha)} \sim k\sqrt{R}$):

$$\omega \, T \, x_a + \sqrt{R} \, (x_a)^2 \approx 0.001,$$

но

$$v_a \, T \approx 0.01.$$

Это позволяет считать интеграл $\int k_{(\alpha)} v^{(\alpha)} \, d\tau$ равным 0 с достаточной степенью точности. Оправданность такого пренебрежения будет видна в дальнейшем из численных расчетов.

Итак, мы имеем две зависимости красного смещения от времени, соответствующие различным видам движения компонента двойной системы. Рассмотрим теперь метод, с помощью которого может быть решена обратная задача для движения центра масс системы. Будем считать, что он движется по геодезической (это предположение, как и следует из (7), выполняется с большой степенью точности). Так как мы рассматриваем черную дыру Шварцшильда, то можно получить зависимость красного смещения от центра масс в

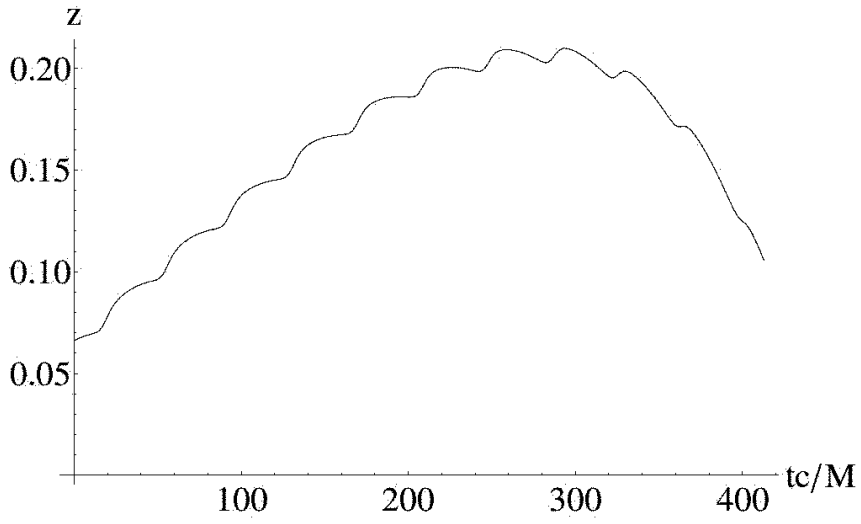


Рис. 1: *Зависимость красного смещения излучения от времени наблюдения. Безразмерной величине времени* $\dfrac{tc}{M}$ *здесь соответствует 20 с.*

аналитическом виде:

$$z_0 = \pm\sqrt{\frac{\epsilon^2}{(1-\frac{2}{r})^2} - \frac{\epsilon^2 d^2}{r^2(1-\frac{2}{r})} - \frac{1+\frac{l^2}{r^2}}{1-\frac{2}{r}} + \frac{d^2}{r^2}(1+\frac{l^2}{r^2}) + \\ + \frac{ld}{r^2}\cos\beta + \frac{\epsilon}{1-\frac{2}{r}} - 1} = F(\epsilon, l, d, r, \beta). \tag{17}$$

Для решения поставленной задачи, воспользуемся следующими уравнениями, которые следуют из (17):

$$\begin{cases} z_0|_n = F(\epsilon, l, d, r, \beta)|_n\,; \\ \dfrac{\mathrm{d}z_0}{\mathrm{d}\tau}\bigg|_n = \left(\dfrac{\mathrm{d}F}{\mathrm{d}d}\,\dfrac{\mathrm{d}d}{\mathrm{d}\tau} + \dfrac{\mathrm{d}F}{\mathrm{d}r}\,\dfrac{\mathrm{d}r}{\mathrm{d}\tau} + \dfrac{\mathrm{d}F}{\mathrm{d}\beta}\,\dfrac{\mathrm{d}\beta}{\mathrm{d}\tau}\right)\bigg|_n\,; \\ \dfrac{\mathrm{d}^2F}{\mathrm{d}\tau^2}\bigg|_n = \left(\dfrac{\mathrm{d}^2F}{\mathrm{d}d^2}\left(\dfrac{\mathrm{d}d}{\mathrm{d}\tau}\right)^2 + \dfrac{\mathrm{d}^2F}{\mathrm{d}r^2}\left(\dfrac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \dfrac{\mathrm{d}^2F}{\mathrm{d}\beta^2}\left(\dfrac{\mathrm{d}\beta}{\mathrm{d}\tau}\right)^2\right. \\ \qquad\qquad \left. + \dfrac{\mathrm{d}F}{\mathrm{d}d}\,\dfrac{\mathrm{d}^2d}{\mathrm{d}\tau^2} + \dfrac{\mathrm{d}F}{\mathrm{d}r}\,\dfrac{\mathrm{d}^2r}{\mathrm{d}\tau^2} + \dfrac{\mathrm{d}F}{\mathrm{d}\beta}\,\dfrac{\mathrm{d}^2\beta}{\mathrm{d}\tau^2}\right)\bigg|_n\,; \\ r(d)|_n = r(\epsilon, l)|_n\,. \end{cases} \tag{18}$$

Здесь n — номер некоторой точки на графике, выбранной произвольно. В нашем случае n принимает значения 1,2. В поле Шварцшильда известны аналитические решения как для времениподобных геодезических, так и для изотропных (см., напр. [10]). Поэтому все производные в (18) легко могут быть найдены. Численное решение полученной системы уравнений и дает нам искомые параметры движения центра масс системы. Результаты таких вычислений по приведенному алгоритму представлены в таблице 2.

Следующей задачей является получение параметров внутреннего движения компонент, из графика соответствующей части красного смещения. Эта задача является более сложной

Величина	Точное значение	Вычисленное значение
Расстояние перицентра, p_1	20.0000	19.9974
Расстояние апоцентра, p_2	30.0000	29.9988
Долгота восходящего узла, Ω	0.90000	0.90003
Параметр луча, p	23.4833	23.4866
Долгота апоцентра, δ	0.00000	-0.00023
Угол в плоскости луча, ϕ_0	0.42900	0.43026

Таблица 2: Результаты вычисления параметров движения центра масс

вследствие того, что уже мы не имеем аналитического решения для данного типа движения (из уравнений (8)). Поэтому алгоритм, аналогичный используемому выше, для такой задачи осуществить нельзя.

Для решения задачи мы воспользуемся тем обстоятельством, что, как видно из численных оценок, добавочные слагаемые, включающие компоненты тензора кривизны, малы. Оставшуюся же часть уравнений, как это видно из (8), можно рассматривать как уравнения для вращающихся кеплеровских орбит (с вектором угловой скорости $\omega^{(\alpha)}$). Тогда опять можно получить приближенную аналитическую формулу для красного смещения.

Орбитальный параметр	Заданное значение	Вычисленное значение
Эксцентриситет,	0.637	0.621
Параметр,	0.000037	0.000021
Расстояние перицентра, p_1	0.000108	0.000127
Расстояние апоцентра, p_2	0.0300	0.0271
Угол между плоскостью относительного движения и плоскостью движения центра масс, β	0.6881	0.6276
Долгота апоцентра, α	1.26	1.15

Таблица 3: Результаты вычисления параметров относительного движения компонент

По причине их громоздкости, мы не будем выписывать здесь эти уравнения. Отметим лишь, что алгоритм решения заключается в том, что угловая скорость кеплеровских орбит учитывается здесь по теории возмущений, как поправка первого порядка малости. Это уже позволяет полностью восстановить все параметры орбиты внутреннего движения компонент. Причем дальнейшая эволюция этих параметров может уже быть найдена на основе более точных уравнений (8), учитывающих внешнее гравитационное поле.

Результаты численных вычислений приведены в таблице 3.

Здесь заданное значение соответствует величинам, которые брались при решении прямой задачи. Сравнивая значения двух последних столбцов таблицы, можно сделать вывод о точности решения задачи представленным методом.

Список литературы

[1] Muno, M. P. A remarkable low-mass x-ray binary within 0.1 parsecs of the galactic center / M. P. Muno and others // The Astrophysical Journal.- 2005.- Vol. 633.- Pp. 228-239.

[2] Wex N. Testing Relativistic Gravity with Radio Pulsars / N. Wex // Journal of Cosmology and Astroparticle Physics — 05/2014 — 2014(08).

[3] Mark, R. The environment of the Galaxy's central black hole l / R. Mark, Morris. // The galactic black hole. Lectures on General Relativity and Astrophysics, edited by H. F. Falcke and F. W. Hehl pp. 95-122.- IOP Publishing Ltd, Bristol, Philadelphia, 2003.

[4] Gillesen S. Monitoring stellar orbits around the massive black hole in the Galactic center / S. Gillesen at al. // The Astrophysical Journal —2009 February 20 — Vol. 692 — Pp. 1075-1109

[5] Мизнер Ч., Торн К., Уилер Дж. Гравитация: В 3 т. / Пер. с англ. Под ред. В. Б. Брагинского, И. Д./Новикова. — М.: Мир, 1977. — Т. 1. —476 с.; Т. 2. — 528 с.; Т. 3. — 512 с.

[6] Deshko, Yu./Yu. Deshko, P. Fortini, A. Giorbatsievich// Proceedings of the 5th International Conference Bolyai-Gauss-Lobachevsky (BGL-5). Minsk, 2006 P. 492 – 497.

[7] Fortini, P. /P.Fortini, A. Giorbatsievich, A. Ortolan //‘ LNL Annual Report 2006. Instituto Nazionale di Fisica Nucleare, La-boratori Nazionale di Legnaro, Italia, 2007, P. 123 -124. Proceedings of the 5th International Conference Bolyai-Gauss-Lobachevsky (BGL-5). Minsk, 2006 P. 492 – 497.

[8] Gorbatsievich, A. The equations of motion of compact binaries in the neighbourhood of supermassive black hole / A. Gorbatsievich, A. Bobrik // The Sun, The Stars, The Universe and General Relativity: International Conference in Honor of Ya. B. Zeldovich's 95th Anniversary. — Vol. 1205. — AIP, 2010. — Pp. 87-96.

[9] Синг, Дж. Общая теория относительности. / Дж. Синг // Москва, 1963.

[10] Чандрасекар, С. Математическая теория черных дыр. Часть 2./ С. Чандрасекар // Москва, "Мир", 1986.

[11] Bozza, V. Gravitational Lensing of stars in the central arcsecond of our Galaxy / V. Bozza, L. Mancini // The Astrophysical Journal.- 2005.- Vol. 627.- P. 790.

[12] Tarasenko, A. Reconstruction of a compact object motion in the vicinity of a black hole by its electromagnetic radiation / A. Tarasenko // Phys. Rev. D. - Vol. 81 - P. 123005

Движение по геодезической в конформно-плоской метрике как задача одночастичной лагранжевой динамики

Л.М. Томильчик
Институт физики им. Б.И. Станановап, НАН Беларуси

Динамические модели с координатно-зависимой массой нашли в настоящее время широкое применение к расчёту наблюдаемых эффектов в квантовых системах. В работе [1] было показано, что картина движения пробного тела по геодезической в конформно-плоской статической метрике допускает трактовку в терминах классической динамики частицы с координатно-зависимой массой. Перспективность такого подхода к описанию астрофизических проявлений скрытой массы продемонстрирована путём расчёта наблюдаемой формы ротационных кривых типичных спиральных галактик.

В настоящей работе вопрос об эквивалентном динамическом описании движения пробных тел рассмотрен для общего, в том числе и нестационарного случая и на примере частных модельных точно решаемых задач продемонстрирована эффективность подхода.

Обычное уравнение геодезической

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\cdot\frac{dx^\beta}{ds} = 0 \tag{1}$$

для случая конформно-плоской метрики вида

$$g_{\mu\nu} = U^2\eta_{\mu\nu}, \quad \eta_{\mu\nu} = \mathrm{diag}\left\{1,-1,-1,-1\right\}, \tag{2}$$

где $U = U(x)$ – скалярная функция четырёх квазидекартовых координат $\{x_0, x_k;\ k = 1,2,3\}$, а линейный элемент ds определён следующим образом:

$$ds = \left\{g_{\mu\nu}dx^\mu dx^\nu\right\}^{\frac{1}{2}} = U\cdot\left(dx^\mu dx_\mu\right)^{\frac{1}{2}}, \tag{3}$$

принимает вид

$$\frac{dx^\mu}{ds^2} + U^{-1}\left\{2\frac{du}{ds}\frac{dx^\mu}{ds} - \frac{du}{dx^\mu}\frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds}\right\}. \tag{4}$$

В целях последующего сопоставления с формализмом одночастичной лагранжевой динамики материальной точки выразим элемент ds, определяемый формулой (3) через время наблюдателя t стандартным образом

$$ds = cUd\tau = cU\gamma^{-1}dt, \tag{5}$$

где

$$\gamma = \left\{1 - \frac{1}{c^2}\left(\frac{dx_k}{dt}\right)^2\right\}^{-\frac{1}{2}} = \left(1 - \dot{r}^2/c^2\right)^{-\frac{1}{2}}.$$

После этого с учётом соотношений

$$\frac{d}{ds} = (cU)^{-1}\frac{d}{d\tau} = (cU)^{-1}\gamma\frac{d}{dt}$$

и условия

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = c^2$$

уравнение геодезической (4) запишется в виде

$$\frac{d}{d\tau}\left(U\frac{dx^\mu}{d\tau}\right) - c^2\frac{\partial U}{\partial x_\mu} = 0, \tag{6}$$

или в нековариантных обозначениях

$$\gamma\frac{d}{dt}\left(U\gamma\frac{dx_k}{dt}\right) + c^2\frac{\partial U}{\partial x_k} = 0, \tag{7}$$

$$\gamma\frac{d}{dt}\left(U\gamma\right) - \frac{\partial U}{\partial t} = 0. \tag{8}$$

Легко видеть, что эти уравнения могут быть получены в качестве уравнений одночастичной лагранжевой динамики на основе функционала действия, заданного стандартным путём с использованием в качестве линейного элемента выражения (5) для ds, т.е.

$$S = -m_0 c\int ds = -m_0 c^2\int U\left(x_k,t\right)\gamma^{-1}dt, \tag{9}$$

что соответствует функции Лагранжа следующего вида:

$$L\left(x_k,\dot{x}_k,t\right) = -m_0c^2\gamma^{-1}U\left(x_k,t\right). \tag{10}$$

Наличие явной временной зависимости свидетельствует о нестационарности (незамкнутости) системы, описываемой таким лагранжианом. Соответствующие определению (10) уравнения Эйлера-Лагранжа имеют вид

$$m_0\frac{d}{dt}\left(\gamma U\frac{dx_k}{dt}\right) + m_0c^2\gamma^{-1}\frac{\partial U}{\partial x_k} = 0. \tag{11}$$

После умножения на $m_0^{-1}\gamma$ это уравнение совпадает с (7).

На основе стандартных определений получаем явное выражение для функции Гамильтона

$$\mathrm{H}\left(x_k,\dot{x}_k,t\right) = m_0c^2\gamma U\left(x_k,t\right) \tag{12}$$

и (с учётом (11)) находим общую связь:

$$\frac{d\mathrm{H}\left(x_k,\dot{x}_k,t\right)}{dt} = -\frac{\partial L\left(x_k,\dot{x}_k,t\right)}{\partial t}. \tag{13}$$

Поскольку явную зависимость от времени содержит только функция U, то, используя (10) и (12), из (13) получаем соотношение

$$m_0c^2\frac{d}{dt}\left(\gamma U\right) = m_0c^2\gamma^{-1}\frac{\partial U}{\partial t}. \tag{14}$$

Видно, что после сокращения на множитель m_0c^2 и умножения на γ соотношение (14) идентично уравнению (8).

Динамический смысл этого уравнения очевиден: скорость изменения энергии (в данном случае: $\mathrm{E} = m_0c^2\gamma U$) определяется видом явной зависимости функции $U\left(x_k,t\right)$ от времени.

В стационарном случае, когда $U = U\left(x_k\right)$, так что $\frac{\partial U}{\partial t} = 0$, величина $\mathrm{H}\left(x_k,\dot{x}_k\right) = m_0c^2\gamma U\left(x_k\right)$ имеет характер интеграла движения. Комбинация $\gamma\left(\dot{x}_k\right)U\left(x_k\right)$ в уравнении (7) сохраняет постоянное значение при каждом фиксированном выборе начальных условий (см. [ ]).

В интересующем нас случае сферической симметрии можно использовать (квази)сферические пространственные координаты r, θ, φ, так что интервал Минковского примет следующий вид:

$$dx^{\mu}dx_{\mu} = c^2dt^2 - dr^2 - r^2d\Omega, \quad d\Omega = d\theta^2 + \sin^2\theta d\varphi^2, \tag{15}$$

и выделить радиально-временную часть метрики (2) путём фиксации произвольного направления в пространстве условиями $d\theta = d\varphi = 0$, как это принято при рассмотрении сферически-симметричных задач в рамках ОТО. При этом широко используется то обстоятельство, что подходящим выбором координат такая (1+1)-метрика всегда приводится к конформно-плоской форме подходящим координатным преобразованием.

В этом случае линейный элемент ds трансформируется следующим образом

$$ds \to ds_{t,r} = c\bar{U}(r,t)\,\bar{\gamma}^{-1}dt, \tag{16}$$

где

$$\bar{\gamma} = \left\{1 + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right\}^{-\frac{1}{2}} = \left(1 + \frac{\dot{r}^2}{c^2}\right)^{-\frac{1}{2}}.$$

Соответственно, функция Лагранжа будет иметь вид:

$$\bar{L}(r,\dot{r},t) = -m_0c^2\bar{U}(r,t)\,\bar{\gamma}^{-1}, \tag{17}$$

а уравнения Эйлера-Лагранжа запишутся следующим образом

$$\bar{\gamma}\frac{d}{dt}\left(\bar{U}\bar{\gamma}\dot{r}\right) - c^2\frac{\partial\bar{U}}{\partial r} = 0, \tag{18}$$

$$\bar{\gamma}\frac{d}{dt}\left(\bar{U}\bar{\gamma}\right) - \frac{\partial\bar{U}}{\partial t} = 0. \tag{19}$$

В стационарном случае $\left(U = \bar{U}(\bar{r})\right)$ одночастичная функция Лагранжа определится выражением

$$\bar{L}(\bar{r},\dot{\bar{r}}) = -m_0c^2\bar{U}(\bar{r})\,\bar{\gamma}^{-1}. \tag{20}$$

Здесь $\bar{r}(r)$, в общем случае, новая радиальная координата.

Уравнение Эйлера-Лагранжа (18) сохранит форму (с заменой $U(r,t) \to \bar{U}(\bar{r})$), величина $\bar{U}(\bar{r})\,\bar{\gamma}(\dot{\bar{r}})$ (энергия $\mathrm{E} = m_0c^2\bar{\gamma}\bar{U}$) будет интегралом движения.

Проиллюстрируем продуктивность данного подхода на примере конкретных физически содержательных моделей.

Рассмотрим с этих позиций внешнее решение Шварцшильда, записанное в координатах $x_0 = ct, r, \theta, \varphi$:

$$dS^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1}dr^2 - r^2d\Omega, \tag{21}$$

где $r_g = \frac{2MG}{c^2}$ – гравитационный радиус, $r_g < r < \infty$.

Для наших целей особый интерес представляют области изменения переменных, допускающие переход к плоскому пределу и, соответственно к Ньютонову приближению.

Радиально-временная (1+1)-часть метрики (21) приводится введением черепашьей (tortoise) координаты r^* к конформно-плоской форме

$$d\bar{S}^2_{r,t} = \left(1 - \frac{r_g}{r}\right)\left\{c^2dt^2 - dr^{*^2}\right\}, \tag{22}$$

где

$$r^* = r^*(r) = r + r_g\ln\left(\frac{r}{r_g} - 1\right). \tag{23}$$

Условие, определяющее распространение сигнала в произвольном но фиксированном радиальном направлении $(\theta_{fix}, \varphi_{fix})$, определяется уравнением:

$$c^2 dt^2 - {dr^*}^2 = 0,$$

или после интегрирования

$$c^2 \left(t_{fin} - t_{in}\right)^2 - \left(r^*_{fin} - r^*_{in}\right)^2 = 0. \tag{24}$$

Поскольку, как это видно из определения (23), при выполнении условия $\frac{r_g}{r} \ll 1$ различие между значениями r^* и r становится пренебрежимо малым по мере увеличения r, то соответственно, уравнение (24) в этой асимптотической области будет практически воспроизводить картину распространения фронта сферической электромагнитной волны в пространстве Минковского.

Рассмотрим теперь описание радиального движения пробных тел в этой области. Исходный линейный элемент определится из соотношения (22), так что соответствующая функция Лагранжа будет иметь следующий вид

$$L\left[r, \dot{r}^*\left(r\right)\right] = -m_0 c^2 \left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}} \left\{1 - \left(\frac{\dot{r}^*}{c}\right)^2\right\}^{\frac{1}{2}}. \tag{25}$$

На основании (23) находим:

$$\left(\frac{\dot{r}^*}{c}\right)^2 = \frac{\dot{r}^2}{c^2}\left[1 + \frac{r_g}{r}\left(1 - \frac{r_g}{r}\right)^{-1}\right]^2. \tag{26}$$

Для перехода к Ньютонову приближению в этих соотношениях необходимо ограничиться членами первого порядка по $\frac{r_g}{r} \ll 1$ и $\left(\frac{\dot{r}}{c}\right)^2 \ll 1$. В итоге, опуская, как обычно, постоянный член $-m_0 c^2$, получим

$$L_N\left(r, \dot{r}\right) = \frac{m_0 \dot{r}^2}{2} - U_N\left(r\right), \tag{27}$$

где U_N – потенциальная энергия

$$U_N\left(r\right) = -m_0 \varphi_N\left(r\right) = -\frac{m_0 MG}{r}. \tag{28}$$

Видно, что в пределе ${}^{r_g}\!/_{r} \ll 1$ из (25) возникает выражение, соответствующее нерелятивистской функции Лагранжа для частицы с массой m_0 в ньютоновом гравитационном поле, создаваемом фиксированным гравитирующим центром (точечной массой M).

В рассматриваемом нами подходе значительный интерес представляет описание тонкого сферического слоя, примыкающего непосредственно к горизонту, которое широко используется в физике (массивных) чёрных дыр. Это – прежде всего, переход к представлению Риндлера, где условие $\frac{r - r_g}{r_g} \ll 1$ достигается путём введения радиальной координаты

$$\rho = 2\left[r_g\left(r - r_g\right)\right]^{\frac{1}{2}}, \tag{29}$$

которая возникает из (23) в первом порядке малости по $\left(\frac{\delta r}{r_g}\right)^{\frac{1}{2}}$.

В квазидекартовых координатах

$$cT = \rho sh\omega,\ Z = \rho ch\omega,\ X = \rho\theta\cos\varphi,\ Y = \rho\theta\sin\varphi,$$

где $\omega = \frac{ct}{2r_g}$, $\theta \ll 1$ – малое фиксированное значение полярного угла, что соответствует выделению малой области горизонта, метрике Шварцшильда придается форма метрики Минковского

$$dS^2 = c^2 dT^2 - dZ^2 - dX^2 - dY^2, \tag{30}$$

Как известно, координаты Риндлера покрывают только часть пространства Минковского (т.н. клин Риндлера).

В частности, из вида радиально-временной части метрики (30), записанной в координатах ρ, ω:

$$dS^2_{\rho,\omega} = \rho^2 d\omega^2 - d\rho^2$$

следует, что значение $\rho = 0$ ($cT = \pm Z$, причём Z>0) ограничивают соответствующую односвязную пространственно-временную область практически так же, как линии сингулярности в преобразованиях SCT.

Обратимся теперь к преобразованиям, обратным (4.1), рассматривая их в качестве общекоординатных. Тогда получим

$$\begin{aligned} x = x\left(x', x'_0\right) &= \frac{x'\left(1-\frac{x'}{2R_0}\right)+\frac{{x'_0}^2}{2R_0}}{\left(1-\frac{x'}{2R_0}\right)^2-\left(\frac{x'_0}{2R_0}\right)^2}, \\ x_0 = x_0\left(x', x'_0\right) &= \frac{x'_0}{\left(1-\frac{x'}{2R_0}\right)^2-\left(\frac{x'_0}{2R_0}\right)^2}. \end{aligned} \tag{31}$$

Область определения переменных x', x'_0 – открытый левый сектор псевдоевклидовой плоскости $\{x', x'_0\}$, ограниченный линиями сингулярности $x'_\pm = 2R_0 \mp x'_0$ (образующими двумерного конуса с вершиной в точке $x' = 2R_0$, $x'_0 = 0$), пересекающими ось x'_0 в точках ${x'_0}^{(\lim)} = \pm 2R_0$.

После несложных вычислений получаем

$$dS^2_L = dx_0^2 - dx^2 = \left\{\left(1-\frac{x'}{2R_0}\right)^2 - \left(\frac{x'_0}{2R_0}\right)^2\right\}^{-1}\left({dx'_0}^2 - {dx'}^2\right).$$

Видно, что лоренцов линейный элемент $dS_L = c\gamma^{-1}dt$ принимает в штрихованных координатах конформно-плоскую форму, так что соответствующий одночастичный модельный лагранжиан запишется в виде

$$L\left(\dot{x}', x', x'_0\right) = -m_0c^2\left[1-\frac{1}{c^2}\left(\frac{dx'}{dt'}\right)^2\right]^{\frac{1}{2}}\left\{\left(1-\frac{x'}{2R_0}\right)^2 - \left(\frac{x'_0}{2R_0}\right)^2\right\}^{-1}. \tag{32}$$

Видно, что в приближении $\frac{\Delta x'}{R_0} \ll 1$, $\left(\frac{x'_0}{R_0}\right)^2 \ll 1$ $\left(\frac{\dot{x}'}{c}\right)^2 \ll 1$ выражение (32) сводится к следующему:

$$L\left(x', \dot{x}', x'_0\right) \to L_N\left(x', \dot{x}'\right) = \frac{m_0{\dot{x}'}^2}{2} - \frac{m_0c^2}{R_0}\cdot x', \tag{33}$$

что соответствует картине движения частицы произвольной массы относительно неинерциальной (равноускоренной) системы отсчёта, которая сама движется вдоль положительного направления оси x IRF с постоянным ускорением $W = \frac{c^2}{R_0}$. Ускорение частицы, измеренное в этой системе отсчёта, отрицательно и совпадает с фоновым ускорением $W_B = -\frac{c^2}{R_0^2}$, полученным в п. 2 чисто кинематическим путём. На основании принципа эквивалентности можно рассматривать такую систему как инерциальную, в которой действует постоянное гравитационное поле с потенциалом

$$\varphi_N\left(x'\right) = \frac{c^2}{R_0}x'. \tag{34}$$

Эта динамическая картина полностью соответствует кинематике Галилея-Ньютона.

Для получения картины, обобщающей этот результат на случай радиальной координаты (Шварцшильда) воспользуемся следующим приёмом. Будем исходить из метрики Минковского (15), выделив её радиально-временную часть $dS^2_{r,t}$ стандартными условиями $d\theta = d\varphi = 0$:

$$dS^2_{r,t} = c^2dt^2 - dr^2, \tag{35}$$

с последующим координатным преобразованием следующего вида:

$$\begin{array}{l} r = r\left(r', t'\right) = \frac{r'\left(1-\frac{r'}{2r_0}\right)+\frac{c^2t'^2}{2r_0}}{\left(1-\frac{r'}{2r_0}\right)^2-\left(\frac{ct'}{2r_0}\right)^2}, \\ t = t'\left(r', t'\right) = \frac{t'}{\left(1-\frac{r'}{2r_0}\right)^2-\left(\frac{ct'}{2r_0}\right)^2}, \end{array} \tag{36}$$

где r_0 – параметр размерности длины.

Область определения переменных r', t' идентична области определения координат x', t' в соответствующем секторе псевдоевклидовой плоскости $\{x'_0, x'\}$ (формула (31)). Эта область фактически соответствует клину Риндлера, вершина которого находится в точке $\{r' = 2r_0, t' = 0\}$. Из (36) без труда находим

$$d\bar{S}^2_L = c^2dt^2 - dr^2 = \left\{\left(1 - \frac{r'}{2r_0}\right)^2 - \left(\frac{ct'}{2r_0}\right)^2\right\}^{-2}\left(c^2dt'^2 - dr'^2\right). \tag{37}$$

Соответствующий одночастичный лагранжиан имеет следующий вид:

$$L\left(r', \dot{r}', t'\right) = -m_0c^2\left[1 - \frac{1}{c^2}\left(\frac{dr'}{dt'}\right)^2\right]^{\frac{1}{2}}\left\{\left(1 - \frac{r'}{2r_0}\right)^2 - \left(\frac{ct'}{2r_0}\right)^2\right\}^{-1}. \tag{38}$$

В приближении $\frac{\Delta r'}{r_0} \ll 1, \frac{c\Delta t'}{r_0} \ll 1$ (small-neigbourhood approximation); $\left(\frac{\dot{r}'}{c}\right)^2 \ll 1$ отсюда находим

$$L\left(r', \dot{r}', t'\right) \to L_N\left(r', \dot{r}'\right) = \frac{m_0\dot{r}'^2}{2} - \frac{m_0c^2}{r_0}\dot{r}', \tag{39}$$

что соответствует функции Лагранжа для нерелятивистского движения точечной массы m_0 в поле Ньютонова гравитационного потенциала

$$\varphi_N\left(r'\right) = \frac{c^2}{r_0}r'. \tag{40}$$

Мы будем полагать параметр r_0 совпадающим с R_0 в (31).

Последнее выражение соответствует ситуации вблизи сферы Шварсшильда большого радиуса и отвечает картине радиального постоянного гравитационного поля в тонком сферическом слое вблизи такой сферы. Если учесть, что $R_0 \sim cH_0^{-1}$ (H_0 – постоянная Хаббла) численно соответствует размеру наблюдаемой Вселенной, то эффективный лагранжиан вида (31) моделирует наличие постоянного ускорения $\sim cH_0$ в системе отсчёта удалённого наблюдателя.

Литература

1. Gorbatsevich A.K., Tomilchik L.M. Proc. of the IX Workshop: Problems of High Energy Physics and Field Theory (Russia, Protvino, July 7-13, 1986). Moskow “Nauka” (1987) 378–383 (in Russian).

The geometry of jet Hamiltonians in Langmuir-Blodgett films

V. Balan, M. Neagu, A.Oană, N.G. Krylova, H.V. Grushevskaya

Abstract

In this paper we develop the geometry (in the sense of nonlinear connection, Cartan linear connection, d-torsions and d-curvatures) of some Hamiltonians, which describe the motion of a particle in compressing Langmuir monolayer under the action of electrocapillary forces. We demonstrate that the strong electrocapillary effects appear as the nonconservation of the mass of a particle in monolayer and as a distortion of the Hamiltonian space.

Mathematics Subject Classification (2010): 53C60, 53C80, 81T13.

Key words and phrases: 1-jet space, 2D-monolayer Hamiltonians, nonlinear connections, Cartan canonical connections, d-torsions and d-curvatures.

1 Introduction

The Langmuir-Blodgett (LB) film is a high-ordered nanostructured material that is formed as the result of a sequential transfer of compressed LB-monolayers of amphiphilic molecules from liquid subphase to a solid support. The formation of LB-monolayer occurs through the self-organizing of molecules on a surface of liquid subphase during Langmuir monolayer compression by a barrier (see [1, 9]). At that, a two-dimensional (2D) first-order phase transition from a 2D liquid state up to a condense state of monolayer occur. Up today, a theoretical description of the phase transition has been performed within the frame of mean field theory [6]. However, such phenomenological description does not allow to obtain some experimentally observable characteristics of the structurization process. For example, the coexisting domains are the structural elements responsible for a nontrivial rheology [2]. Brewster angle microscopy has allowed studying the kinetic mechanism for domain growth. It has been revealed the existence of various complex domains shapes which change from circular domains to labyrinthine finger-shaped ones during the phase transition. The shape of liquid-condensed domains depends on the structure of amphiphilic molecule, including chirality, parameters of subphase, temperature and rate of compression [7, 5]. At present, the phase coexistence in 2D systems is described by introducing model electric dipole-dipole interactions [7]. However, this model is not able to reproduce the complex domain shapes seen in actual phase domains, and does not explain the dependencies on the monolayer formation parameters. For the correct description of such peculiarities of the process it is necessary to use kinetic and dynamical models.

In the paper [4] it has been proposed to examine the LB-monolayer structurization process by taking into account the electrocapillarity phenomenon on the air/water interface, and the 3D Lagrange-Finsler metric of space-time, in which a moving particle of the monolayer has been constructed. In the paper [10], it has been assumed that an expansion near singular points for such geometrical Lagrangian describes phase transitions of the LB-monolayer. Trajectories of particles in a field of electrocapillarity forces of the monolayer have been calculated in a resonant approximation using a Jacobi equation. The obtained Lagrangian will be used further to develop the

geometry of Hamiltonian space associated with the dynamics of the 2D Langmuir monolayer under the action of electrocapillarity forces.

We shall further consider the usual physical time defined by the 1-dimensional Euclidian manifold ($T = [0, \infty), h_{11}(t) = 1$), whose Christoffel symbol is $\kappa^1_{11}(t) = 0$. We consider as well the plane manifold $\mathbb{R}^2$ endowed with the polar coordinates (r, φ), where $r > 0$ and $\varphi \in [0, 2\pi)$. These ingredients build the corresponding 1-jet space and its dual, whose local coordinates are:

$$\begin{aligned} J^1(T, \mathbb{R}^2) &\rightsquigarrow (t, x^1, x^2, y^1_1, y^2_1) = (t, r, \varphi, \dot{r}, \dot{\varphi}), \\ J^{1*}(T, \mathbb{R}^2) &\rightsquigarrow (t, x^1, x^2, p^1_1, p^1_2) = (t, r, \varphi, R, \Phi). \end{aligned}$$

In this context, we use the special function

$$f(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$$

and the following entities with physical meaning:

- m is the mass of the particle;
- V is the monolayer compression speed;
- p is a constant monolayer parameter given by the physics formula

$$p = \frac{\pi^2 q^2}{\varepsilon \varepsilon_0} \frac{\rho_0^2}{R_0^2};$$

- A and U are the following monomolecular layer functions:

$$\begin{aligned} A(t, r) = &\quad p r^5 |V| e^{\frac{2|V|t}{r}}, \\ U(t, r) = &\quad p \left\{ \left[-\frac{4}{3} r^5 + \frac{16}{15} (|V|t) r^4 + \frac{1}{30} (|V|t)^2 r^3 + \frac{1}{45} (|V|t)^3 r^2 \right.\right. \\ &\left.\left. + \frac{1}{45} (|V|t)^4 r + \frac{2}{45} (|V|t)^5 \right] e^{\frac{2|V|t}{r}} - \frac{4}{45} \frac{(|V|t)^6}{r} f\left(\frac{2|V|t}{r} \right) \right\}. \end{aligned}$$

The 2D-motion of a particle of the monolayer is governed by the following jet Lagrangian function:

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m\dot{r}^2}{2} + \frac{mr^2}{2} \dot{\varphi}^2 - A\dot{r}^{-1} + U. \tag{1.1}$$

In what follows, we try to construct the approximate Hamiltonian which characterizes the Lagrangian (1.1). In this sense, we note that the value of momentum components induced by the Lagrangian (1.1) are given by the formulas: $p^1_i = (\partial L) / \left(\partial y^i_1\right)$, that is,

$$\begin{aligned} p^1_1 &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} + A\dot{r}^{-2} =: R, \\ p^1_2 &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \dot{\varphi} =: \Phi. \end{aligned} \tag{1.2}$$

Remark 1 *Throughout this paper, the Latin indices $i, j, k, l, \ldots$ run from 1 to 2. The Einstein convention of summation is also assumed in this work.*

Considering the first equation of the equalities (1.2) and supposing that the discriminant $\Delta = 4R^3 - 27m^2A$ is negative, that is,

$$R < 3\sqrt[3]{\frac{m^2A}{4}},$$

we find the following Legendre transformations:

$$\dot{r} = \frac{1}{3}\frac{R}{m} + \sqrt[3]{\sqrt{\frac{1}{4}\frac{A^2}{m^2} - \frac{A}{27}\frac{R^3}{m^4}} - \frac{1}{2}\frac{A}{m} + \frac{1}{27}\frac{R^3}{m^3}} + \frac{1}{9}\frac{R^2}{m^2\sqrt[3]{\sqrt{\frac{1}{4}\frac{A^2}{m^2} - \frac{A}{27}\frac{R^3}{m^4}} - \frac{1}{2}\frac{A}{m} + \frac{1}{27}\frac{R^3}{m^3}}},$$
$$\dot{\varphi} = \frac{\Phi}{mr^2}.$$

Because the Legendre transformations are very complicated, two limiting cases will be further considered.

The first limiting case occurs for extremely small perturbations, when the compression rate $|V|$ is very small and $A \to 0$. Then the corresponding approximate Legendre transformations take the form:

$$\dot{r} \approx \frac{R}{m}, \qquad \dot{\varphi} = \frac{\Phi}{mr^2}. \tag{1.3}$$

The second limiting case is the case of strong electrocapillary effects during the Langmuir monolayer formation when $|V|$ is high and $A \to \infty$. We approximate, by using power series of second order in $\frac{R^3}{Am^2}$, which infers the following approximate Legendre transformations for the second case:

$$\dot{r} \approx \frac{1}{3}\frac{R}{m} - \left(\frac{A}{m}\right)^{1/3}, \qquad \dot{\varphi} = \frac{\Phi}{mr^2}. \tag{1.4}$$

Then the standard expression of the Hamiltonian has the form:

$$H = y_1^i p_i^1 - L = \dot{r}R + \dot{\varphi}\Phi - L = \frac{m\dot{r}^2}{2} + \frac{mr^2}{2}\dot{\varphi}^2 + 2A\dot{r}^{-1} - U, \tag{1.5}$$

which leads to the following approximate Hamiltonians:

$$H_I \approx \frac{1}{2m}R^2 + \frac{1}{2mr^2}\Phi^2 + 2AmR^{-1} - U, \qquad A \to 0; \tag{1.6}$$

$$H_{II} \approx \frac{1}{2m}\left(\frac{R}{3} - (Am^2)^{1/3}\right)^2 + \frac{2Am}{\frac{R}{3} - (Am^2)^{1/3}} + \frac{1}{2mr^2}\Phi^2 - U, \qquad A \to \infty. \tag{1.7}$$

The differential geometry (in the sense of nonlinear connection, Cartan linear connection, d-torsions and d-curvatures) produced by an arbitrary Hamiltonian on the dual 1-jet space was completely developed by Atanasiu, Neagu and Oană in the monograph [3] [1]. In this framework, the present work is devoted to the development of the Hamiltonian geometries [3] produced by the 2D-monolayer Hamiltonians (1.6) and (1.7) on the dual 1-jet vector bundle $J^{1*}(T, \mathbb{R}^2)$.

2 Jet Hamiltonian geometry of the Langmuir monolayer space

2.1 The case of Hamiltonian H_I

In the case when the compression rate $|V|$ is very small and $A \to 0$, the Legendre transformations (1.3) exhibit an ordinary form in the momentum components. By the approximation $A \to 0$, the Hamiltonian H_I (1.6) becomes the classic Hamiltonian of a particle movement in the potential field U:

$$H_{I0} \approx \frac{1}{2m}R^2 + \frac{1}{2mr^2}\Phi^2 - U.$$

[1] The geometrical ideas from [3] are similar (but however distinct ones) to those exposed by Miron et al. in [8].

We will further consider the metric induced by the Hamiltonian H_I (1.6) at extremely small, but non-zero values of A. The local *fundamental metrical d-tensor* produced by the Hamiltonian (1.6) is given by the formula:

$$g^{ij} = \frac{1}{2}\frac{\partial^2 H_I}{\partial p_i^1 \partial p_j^1}.$$

By direct computation, we obtain the components g^{ij} of the fundamental metrical d-tensor in matrix form:

$$\left(g^{ij}\right)_{i,j=\overline{1,2}} = \begin{pmatrix} \dfrac{1+4Am^2R^{-3}}{2m} & 0 \\ 0 & \dfrac{1}{2mr^2} \end{pmatrix}.$$

By assuming that $1+4Am^2R^{-3} \neq 0$, we infer the inverse of the fundamental metrical d-tensor,

$$\left(g_{ij}\right)_{i,j=\overline{1,2}} = \begin{pmatrix} \dfrac{2m}{1+4Am^2R^{-3}} & 0 \\ 0 & 2mr^2 \end{pmatrix}.$$

Proposition 2 *The* ***canonical nonlinear connection*** *of the 2D-monolayer Hamiltonian (1.6) is given by the local coefficients*

$$N = \left(N_{1\ (i)1}^{(1)} = 0,\ N_{2\ (i)j}^{(1)}\right),$$

where

$$N_{2\ (1)1}^{(1)} = \frac{2m^2R}{R^3+4Am^2}\left[-2\frac{\partial A}{\partial r} + \frac{3mRA}{R^3+4Am^2}\left(\frac{1}{mr^3}\Phi^2 + \frac{\partial U}{\partial r}\right)\right],$$

$$N_{2\ (1)2}^{(1)} = N_{2\ (2)1}^{(1)} = -\tfrac{1}{r}\Phi, \qquad N_{2\ (2)2}^{(1)} = r\left(R - 2Am^2R^{-2}\right).$$

Proof. The general formula of the nonlinear connection is (see [3]):

$$N_{2\ (i)j}^{(1)} = \frac{1}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p_k^1} - \frac{\partial g_{ij}}{\partial p_k^1}\frac{\partial H}{\partial x^k} + g_{ik}\frac{\partial^2 H}{\partial x^j \partial p_k^1} + g_{jk}\frac{\partial^2 H}{\partial x^i \partial p_k^1}\right],$$

where $(t, x^1, x^2, p_1^1, p_2^1) = (t, r, \varphi, R, \Phi)$. □

The above canonical nonlinear connection of the 2D-monolayer Hamiltonian (1.6) produces the following local adapted basis of derivation operators:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{2\ (i)j}^{(1)}\frac{\partial}{\partial p_j^1},$$

where $(t, x^1, x^2, p_1^1, p_2^1) = (t, r, \varphi, R, \Phi)$. In this context, we have

Proposition 3 *The* ***Cartan canonical linear connection*** *of the 2D-monolayer Hamiltonian (1.6) is given by the local adapted coefficients*

$$C\Gamma(N) = \left(\kappa_{11}^1 = 0, A_{j1}^i, H_{jk}^i, C_{i(1)}^{j(k)}\right),$$

where

$$A_{j1}^i = \delta^{i1}\delta_{j1}\tfrac{-2m^2}{R^3+4Am^2}\frac{\partial A}{\partial t}, \quad C_{i(1)}^{j(k)} = \delta_{i1}\delta^{j1}\delta^{k1}\tfrac{6Am^2R^{-4}}{1+4Am^2R^{-3}},$$

$$H_{11}^1 = -\tfrac{2m^2}{(R^3+4Am^2)^2}\left[\left(R^3 - 8Am^2\right)\frac{\partial A}{\partial r} + \frac{18m^3RA^2}{R^3+4Am^2}\left(\frac{1}{mr^3}\Phi^2 + \frac{\partial U}{\partial r}\right)\right],$$

$$H_{12}^1 = H_{21}^1 = \tfrac{6Am^2R^{-4}\Phi}{r(1+4Am^2R^{-3})}, \quad H_{22}^1 = -r\left(1+4Am^2R^{-3}\right),$$

$$H_{11}^2 = -\tfrac{6Am^2R^{-4}\Phi}{r^3(1+4Am^2R^{-3})^2}, \qquad H_{12}^2 = H_{21}^2 = \tfrac{1}{r}, \quad H_{22}^2 = 0.$$

Proof. We use the general formulas of the coefficients of the Cartan canonical linear connection, which are given by (see [3])

$$A^i_{j1}=\frac{g^{il}}{2}\frac{\partial g_{lj}}{\partial t},\qquad C^{j(k)}_{i(1)}=-\frac{g_{ir}}{2}\left(\frac{\partial g^{jr}}{\partial p^1_k}+\frac{\partial g^{kr}}{\partial p^1_j}-\frac{\partial g^{jk}}{\partial p^1_r}\right)=-\frac{g_{ir}}{2}\frac{\partial g^{jr}}{\partial p^1_k},$$

$$\begin{aligned}H^i_{jk} &= \frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k}+\frac{\delta g_{kr}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^r}\right)=\\ &= \Gamma^i_{jk}-N^{(1)}_{2\,(1)k}C^{i(1)}_{j(1)}-N^{(1)}_{2\,(1)j}C^{i(1)}_{k(1)}+\frac{g^{ir}}{2}N^{(1)}_{2\,(s)r}\frac{\partial g_{jk}}{\partial p^1_s},\end{aligned}$$

where

$$\Gamma^i_{jk}=\frac{g^{ir}}{2}\left(\frac{\partial g_{jr}}{\partial x^k}+\frac{\partial g_{kr}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^r}\right),$$

for $(t,x^1,x^2,p^1_1,p^1_2)=(t,r,\varphi,R,\Phi)$. □

Proposition 4 *The Cartan canonical connection $C\Gamma(N)$ of the 2D-monolayer Hamiltonian (1.6) has the following* ***adapted local d-torsions****:*

$$T^r_{1j}=-\overline{P}^{(1)\ (j)}_{(r)1(1)}=-A^j_{r1},\quad P^{r(j)}_{i(1)}=C^{r(j)}_{i(1)},$$

$$P^{(1)\ (j)}_{(r)i(1)}=\frac{\partial N^{(1)}_{2\,(r)i}}{\partial p^1_j}+H^j_{ri},\qquad \overline{R}^{(1)}_{(r)1j}=-\frac{\partial N^{(1)}_{2\,(r)j}}{\partial t},$$

$$R^{(1)}_{(r)ij}=\frac{\delta N^{(1)}_{2\,(r)i}}{\delta x^j}-\frac{\delta N^{(1)}_{2\,(r)j}}{\delta x^i},$$

that is, we have

$$P^{(1)\ (1)}_{(1)1(1)}=\frac{6m^2R}{(R^3+4Am^2)^2}\left[\frac{\partial A}{\partial r}R^2-\frac{2mRA\left(2R^3-Am^2\right)}{(R^3+4Am^2)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right],$$

$$P^{(1)\ (2)}_{(1)1(1)}=\frac{6Am^2R^{-4}\Phi}{r^3(1+4Am^2R^{-3})^2},\quad P^{(1)\ (1)}_{(2)1(1)}=P^{(1)\ (1)}_{(1)2(1)}=\frac{6Am^2R^{-4}\Phi}{r(1+4Am^2R^{-3})},$$

$$P^{(1)\ (2)}_{(2)1(1)}=P^{(1)\ (2)}_{(1)2(1)}=0,\quad P^{(1)\ (2)}_{(2)2(1)}=0,\quad P^{(1)\ (1)}_{(2)2(1)}=0,\ \textit{and}$$

$$R^{(1)}_{(1)12}=-R^{(1)}_{(1)21}=-\frac{4m^2\Phi\left(R^3-2Am^2\right)}{r(R^3+4Am^2)^2}\Omega,$$

$$R^{(1)}_{(1)11}=R^{(1)}_{(2)11}=R^{(1)}_{(1)22}=R^{(1)}_{(2)22}=0,$$

$$R^{(1)}_{(2)12}=-R^{(1)}_{(2)21}=\frac{2m^2r}{R^2}\left[-\frac{\partial A}{\partial r}+\frac{3mRA}{(R^3+4Am^2)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right],$$

where

$$\Omega=\left[\frac{3A}{r}-2\frac{\partial A}{\partial r}+\frac{6mRA}{(R^3+4Am^2)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right].$$

It is easy to see that, when $A\to 0$, and $\frac{\partial A}{\partial r}\to 0$, all the components of the d-torsion tend to zero too.

Using the general formulas of d-curvatures from [3], and using the general expressions of the coefficients A^i_{j1} and $C^{j(k)}_{i(1)}$ for $C\Gamma(N)$, we find the following geometrical result:

Proposition 5 *The Cartan canonical connection $C\Gamma(N)$ of the 2D-monolayer Hamiltonian (1.6) has the following* ***adapted local d-curvatures:***

$$\begin{aligned}
\overline{R}^l_{i1k} &= \tfrac{\delta A^l_{i1}}{\delta x^k} - \tfrac{\partial H^l_{ik}}{\partial t} + A^1_{i1}H^l_{1k} - H^1_{ik}A^l_{11} + C^{l(1)}_{i(1)}\overline{R}^{(1)}_{(1)1k},\\
R^l_{ijk} &= \tfrac{\delta H^l_{ij}}{\delta x^k} - \tfrac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij}H^l_{rk} - H^r_{ik}H^l_{rj} + C^{l(1)}_{i(1)}R^{(1)}_{(1)jk},\\
\overline{P}^{l\ (k)}_{i1(1)} &= \tfrac{\partial A^l_{i1}}{\partial p^1_k} - C^{l(k)}_{i(1)/1} + C^{l(1)}_{i(1)}\overline{P}^{(1)\ (k)}_{(1)1(1)},\\
P^{l\ (k)}_{ij(1)} &= \tfrac{\partial H^l_{ij}}{\partial p^1_k} - C^{l(k)}_{i(1)|j} + C^{l(1)}_{i(1)}P^{(1)\ (k)}_{(1)j(1)},\\
S^{l(j)(k)}_{i(1)(1)} &= \tfrac{\partial C^{l(j)}_{i(1)}}{\partial p^1_k} - \tfrac{\partial C^{l(k)}_{i(1)}}{\partial p^1_j} + C^{1(j)}_{i(1)}C^{l(k)}_{1(1)} - C^{1(k)}_{i(1)}C^{l(j)}_{1(1)},
\end{aligned}$$

where $(t, x^1, x^2, p^1_1, p^1_2) = (t, r, \varphi, R, \Phi)$, and we have

$$\begin{aligned}
C^{l(k)}_{i(1)/1} &= \tfrac{\partial C^{l(k)}_{i(1)}}{\partial t} + C^{1(k)}_{i(1)}A^l_{11} + C^{l(1)}_{i(1)}A^k_{11} - C^{l(k)}_{1(1)}A^1_{i1},\\
C^{l(k)}_{i(1)|j} &= \tfrac{\delta C^{l(k)}_{i(1)}}{\delta x^j} + C^{1(k)}_{i(1)}H^l_{1j} + C^{l(1)}_{i(1)}H^k_{1j} - C^{l(k)}_{1(1)}H^1_{ij}.
\end{aligned}$$

2.2 The case of Hamiltonian H_{II}

In the case when the compression rate $|V|$ is very large and $A \to \infty$, the approximation of the Hamiltonian H_{II} (1.7) leads to:

$$H_{II0} \approx \frac{1}{18m}\left(R - 9\left(Am^2\right)^{1/3}\right)^2 + \frac{1}{2mr^2}\Phi^2 - U - 7m^{1/3}A^{2/3}.$$

This means that the mass of particle is not preserved and, in addition to potential field U, a particle is under the influence of the potential $9\left(Am^2\right)^{1/3}$ of a gauge field. Such anisotropic effects are specific to Finsler-Hamilton geometry models.

Let us denote

$$\left(\frac{R}{3} - \left(Am^2\right)^{1/3}\right) =: R_1, \qquad R_1 < 0.$$

The local *fundamental metrical d-tensor* produced by H_{II} (1.7) is given by the formula:

$$\left(g^{ij}\right)_{i,j=\overline{1,2}} = \begin{pmatrix} \dfrac{1 + 4Am^2R_1^{-3}}{18m} & 0 \\ 0 & \dfrac{1}{2mr^2} \end{pmatrix}.$$

Assuming that $1 + 4Am^2R_1^{-3} \neq 0$, the inverse of the fundamental metrical d-tensor is given by

$$\left(g_{ij}\right)_{i,j=\overline{1,2}} = \begin{pmatrix} \dfrac{18m}{1 + 4Am^2R_1^{-3}} & 0 \\ 0 & 2mr^2 \end{pmatrix}.$$

The *canonical nonlinear connection* is given by the local coefficients

$$N^{(1)}_{2\ (1)1} = \frac{m^2\Theta}{R_1^3 + 4Am^2}, \quad N^{(1)}_{2\ (1)2} = N^{(1)}_{2\ (2)1} = -\frac{1}{r}\Phi, \quad N^{(1)}_{2\ (2)2} = \frac{r}{3}\left(R_1 - \frac{2Am^2}{R_1^2}\right), \tag{2.8}$$

where we denoted

$$\Theta = -\frac{4Am^2 + 12\left(Am^2\right)^{2/3}R_1 + R_1^3}{\left(Am^2\right)^{2/3}}\frac{\partial A}{\partial r} + \frac{18mAR_1^2}{R_1^3 + 4Am^2}\left(\frac{1}{mr^3}\Phi^2 + \frac{\partial U}{\partial r}\right).$$

The *Cartan canonical linear connection* is given by the local adapted coefficients

$$A^i_{j1}=\delta^{i1}\delta_{j1}\frac{-2m^2\left(R_1+\left(Am^2\right)^{1/3}\right)}{R_1\left(R_1^3+4Am^2\right)}\frac{\partial A}{\partial t},\quad C^{j(k)}_{i(1)}=\delta_{i1}\delta^{j1}\delta^{k1}\frac{2Am^2}{R_1\left(R_1^3+4Am^2\right)},$$
$$H^1_{11}=-\frac{2m^2}{\left(R_1^3+4Am^2\right)^2}\left[\left(R_1^3-8Am^2\right)\frac{\partial A}{\partial r}+\frac{18m^3A^2R_1}{R_1^3+4Am^2}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right],$$
$$H^1_{12}=H^1_{21}=\frac{2Am^2\Phi}{rR_1\left(R_1^3+4Am^2\right)},\quad H^1_{22}=-\frac{r}{9}\left(1+4Am^2R_1^{-3}\right),$$
$$H^2_{11}=-\frac{18Am^2\Phi R_1^2}{r^3\left(R_1^3+4Am^2\right)^2},\quad H^2_{12}=H^2_{21}=\frac{1}{r},\ \text{and}\ H^2_{22}=0.$$

The Cartan canonical connection $C\Gamma(N)$ has the following *adapted local d-torsion* components:

$$P^{(1)\ (1)}_{(1)1(1)}=\frac{6m^2R_1}{(R_1^3+4Am^2)^2}\left[R_1^2\frac{\partial A}{\partial r}-\frac{2mA\left(2R_1^3-Am^2\right)}{\left(R_1^3+4Am^2\right)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right],$$
$$P^{(1)\ (2)}_{(1)1(1)}=\frac{18Am^2\Phi R_1^2}{r^3\left(R_1^3+4Am^2\right)^2},\quad P^{(1)\ (1)}_{(2)1(1)}=P^{(1)\ (1)}_{(1)2(1)}=\frac{2Am^2\Phi}{rR_1\left(R_1^3+4Am^2\right)},$$
$$P^{(1)\ (2)}_{(2)1(1)}=P^{(1)\ (2)}_{(1)2(1)}=0,\quad P^{(1)\ (2)}_{(2)2(1)}=0,\quad P^{(1)\ (1)}_{(2)2(1)}=0,$$

and

$$R^{(1)}_{(1)12}=-R^{(1)}_{(1)21}=-\frac{4m^2\Phi\left(R_1^3-2Am^2\right)}{r(R_1^3+4Am^2)^2}\cdot\Theta,$$
$$R^{(1)}_{(1)11}=R^{(1)}_{(2)11}=R^{(1)}_{(1)22}=R^{(1)}_{(2)22}=0,$$
$$R^{(1)}_{(2)12}=-R^{(1)}_{(2)21}=\frac{2m^2r}{3R_1^2}\left[-\frac{\partial A}{\partial r}+\frac{3mAR_1}{\left(R_1^3+4Am^2\right)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right)\right],$$

where

$$\Theta=\frac{3A}{r}-2\frac{\partial A}{\partial r}+\frac{6mAR_1}{\left(R_1^3+4Am^2\right)}\left(\frac{1}{mr^3}\Phi^2+\frac{\partial U}{\partial r}\right).$$

At the limit, for $A\to\infty$, we get the approximation

$$P^{(1)\ (1)}_{(1)1(1)}=-\frac{1}{A}\frac{\partial A}{\partial r},\quad P^{(1)\ (1)}_{(2)1(1)}=P^{(1)\ (1)}_{(1)2(1)}=P^{(1)\ (1)}_{(2)2(1)}=0,$$
$$P^{(1)\ (2)}_{(1)1(1)}=P^{(1)\ (2)}_{(2)1(1)}=P^{(1)\ (2)}_{(1)2(1)}=P^{(1)\ (2)}_{(2)2(1)}=0,$$

$$R^{(1)}_{(1)12}=-R^{(1)}_{(1)21}=\frac{4}{r}-\frac{8}{3}\frac{1}{A}\frac{\partial A}{\partial r},\quad R^{(1)}_{(2)12}=-R^{(1)}_{(2)21}=-\frac{2}{9}r\frac{\left(Am^2\right)^{1/3}}{A}\frac{\partial A}{\partial r},$$
$$R^{(1)}_{(1)11}=R^{(1)}_{(2)11}=R^{(1)}_{(1)22}=R^{(1)}_{(2)22}=0.$$

3 Discussion and conclusions

We conclude that an increase in the Langmuir monolayer compression rate $|V|$ leads to the strong electrocapillary effects that appear as a gauge field, and, respectively, as a Finsler-Lagrange-space distortion. Though the particle stays in a coordinate plate of the monolayer, it undergoes a torsion in the dual 1-jet space, transiting from one section to another and, respectively, leaves the 1-jet slice of Langmuir monolayer, ceasing to exist. An apparent deficiency of mass m_{LB} owing the non-conservation of the particle mass is equal to $m_{LB}=(R_I-R_{II})/\dot{r}=-2m$. At this moment

the Langmuir monolayer of hydrated particles becomes LB-monolayer of particles which throw off their hydrated shells.

Acknowledgments. The present work was developed under the auspices of the Grant No. 11962012-BRFFR-RAF14RA-006, within the cooperation framework between the Romanian Academy and the Belarussian Republican Foundation for Fundamental Research.

References

[1] S. Acharya, J.P. Hill, K. Ariga, *Soft Langmuir–Blodgett technique for hard nanomaterials*, Adv. Mater., **21**(2009), 2959-2981.

[2] L.R. Arriaga, I. López-Montero, J. Ignes-Mullol, F. Monroy, *Domain-growth kinetic origin of nonhorizontal phase coexistence plateau in Langmuir monolayers*: Compression Rigidity of a Raft-Like Lipid Distribution J. Phys. Chem. B, 2010, **114** (**13**), 4509–4520.

[3] Gh. Atanasiu, M. Neagu, A. Oană, *The Geometry of Jet Multi-Time Lagrange and Hamiltonian Spaces: Applications in Theoretical Physics*, Fair Partners Publishers, Bucharest, 2013.

[4] H.V. Grushevskaya, N.G. Krylova, *Effects of Finsler geometry in physics surface phenomena: case of monolayer systems*, Hypercomplex Numbers in Geom. and Phys., **1** (**15**), vol. **8** (2011), 128–146.

[5] V.V. Hrushevsky, H.V. Krylova, *Thermodynamics of phase states in Langmuir Blodgett monolayers*, In: "*Low-dimensional systems-2*" (Eds. S.A. Maskevich et al.), Grodno State University, Grodno, no. **4** (2005), 30-36.

[6] V.M. Kaganer, E.B. Loginov, 1995, Phys. Rev. E 51, 2237.

[7] S.A. Kane, M. Compton, N. *Wilder interactions determining the growth of chiral domains in phospholipid monolayers*, Experimental Results and Comparison with Langmuir Theory, Vol. **16**, No. **22**, (2000), 8447-8455.

[8] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[9] H. Motschmann, H. Möhwald, *Langmuir–Blodgett Films*, Handbook of Applied Surface and Colloid Chemistry (Ed. Krister Holmberg), Ch. **28** (John Wiley & Sons, 2001), 629-648.

[10] M. Neagu, N.G. Krylova, H.V. Grushevskaya, *Jet theoretical Yang-Mills energy in the geometric dynamics of 2D-monolayer*, Journal of Mathematical Physics, **54** (2013), 031508, 14 pages; doi: 10.1063/1.4795715.

Vladimir Balan
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics,
313 Splaiul Independentei, RO-060042, Bucharest, Romania.
E-mail: vladimir.balan@upb.ro

Mircea Neagu and Alexandru Oană
University Transilvania of Braşov, Faculty of Mathematics and Informatics,
Department of Mathematics and Informatics,
50 Iuliu Maniu Ave., RO-500091, Braşov, Romania.
E-mails: mircea.neagu@unitbv.ro, alexandru.oana@unitbv.ro

Nina G. Krylova and Halina V. Grushevskaya
Physics Department, Belarusian State University,
4 Nezavisimosti Ave., 220030 Minsk, Belarus
E-mails: nina-kr@tut.by, grushevskaja@bsu.by.

The det-induced $m-$th root Finsler geometry of Mueller-type submanifolds

Vladimir Balan
University Politehnica of Bucharest, Romania; vladimir.balan@upb.ro
V.M. Red'kov
redkov@dragon.bas-net.by; B.I. Stepanov Institute of Physics
Mircea Neagu
University Transilvania of Braşov, Romania. E-mail: mircea.neagu@unitbv.ro

Abstract. The theory of the Lorentz group provides clues for approaching problems of light polarization optics in the frames of the vector Mueller and spinor Jones formalisms. Differences in describing completely polarized vs. partly polarized light correlate with the properties of isotropic and time-like vectors in Special Relativity. The enveloping framework for the involved geometric objects is the matrix ansatz $M = \mathcal{M}(4, \mathbb{R})$. A canonic parametrization by means of Dirac matrices is used, where four real 4-vectors of the form (k, m, n, l) appear as parameters for candidate Mueller matrices. In this realization, the determinant represents a 4-th order homogeneous polynomial in the 16 parameters, hence naturally providing on the smooth manifold M a locally-Minkowski $m-$th root metric of Finsler type. The main Einstein and Maxwell-deflection type equations are described, and the KCC Jacobi stability theory related to the induced structures, is presented. It is notable that, unlike the extrinsic Finsler norm, the induced one is no longer of locally-Minkowski type, and provides - besides a non-trivial palette of geometric objects - a substantial h-curvature theory. In the present work, we study the corresponding det-induced Finsler structures on several notable low-dimensional submanifolds of Mueller type.

MSC2010: 54B40, 53C60.

Keywords: Mueller matrices; Stokes formalism; Finsler structure; $m-$th root metrics; KCC invariants; torsion; curvature; Einstein-type equations; Maxwell-type equations; Finsler submanifolds.

1 Introduction. Mueller matrices in spinorial representation

The classic Mueller framework is represented by a series of pioneering papers from the beginning of the XX-th century [1, 2, 3, 4, 5, 6]. More recently, fruitful developments extended this topic, and pointed out sound possible applications in polarized optics [7, 8, 9, 10, 11, 12, 13, 14].

We shall further briefly describe the class of Mueller matrices, the pseudo-Finsler ambient $m-$th root structure of determinant type on the set of matrices $\mathcal{M}_{2\times 2}(\mathbb{C})$ and its properties, and then exemplify the corresponding induced Finsler structures on the Mueller classes, in order to emphasize their richness within the enveloping Finslerian ansatz.

We firstly consider the spinorial representation of $\mathbb{R}^4$ inside $\mathcal{M}_{2\times 2}(\mathbb{C})$,

$$(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \longleftrightarrow (a_0, a_1, -ia_2, a_3) \in \mathbb{R} \times \mathbb{R} \times i\mathbb{R} \times \mathbb{R}$$
$$\leftrightarrow A = a_0 I_2 + a_1\sigma_1 - ia_2\sigma_2 + a_3\sigma_3 = \left(\begin{smallmatrix} a_0+a_3 & a_1-a_2 \\ a_1+a_2 & a_0-a_3 \end{smallmatrix}\right) \equiv a_0 + \mathbf{a}\vec{\sigma},$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

are the Pauli matrices.

This allows the rewriting of the matrices in $\mathcal{M}_{4\times 4}(\mathbb{R}) \equiv \mathbb{R}^{16}$ in the form:

$$S \equiv S_x = \begin{pmatrix} k_0 + k_3 & k_1 - k_2 & n_0 + n_3 & n_1 - n_2 \\ k_1 + k_2 & k_0 - k_3 & n_1 + n_2 & n_0 - n_3 \\ \ell_0 + \ell_3 & \ell_1 - \ell_2 & m_0 + m_3 & m_1 - m_2 \\ \ell_1 + \ell_2 & \ell_0 - \ell_3 & m_1 + m_2 & m_0 - m_3 \end{pmatrix}, \tag{1}$$

which is in one-to-one linear correspondence to the vector

$$x = (k_0, k_1, k_2, k_3, m_0, m_1, m_2, m_3, n_0, n_1, n_2, n_3, \ell_0, \ell_1, \ell_2, \ell_3) \in \mathbb{R}^{16}.$$

The determinant of S_x is a homogeneous of degree four polynomial in the coefficients of x, having the form

$$\begin{aligned}
&\det(S_x) = \\
&(+k_0^2m_2^2 - k_0^2m_1^2 - k_0^2m_3^2 + k_0^2m_0^2) + (-k_1^2m_2^2 + k_1^2m_1^2 + k_1^2m_3^2 - k_1^2m_0^2) \\
&+(+k_2^2m_0^2 - k_2^2m_1^2 + k_2^2m_2^2 - k_2^2m_3^2) + (-k_3^2m_2^2 + k_3^2m_1^2 + k_3^2m_3^2 - k_3^2m_0^2) \\
&+(+\ell_0^2n_0^2 - \ell_0^2n_1^2 - \ell_0^2n_3^2 + \ell_0^2n_2^2) + (-\ell_1^2n_2^2 + \ell_1^2n_1^2 - \ell_1^2n_0^2 + \ell_1^2n_3^2) \\
&+(+\ell_2^2n_0^2 - \ell_2^2n_3^2 - \ell_2^2n_1^2 + \ell_2^2n_2^2) + (-\ell_3^2n_2^2 + \ell_3^2n_1^2 - \ell_3^2n_2^2 + \ell_3^2n_3^2) \\
&+2(-k_3\ell_3n_1m_1 - k_3\ell_2n_2m_3 + k_3\ell_3m_0n_0 + k_3\ell_2m_2n_3 - k_3\ell_2m_1n_0 - k_2\ell_2n_2m_2 - k_0\ell_3m_0n_3 + k_0\ell_2m_1n_3 \\
&+k_0\ell_3n_1m_2 - k_0\ell_2m_2n_0 + k_0\ell_0n_1m_1 - k_0\ell_3n_2m_1 + k_0\ell_1m_1n_0 + k_0\ell_2n_2m_0 + k_0\ell_3m_3n_0 - k_3\ell_3m_3n_3 \\
&+k_3\ell_1n_1m_3 - k_3\ell_0n_1m_2 + k_1\ell_0n_1m_0 + k_1\ell_0m_2n_3 - k_1\ell_2m_0n_3 + k_1\ell_2m_3n_0 + k_1\ell_2n_1m_2 - k_1\ell_2n_2m_1 \\
&-k_1\ell_1m_3n_3 + k_1\ell_1m_0n_0 - k_0\ell_0m_0n_0 - k_1\ell_0m_1n_0 + k_1\ell_1n_2m_2 - k_1\ell_1n_1m_1 + k_0\ell_1n_2m_3 - k_0\ell_1m_2n_3 \\
&+k_2\ell_3n_2m_3 - k_2\ell_3n_1m_0 - k_2\ell_3m_2n_3 - k_3\ell_0m_3n_0 + k_3\ell_1m_2n_0 + k_3\ell_0n_2m_1 - k_0\ell_2n_1m_3 + k_2\ell_0m_2n_0 \\
&+k_2\ell_2m_3n_3 - k_2\ell_2m_0n_0 + k_2\ell_0n_1m_3 - k_2\ell_0n_2m_0 - k_2\ell_0m_1n_3 - k_0\ell_1n_1m_0 - k_0\ell_0n_2m_2 - k_3\ell_1m_1n_3 \\
&-k_3\ell_1n_2m_0 + k_3\ell_2n_1m_0 + k_0\ell_0m_3n_3 + k_2\ell_2n_1m_1 + k_2\ell_1m_0n_3 - k_2\ell_1n_1m_2 + k_2\ell_1n_2m_1 + k_2\ell_3m_1n_0 \\
&-k_2\ell_1m_3n_0 + k_3\ell_3n_2m_2 + k_3\ell_0m_0n_3 + k_1\ell_3n_2m_0 + k_1\ell_3m_1n_3 - k_1\ell_3m_2n_0 - k_1\ell_0n_2m_3 - k_1\ell_3n_1m_3).
\end{aligned} \tag{2}$$

This large expression was shown to reduce to a considerably abbreviated form, by involving the formal cross product in $\mathbb{C}^3$ and the formal Minkowski quadratic form in $\mathbb{C}^4$.

Using the representation (1), a classification can be provided [9, 14, 18] for the so-called *Mueller matrices*, the subset of $\mathcal{M}_* \subset \mathcal{M}_{4\times 4}(\mathbb{R})$ formed by matrices $S \in \mathcal{M}_{4\times 4}(\mathbb{R})$ which satisfy the following properties:

$$\begin{cases} \forall v = (v^0, \underbrace{v^1, v^2, v^3}_{\mathbf{v}})^t \in \mathbb{R}^4, \quad v^0 > 0 \Rightarrow (Sv)^0 > 0, \\ \\ \forall v = (v^0, v^1, v^2, v^3)^t \in \mathbb{R}^4, \quad Q_{3,1}(v) > 0 \Rightarrow Q_{3,1}(Sv) > 0, \end{cases}$$

where we denoted $Q_{3,1}(v) = (v^0)^2 - \mathbf{v}^2 \equiv (v^0)^2 - [(v^1)^2 + (v^2)^2 + (v^3)^2]$[1]. In the following we shall use the set of real matrices $\mathcal{M}_{4\times 4}(\mathbb{R})$ described in spinorial form as ambient manifold for the subclasses represented by the Mueller submanifolds.

2 The det-ambient m-th root Finsler structure

In order to construct the det-induced pseudo-Finsler structure on the Mueller sets, we first note that Finsler Geometry appears as a natural extension of the Euclidean $(\mathbb{R}^n, \delta_{ij})$ and further, Riemannian $(M, g_{ij(x)})$ imbricated structures, and can be regarded as a particular case of the Lagrange $(M, L(x, y))$, Generalized Lagrange $(M, g_{ij}(x, y))$ and (h, v)-metric structures, all constructed on the tangent bundle of a manifold [22, 30, 31].

From this perspective, a Finsler structure is a couple (M, F), such that:
M real n-dimensional C^∞ manifold; $F : TM \to [0, \infty)$ is a mapping (*Finsler fundamental function*), which satisfies:

[1] We have denoted here $||v||_{3,1} = \sqrt{(v^0)^2 - \mathbf{v}^2}$ and $\mathbf{v}^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$.

1. F smooth on the slit tangent space $T_0M = \{(x, y)|x \in M, y \in T_xM, y \neq 0\}$ continuous on the null section;
2. F positive 1-homogeneous in y, i.e., $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
3. F defines the smooth maps $g_{ij} : TM\setminus\{0\} \to \mathbb{R}$, $i, j \in \overline{1, n}$ components of the metric Finsler tensor field $g = g_{ij}dx^i \otimes dx^j$, which form a symmetric positive definite matrix, $(g_{ij})_{i,j\in\overline{1,n}}$, $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.

One can note that the Finsler axioms may be relaxed, and variations may occur, as follows:

- the positivity of the norm F may be dropped;
- the positive definiteness of the metric tensor g may be replaced by its maximal constant signature;
- the domain of F may be restricted to less than TM;
- the smoothness domain of F, may be restricted to less than T_0M;
- the positive 1-homogeneity may be replaced by overall 1-homogeneity.

As main examples of Finsler norms, the following Finsler-type fundamental functions are established, leading to fruitful geometric models for Physics:

a) Riemannian: $F(x, y) = \sqrt{\gamma_x(y, y)}$, where (M, γ) is a Riemannian space.
b) Randers: $F = \alpha + \beta$, where (M, γ) is a Riemann space and b is a differential 1-form on M, denoting $\alpha = \sqrt{\gamma_x(y, y)}$ and $\beta = b(y)$.
c) Kropina: $F = \alpha^2/\beta$, with the same notations as above.
d) (α, β)-metrics: $F = \varphi(\alpha, \beta)$, which satisfies (1-4), where α and β are as above.
e) m-th root metrics: $F(x, y) = \sqrt[m]{a_{i_1\dots y_m}(x)\, y^{i_1} \cdot \ldots \cdot y^{i_m}}$, where a is a $(0, m)$-tensor field on M, which is symmetric in all its indices.
f) $(\alpha, \mu)-$metrics: $F = \psi(\alpha, \mu)$ satisfying (1-4), where α is as above, and μ is an m-th root Finsler metric.

The most prominent classic contributions to Finsler geometry were provided by[2]: Bernhard Riemann[3], Paul Finsler (doctoral dissertation presented in 1918), J.L.Synge (1925, 1936), J.H. Taylor (1925, 1927), L. Berwald (1926, 1935), Elie Cartan (1933), O. Varga, H. Busemann, H. Rund (after 1940), a.o.

We further assume given a Finsler structure (M, F). We note that in the position-dependent Finsler case, the geodesics cease to be straight lines. Their trajectories are governed by five invariants, which emerge from the Finsler spray. The natural Kern 2-homogeneous Finslerian spray lives in $\Gamma(TTM)$ [24, 31]:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \quad G^j = \frac{1}{2} g^{ji} \left(\frac{\partial^2 F^2}{\partial y^i \partial x^k} y^k + \frac{\partial F^2}{\partial x^i} \right),$$

and produces the Cartan nonlinear connection, of coefficients $N_i^a = \frac{\partial G^a}{\partial y^i}$.
The spray coefficients G^i define the dynamical covariant derivative $\nabla X^i = S(X^i) + N_j^i X^j$, essential in defining the KCC invariants, introduced by Antonelli and Bucataru [15], and studied by Sabau, Neagu a.o. The geodesic system of the Finsler structure (M, F) has the form,

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0 \Leftrightarrow \begin{cases} \dot{x}^i = y^i \\ \dot{y}^i = -2G^i(x, y). \end{cases}$$

[2]We indicate the years of major advances
[3]In "Habilitation address" (1854 - Göttingen), "Uber die Hypothesen, welche der Geometrie zu Grunde liegen" (On the hypotheses lying at the basis of Geometry), he provided the first Finsler norm $F(y) = \sqrt[4]{(y^1)^4 + (y^2)^4 + \ldots + (y^n)^4}$, for all $y = (y^1, \ldots, y^n) \in T_xM$ and $x \in M = \mathbb{R}^n$

The straightening of geodesics - an issue of vanishing of the *KCC invariants*, used to study the structural (Jacobi) stability [15]:

- The first KCC invariant $\varepsilon^i(x,y) = N^i_j(x,y)y^j - 2G^i(x,y)$ leads to a semispray, $S = y^i \frac{\delta}{\delta x^i} + \varepsilon^i \frac{\partial}{\partial y^i}$, homogeneous *only* for Finsler structures.
- The second KCC invariant detects gathering/dissipating of geodesic sheaves:

$$S^i_j = R^i_{jk}y^k - \varepsilon^i_{|j} = 2\frac{\partial G^i}{\partial x^j} + 2G^s F^i_{js} - \frac{\partial N^i_j}{\partial x^s}y^s - N^i_s N^s_j - \frac{\partial N^i_j}{\partial x^s}y^s,$$

  where R^i_{jk} is the the torsion of the Berwald connection,

$$\Gamma_B = \left(N^a_i = \frac{\partial G^a}{\partial y^i},\ F^i_{jk} = \frac{\partial N^i_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k},\ C^i_{jk} = 0 \right).$$

- the third KCC invariant, (the curvature of the nonlinear connection), $B^i_{jk} = R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k}$.
- the fourth KCC invariant is the Riemann-Christoffel curvature tensor of the Berwald connection, $B^i_{jkl} = \frac{\delta F^i_{jk}}{\delta x^l} - \frac{\delta F^i_{jl}}{\delta x^k} + F^i_{ml}F^m_{jk} - F^i_{mk}F^m_{jl}$.
- the fifth KCC invariant is the Douglas tensor, the last curvature of the Berwald connection, $D^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$.

The KCC invariants have a special meaning in the Finsler framework, which is particularly convenient to describe phenomena where the dynamics depends on the present state and on the direction of change. In particular, the vanishing of the invariants infers affine solutions. Numerous open problems occur, regarding the relation between the behavior of paths and the properties of the KCC invariants.

In Finsler geometry, the non-linear connection can be regarded as a complement $HTM = Ker(N)$ of the vertical sub-bundle $VTM = Ker(d\pi)$ of the vector bundle $(TTM, d\pi, TM)$; it provides the *Whitney decomposition* $TTM = HTM \oplus VTM$, and the sections of the sub-bundles locally spanned by *adapted bases*, $\left\{\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}\right\} \subset \Gamma(HTM)$, $\left\{\dot\partial_i = \frac{\partial}{\partial y^i}\right\} \subset \Gamma(VTM)$; [30, 31]. As well, the *dual splitting* $T^*TM = H^*TM \oplus V^*TM$ similarly leads to the *dual adapted basis*: $\{dx^i\} \subset \Gamma(H^*TM)$, $\left\{\delta y^i = dy^i + N^i_j dx^j\right\} \subset \Gamma(V^*TM)$.

We shall further consider the pseudo-Finsler geometry induced by the determinant (2) on Finsler-Mueller structures. The quartic form $Q(y) = \det S_y$, produces a pseudo-Finsler structure of locally Minkowski type $(M = \mathcal{M}_{4\times 4}(\mathbb{R}), F_v)$ endowed with the $4-$th root pseudo-Finsler function of locally Minkowski type

$$F_v(y) = \sqrt[4]{\det S_y},\ \ \forall y \in T_xM,\ x \in M.$$

As well, we note that in each fiber T_xM there lives the flat Frobenius Euclidean norm

$$||y||_h = \sqrt{Tr(y^t \cdot y)},\ \ \forall y \in T_xM,$$

and further, one may consider its conformal deformation:

$$F_h(x,y) = e^{Q(x)} \cdot ||y||_h,\ \ \forall (x,y) \in TM,$$

which provides a genuine Riemannian metric tensor on M.

We note that the ambient vertical norm is KCC-flat, since the KCC invariants for F_v emerge from
$$G^i = 0,\ N^i_j = 0 \text{ and } F^i_{jk} = 0.$$
these are flat, and all the equations of geodesics are trivial, with solutions consisting of segments of line.

3 Finsler linear d-connections. Einstein-like and Maxwell-like equations

In order to determine the generalized formal Maxwell and Einstein field equations, related to the vertical det-generated m-th root Finsler structure, we fix a non-linear connection $N \equiv \{N_i^a\}$, and a linear d-connection D on TM, described by its adapted coefficients $D\Gamma(N) = \{L^i{}_{jk}, C^a{}_{bc}\}$, with:

$$D_{\delta_k}\delta_j = L^i{}_{jk}\delta_i, \quad D_{\delta_k}\dot\partial_b = L^a{}_{bk}\dot\partial_a,$$
$$D_{\dot\partial_c}\delta_j = C^i{}_{jc}\delta_i, \quad D_{\dot\partial_c}\dot\partial_b = C^a{}_{bc}\dot\partial_a,$$

where the connection is *normal*, i.e.,

$$L^i{}_{jk} = \delta_j^b\delta_a^i L^a{}_{bk}, \;\; C^a{}_{bc} = \delta_i^a\delta_b^j C^i{}_{jc}.$$

We shall further denote the related $h-$ and $v-$ covariant derivatives induced by D by $_|$ and $|$, respectively. The torsion T of the linear connection D has the adapted components

$$\begin{array}{ll} hT(\delta_k,\delta_j) = T^i{}_{jk}\delta_i, & vT(\delta_k,\delta_j) = R^a{}_{jk}\dot\partial_a, \\ hT(\dot\partial_c,\delta_j) = C^i{}_{jc}\delta_i, & vT(\dot\partial_c,\delta_j) = P^a{}_{jc}\dot\partial_a, \\ hT(\dot\partial_c,\dot\partial_b) = 0, & vT(\dot\partial_c,\dot\partial_b) = S^a{}_{bc}\dot\partial_a, \end{array}$$

and the curvature R has essentially the component sets $R, \overset{1}{P}, \overset{2}{P}, S,$:

$$\begin{array}{ll} R(\delta_l,\delta_k)\delta_j = R^i_{jkl}\delta_i, & R(\delta_l,\delta_k)\dot\partial_b = R^a_{bkl}\dot\partial_a, \\ R(\dot\partial_c,\delta_k)\delta_j = \overset{1}{P}{}^i_{jkc}\delta_i, & R(\dot\partial_c,\delta_k)\dot\partial_b = \overset{2}{P}{}^a_{bkc}\dot\partial_a, \\ R(\dot\partial_c,\dot\partial_b)\delta_j = S^i_{jbc}\delta_i, & R(\dot\partial_d,\dot\partial_c)\dot\partial_b = S^a_{bcd}\dot\partial_a, \end{array}$$

where $R^a_{bkl} = \delta_b^j\delta_i^a R^i_{jkl}$, and $S^i_{jbc} = \delta_a^i\delta_j^b S^a_{bcd}$. The explicit components of the torsion are explicitly given by

$$T^i_{jk} = L^i_{[jk]}, \;\; R^a_{jk} = \delta_{[j}N^a_{k]}, \;\; P^i_{ja} = C^i_{ja}, \;\; P^a_{jb} = \frac{\partial N^a_j}{\partial y^b} - L^a_{bi}, \;\; S^a_{bc} = C^a_{[bc]},$$

while the curvature coefficients are [30, 31]

$$\left\{\begin{array}{l} R_j{}^i{}_{kl} = \delta_{[l}L^i_{jk]} + L^s_{j[k}L^i_{sl]} + C^i_{je}R^e_{kl}, \\ \overset{1}{P}{}_j{}^i{}_{kc} = \dot\partial_c L^i_{jk} - C^i_{jc|k} + C^i_{jb}P^b_{kc}, \\ \overset{2}{P}{}_b{}^a{}_{kc} = \dot\partial_c L^a_{bk} - C^a_{bc|k} + C^a_{be}P^e_{kc}, \\ S_b{}^a{}_{cd} = \dot\partial_{[d}L^a_{bc]} + C^e_{j[c}C^a_{ed]}, \end{array}\right.$$

where $\tau_{...[i...j]...} = \tau_{...i...j...} - \tau_{...j...i...}$. The generalized Ricci tensor has the components:

$$R_{jk} = R^s_{jks}, \;\; \overset{1}{P}{}_{bj} = P^e_{bje}, \;\; \overset{2}{P}{}_{jb} = P^k_{jkb}, \;\; S_{ab} = S^e_{abe},$$

and the scalars of curvature

$$R = g^{ij}R_{ij}, \;\; S = h^{ab}S_{ab}.$$

In particular, one may consider the Cartan *canonical* metrical $d-$connection, while the nonlinear connection, for $L = F^2$ can be produced via the Kern Theorem [24, 31],

$$\left\{\begin{array}{l} N^i_j = \dfrac{\partial G^i}{\partial y^j}, \\ L^i{}_{jk} = \dfrac{1}{2}g^{ih}\left(\dfrac{\delta g_{hj}}{\delta x^k} + \dfrac{\delta g_{hk}}{\delta x^j} - \dfrac{\delta g_{jk}}{\delta x^h}\right), \\ C^a{}_{bc} = \dfrac{1}{2}g^{ad}\left(\dfrac{\partial g_{db}}{\partial y^c} + \dfrac{\partial g_{dc}}{\partial y^b} - \dfrac{\partial g_{bc}}{\partial y^d}\right). \end{array}\right.$$

Then the Einstein equations of the canonical d-connection are

$$\begin{cases} R_{ij} - \frac{1}{2}(R+S)g_{ij} = \kappa T_{ij}, \\ S_{ab} - \frac{1}{2}(R+S)h_{ab} = \kappa T_{ab}, \\ \overset{1}{P}_{bj} = \kappa T_{bj}, \ \ \overset{2}{P}_{jb} = -\kappa T_{jb}, \end{cases}$$

where κ is a constant and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. and the conservation laws for these equations have the form

$$\begin{cases} \left(R^i_j - \frac{1}{2}(R+S)\delta^i_j\right)_{|i} + \overset{1}{P}{}^a_j\Big|_a = 0, \\ \left(S^a_b - \frac{1}{2}(R+S)\delta^a_b\right)\Big|_a + \overset{2}{P}{}^i_{b|i} = 0. \end{cases}$$

The ambient metric structure is of locally Minkowski type: the Finsler norm F_v produces the Finsler metric tensor

$$g_{ab} = \frac{1}{2}\frac{\partial^2 F_v^2}{\partial y^a \partial y^b}.$$

which further determines on TM the special Generalized Lagrange metric on TM

$$\mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + g_{ab}(y)\delta y^a \otimes \delta y^b.$$

In the Kern nonlinear connection case, the nonlinear connection N of the vertical Finsler function F_v is trivial, $N^a_i(x,y) = 0, \ \ i,a = \overline{1,4}$. This infers the following consequences:

- the equality $N^a_j = 0$ yields $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i}$;
- the torsion of the canonic linear d-connection has a single non-vanishing component, $\overset{1}{P}{}^a_{bk} = \gamma^a_{bk}$ of $hT(\dot\partial_a, \delta_j)$;
- the curvature tensor fields are related to the torsion field via

$$S^a_{d\ bc}y^d = S^a_{bc} = 0, \ \ y^d R^a_{d\ jk} = R^a_{jk} = 0, \ \ y^d \overset{2}{P}{}^a_{d\ kc} = P^a_{kc} = r^a_{bkc}y^c.$$

Then the curvature sets of the canonical metrical linear d-connection associated to $\mathcal{G}$ are:

$$\begin{cases} R_j{}^i{}_{kh} = 0, \ \ R_b{}^a{}_{kh} = 0, \ \ P_j{}^i{}_{kc} = 0, \ \ P_b{}^a{}_{kc} = 0, , \\ S^a_{b\ cd} = \dfrac{\partial C^a_{bc}}{\partial y^d} - \dfrac{\partial C^a_{bd}}{\partial y^c} + C^f_{bc}C^a_{fd} - C^f_{bd}C^a_{fc}, \end{cases}$$

and the Einstein-like equations of the canonical metrical linear d-connection $C\Gamma(N_{Kern})$ reduce to

$$S_{ab} - \frac{1}{2}Sg_{ab} = T^V_{ab}.$$

As well, the deflections and Maxwell-like equations in the ambient vertical model are, for the Kern canonic d-linear connection attached to the Lagrangian F_v^2,

$$D^a_j = 0, \quad d^a_b \equiv y^a|_b = \delta^a_b + y^c C^a_{cb},$$

and for the F_v Finslerian linear d-connection *of Cartan type*, the *covariant* deflection tensors are

$$D_{ij} = 0, \quad d_{ab} = h_{ab}.$$

Then, the *electromagnetic-like tensors identically vanish*, whence one obtains trivial Maxwell-like equations

$$F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}) \equiv 0, \quad f_{ab} = \frac{1}{2}(d_{ab} - d_{ba}) \equiv 0.$$

4 Induced det - Finsler structures on Mueller submanifolds

We further consider Mueller subclasses of embedded sets N of matrices in $M = \mathcal{M}_{4\times 4}(\mathbb{R})$ described in [9, 14]. In general, for a general Finsler structure (M, F) with M an $m-$dimensional manifold, we shall and let N be an $n-$dimensional submanifold embedded in M via $r : N \to M$, locally given by

$$r(u) = (x^1(u), \dots, x^m(u)), \qquad u = (u^1, \dots, u^n) \in D \subset \mathbb{R}^n,$$

where r is assumed to be an injective immersion, homeomorphism on its image.

We note that N is canonically endowed by *induced Finsler norm* $F_* : TN \to \mathbb{R}$,

$$F_*(u, v) = F(r(u), r_*(v)), \ \ \forall (u, v) \in TN.$$

We shall consider as particular cases the Finlser norms which are canonically induced on the Mueller subspaces by the Finsler F_v m-th root Finsler norm. Among the submanifolds N of the ambient space $\mathcal{M}_{4\times 4}(\mathbb{R}) \equiv \mathbb{R}^{16}$, there exist several remarkable low-dimensional surfaces, as follows[4]:

- $KM-5$, for fixed $K = I_2, M = O_2$. This is parametrized by

  $$A = \left(\begin{smallmatrix} K & N \\ L & M \end{smallmatrix}\right) = \left(\begin{smallmatrix} K & tK \\ sK & 0 \end{smallmatrix}\right) \equiv (1, 0, 0, 0; s, 0, 0, 0; t, 0, 0, 0; 0, 0, 0, 0),$$

  essentially a plane embedded in $\mathbb{R}^{16}$, and there exists no det-induced Finsler pseudo-norm since $F_* = -(v^1v^2)^2 \leq 0$, and the only nontrivial direction components are $y^4 = v^2$ and $y^9 = v^1$.
- $KM-1$, for $K = sI_2$ and $M = tI_2$,

  $$A = \left(\begin{smallmatrix} K & N \\ L & M \end{smallmatrix}\right) = \left(\begin{smallmatrix} sI_2 & 0 \\ 0 & tI_2 \end{smallmatrix}\right) \equiv (s, 0, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0; t, 0, 0, 0)$$

  a plane embedded in $\mathbb{R}^{16}$, with the non-trivial directional components $y^1 = v^1, y^{13} = v^2$ and norm of 2D-Minkowski type, whose geodesics are the cone-lightline generators

  $$F_* = \sqrt{|v^1v^2|}.$$
- $KM-1$, for $K = diag(s+t, s-t)$ and $M = stI_2$, provides

  $$A = \left(\begin{smallmatrix} K & N \\ L & M \end{smallmatrix}\right) \equiv (s, 0, 0, t; s, 0, 0, 0; 0, 0, 0, 0; st, 0, 0, 0),$$

  essentially a saddle, along which the nontrivial direction components are $y^1 = v^1, y^3 = v^2, y^{11} = u^1v^2 + u^2v^1$, whence

  $$F_* = \sqrt[4]{(v^1)^2 - (v^2)^2} \cdot \sqrt{u^1v^2 + u^2v^1}.$$

  We note that this is an $m-$th root type Finsler metric of 4-th order, which is *not* locally Minkowskian and whose geodesics are not straight lines.
- $KM-1$, for $K = \left(\begin{smallmatrix} s & -t \\ t & s \end{smallmatrix}\right) = M$, one gets

  $$A \equiv (s, 0, t, 0; 0, 0, 0, 0; 0, 0, 0, 0; s, 0, t, 0),$$

  the nontrivial direction components $y^1 = y^6 = y^{11} = y^{16} = v^1$ and $-y^2 = y^5 = -y^{12} = y^{15} = v^2$, with the Euclidean norm,

  $$F_* = \sqrt{(v^1)^2 + (v^2)^2}.$$
- $KM-1$, for $K = \left(\begin{smallmatrix} a+b & 0 \\ 0 & a-b \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} c+d & 0 \\ 0 & c-d \end{smallmatrix}\right)$ with $a = \frac{u^1+u^2}{2}, b = \frac{u^1-u^2}{2}$, $c = \frac{u^3+u^4}{2}, d = \frac{u^3-u^4}{2}$ provides a parametrized 4-dimensional submanifold, with

  $$A = diag(u^1, u^2, u^3, u^4) \equiv (a, 0, 0, b; 0, 0, 0, 0; 0, 0, 0, 0; c, 0, 0, d),$$

  a 4-hyperplane endowed with the Berwald-Moor Finsler metric whose geodesics are straight lines,

  $$F_* = \sqrt[4]{v^1v^2v^3v^4}.$$

[4] We shall use the notations for the Mueller classes described in [9, 14, 18]

- $KM-1$, for $K=diag(s+t,s-t)$ and $M=stI_2$, provides

$$A=\left(\begin{smallmatrix} K & N \\ L & M \end{smallmatrix}\right)\equiv(s,0,0,t;s,0,0,0;0,0,0,0;st,0,0,0),$$

essentially a saddle, along which the nontrivial direction components are $y^1=v^1, y^3=v^2, y^{11}=u^1v^2+u^2v^1$, whence

$$F_*=\sqrt[4]{(v^1)^2-(v^2)^2}\cdot\sqrt{u^1v^2+u^2v^1}.$$

We note that this is an $m-$th root type Finsler metric of 4-th order, which is *not* locally Minkowskian and whose geodesics are not straight lines.

5 Conclusions

The Finslerian extensions of classic General Relativity and Electromagnetism provide natural models while anisotropy phenomena are considered; within this framework, the m-th root metrics are at present intensively investigated as possible alternative models for Special Relativity. We presented the basic KCC stability framework for Finslerian-type geodesics, and the Finsler metric formalism was used to derive extended Einstein-like and Maxwell-like equations on the toy-model of Stokes-Mueller classes in the ambient for Mueller classes matrix framework, for the det-generated pseudo-Finsler structure. We have also emphasized that unlike the matrix ambient structure, the Finsler induced structures on Muller classes are extremely rich, which leads for further research to consider the Physics relevance of the objects on the Mueller classes, regarded as low-dimensional Mueller submanifolds of the $M=\mathcal{M}_{4\times4}(\mathbb{R})\equiv\mathbb{R}^{16}$ space.

Acknowledgments. The present work was developed under the auspices of the Grant No. 11962012-BRFFR-RAF14RA-006, within the cooperation framework between the Romanian Academy and the Belarussian Republican Foundation for Fundamental Research.

References

[1] G.G. Stokes, *On the composition and resolution of streams of polarized light from different sources*, Trans. Cambridge Phil. Soc. **9** (1852), 399–416.

[2] H. Mueller, *Memorandum on the polarization optics of the photo-elestic shutter*; Report 2 of the OSRD project OEMsr576, Boston, MA, USA, 1943.

[3] R.C. Jones, *New calculus for the treatment of optical systems. I. Description and discussion of the calculus*, J. Opt. Amer. Soc. **31** (1941), 488–493.

[4] H. Hurwitz, R.C. Jones, *A new calculus for the treatment of optical systems. II. Proof of three general equivalence theorems*, J. Opt. Amer. Soc. **31** (1941), 493–499.

[5] R.C. Jones, *A new calculus for the treatment of optical systems. III. The Sohncke Theory of optical activity*, J. Opt. Amer. Soc. **31** (1941), 500–503.

[6] R.C. Jones, *A new calculus for the treatment of optical systems, IV.*, J. Opt. Amer. Soc. **32** (1942), 486–493.

[7] E.M. Ovsiyuk, *Transitivity in tree-dimensional rotation group and Stokes–Mueller formalism in polarization optics*, Vestn. Mogilev State Univ. A.A. Kuleshov, Ser.Nat.Sci.: Mat.Fiz.Biol. 1 (37) (2011), 69–75.

[8] E.M. Ovsiyuk, V.M. Red'kov, *Degenerate 4-dimensional matrices with semi-group structure and polarization optics*, XLVIII All-Russia Conf., Russia, Moscow 2012, Vestn. RUDN, Ser. Math., Inform. and Phys. **1** (2013), 245–259.

[9] E.M. Ovsiyuk, V.M. Red'kov, *Is the Finsler geometrization of polarization optics possible?*, Proc. of the VIII-th Int. Conf. FERT-2012, Moscow-Fryazino, Russia, HNGP, **9** 1 (17) (2012), 106–161.

[10] E.M. Ovsiyuk, O.V. Veko, V.M. Red'kov, *Mueller's semi-groups of the rank 1 and 2*, Probl. of Phys., Math. and Technics 2 (11) (2012), 34–40.

[11] V.M. Red'kov, *Lorentz group and polarization of the light*, Adv. in App. Clifford Algebras, 21 (2011), 203–220.

[12] V.M. Red'kov, A.A. Bogush, N.G. Tokarevskaya, 4×4 *matrices in Dirac parametrization: inversion problem and determinant*, arXiv:0709.3574v2 [hep-th] 15 Feb 2008.

[13] V.M. Red'kov, E.M. Ovsiyuk, *Transitivity in the theory of the Lorentz group and Stokes–Mueller formalism in polarization optics*, Vestn. Brest Univ., Ser. Fiz.-Mat. 1 (2012), 18–23.

[14] O.V. Veko, E.M. Ovsiyuk, A. Oana, M. Neagu, V. Balan, V.M. Redḱov; *Spinor Structures in Polarization Optics* (in English), Nova Publishers, U.S., 2015, ISBN 978-1-63463-625-4, 416 pp.

[15] P.L. Antonelli, I. Bucataru, *New results about the geometric invariants in KCC-theory*, An.St. Univ. "Al.I.Cuza" Iasi. Mat. N.S. 47 (2001), 405-420.

[16] V. Balan, *Generalized Einstein - Yang Mills equations for the space* $GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$, Tensor N.S., v.52 (**1993**), 199-203.

[17] V. Balan, G. Bogoslovsky, S. Kokarev, D. Pavlov, S. Syparov, N. Voicu, *Geometrical models of the locally anisotropic Space-Time*, Hypercomplex Numbers in Geometry and Physics, Moscow-Russia, 1 (15), 8 (**2011**), 4-37.

[18] V. Balan, V. Red'kov and A. Oana, *Finsler-type structures and det-based classification of Mueller-type submanifolds*, Diff. Geom. - Dyn. Sys. (DGDS), 16 (2014), 63-84.

[19] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.

[20] A.V. Berezin, Yu.A. Kurochkin, E.A. Tolkachev, *Quaternions in Relativistic Physics*, Minsk 1989.

[21] A.A. Bogush, V.M. Red'kov, *On unique parametrization of the linear group* $GL(4,\mathbb{C})$ *and its subgroups by using the Dirac algebra basis*, NPCS 11, 1 (2008), 1–24.

[22] I. Bucataru, R. Miron, *Finsler-Lagrange geometry. Applications to dynamical systems*, Editura Academiei Romane, Bucuresti, 2007.

[23] F.B. Fedorov, *The Lorentz Group*, Moscow 1979.

[24] J. Kern, *Lagrange geometry*, Arch. Math. 25 (1974), 483-443.

[25] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Publishers, New York, 1963, 1969.

[26] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Kyoto, 1986.

[27] M. Matsumoto, K. Okubo, *Theory of Finsler spaces with* $m-$*th root metric: connections and main scalars*, Tensor N.S., 56 (1995), 9-104.

[28] M. Matsumoto, S. Numata, *On Finsler spaces with a cubic metric*, Tensor N.S., 33 (1979), 153-162.

[29] M. Matsumoto, H. Shimada, *On Finsler spaces with* $1-$*form metric*, Tensor N.S., 32 (1978), 161-169.

[30] R. Miron, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, 1994.

[31] R. Miron, M.Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, Geometry Balkan Press, 1996.

[32] Gh. Munteanu, V. Balan, *Lectures of Relativity Theory* (Romanian), Bren Eds., Bucharest, 2000.

[33] D.G. Pavlov, S.S. Kokarev, *Essentials of Polynumbers Field Theory*, Chapter 7 in "Advances in General Relativity Research" (Ed. Cameron Williams), Series: Physics Research and Technology, NOVA Publishers, 2015, ISBN: 978-1-63483-120-8.

[34] D.G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 19-30.

[35] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geom. Phys. 1 (1) (2004), 31–39.

[36] D.G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geom. Phys. 1 (1) (2004), 5-18.

[37] H. Rund, *The Geometry of Finsler Spaces*, Springer, Berlin 1959.

[38] H. Shimada, *On Finsler spaces with the metric* $L = \sqrt[n]{a_{i_1 i_2 \dots i_n}(x) y^{i_1} y^{i_2} \dots y^{i_n}}$, Tensor N.S., 33 (1979), 365-372.

[39] H. Shimada, *On Finsler spaces with* $1-$*form metric II. Berwald-Moor's metric* $L = \sqrt[n]{y^1 y^2 \dots y^n}$, Tensor N.S., 32 (1978), 375-278.

[40] K. Trencevski, E. Celakoska, V. Balan, *Research of gravitation in flat Minkowski space*, Int J. Theor. Phys. 50 (**2011**), 1–26.

The distortion and S-curvature of Langmuir monolayer space under the action of electrocapillary forces

H.V. Grushevskaya, N.G. Krylova
grushevskaja@bsu.by, nina-kr@tut.by; Belarusian State University, Minsk, Belarus

Abstract

The geometro-dynamic approach is proposed to describe two-dimensional phase transitions of first order in the Langmuir monolayer on air/water interface under the action of electrocapillary forces. We revealed a range of particle velocity values at that the time and space components of distortion are different from zero, that proves the emergence of space-time structures at Langmuir monolayer compression. It was shown also that time component of mean Cartan tensor and S-curvature is not zero at large values of displacements and, respectively, the Finsler corrections are significant when the phase transition occurs.

1 Introduction

The Langmuir - Blodgett (LB) technique is a perspective method for fabricating of perfect organic nanostructured materials. A study of formation of molecular LB-monolayers is of great interest not only because electronics applications, but also Langmuir monolayers are suitable models of membranes of living cells. The formation of LB-monolayer occurs through the self-organizing of molecules on a surface of liquid subphase during Langmuir monolayer compression by a barrier (see [1, 2]). At that, a two-dimensional (2D) first-order phase transition from a 2D gas up to a condensed state of monolayer occurs. The phase transition in Langmuir monolayer depends on parameters of monolayer formation, such as a temperature and a rate of compression, structure of amphiphilic molecules and subphase composition.

In experiments performed with the help of Brewster angle microscopy the presence of domains of co-existing phases, shapes of which changes during the phase transition process from circular to labyrinthine-like one and depends on monolayer formation parameters, has been observed [3, 4]. X-ray diffraction of LB-films revealed the mound-shaped structures of the compressed monolayer for liquid-crystal phases L_2, L_2', Ov, S possessing a tilted tail structure and for two-dimensional untilted crystalline phases L_2'', CS [5]. The mounds observed in the experiments are the water expelled out of the liquid crystal and solid phases as a result of phase foliation [4, 5, 6, 7]. The phase foliation parameters depend on the speed of compression too.

To date, the phenomenological models do not correct describe the domain shapes and phase foliation process and their dependencies on the monolayer formation parameters. Previously we proposed a geometric approach consisted in an examination of three dimensional (3D) monolayer Finsler space, metric function of which determined by a surface tension potential and depends on monolayer formation parameters [8, 9, 10]. In such approach one can assume that the experimental facts, considered above, have to reveal as a distortion of model monolayer space. The goal of the paper is to describe a dependence of first-order 2D-phase transitions on monolayer formation parameters in a framework of Finsler geometry approach.

2 A Finsler space of Langmuir monolayer under the action of electrocapillary forces

In the paper [8] a phenomenological Lagrangian L for the two-dimensional (2D) motion of a particle in the Langmuir monolayer at its compression and under the action of electrocapillary forces has been proposed:

$$L(t,r,\dot{r},\dot{\varphi}) = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\varphi}^2 \underbrace{-pr^5|v|e^{\frac{2|v|t}{r}}\cdot\dot{r}^{-1} + U(t,r)}_{U_s(t,r)}, \tag{1}$$

where • m is the *mass* of the particle; • v is the *compressing speed*; • p is a *constant monolayer parameter*

$$p = \frac{\pi^2 q^2}{\varepsilon\varepsilon_0}\frac{\rho_0^2}{R_0^2};$$

• $U_s(t,r)$ is an *electrocapillary potential energy* including the monomolecular layer function

$$\begin{aligned} U(t,r) &= p\left\{\left[-\frac{4}{3}r^5 + \frac{16}{15}(|v|t)\,r^4 + \frac{1}{30}(|v|t)^2\,r^3 + \frac{1}{45}(|v|t)^3\,r^2 + \right.\right. \\ &\left.\left. + \frac{1}{45}(|v|t)^4\,r + \frac{2}{45}(|v|t)^5\right]e^{\frac{2|v|t}{r}} - \frac{4}{45}\frac{(|v|t)^6}{r}\,\mathrm{Ei}\left(\frac{2|v|t}{r}\right)\right\}. \end{aligned} \tag{2}$$

where

$$\mathrm{Ei}\,(z) \stackrel{def}{=} -\int_{-z}^{\infty}\frac{e^{-t}}{t}dt,$$

Based on the Lagrangian L on can determine the parameterized action of the problem (1) as

$$dl_F = mc^2\dot{\xi}d\tau - L(\tau)\dot{\xi}d\tau = \left(A\frac{\dot{\xi}^2}{\dot{r}} + B\dot{\xi} - C\frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{2c^2\dot{\xi}}\right)d\tau, \tag{3}$$

where the parameters $A,\ B$, and C are given by

$$\begin{aligned} &A = p\,|v|\,r^5 e^{\frac{2|v|t}{r}}, \\ &B = mc^2 - p\left(\left(-\tfrac{4}{3}r^5 + \tfrac{16}{15}(|v|t)r^4 + \tfrac{1}{30}(|v|t)^2r^3 + \tfrac{1}{45}(|v|t)^3r^2 + \tfrac{1}{45}(|v|t)^4r + \right.\right. \\ &\left.\left.\tfrac{2}{45}(|v|t)^5\right)e^{\frac{2|v|t}{r}} - \tfrac{4}{45}\tfrac{(|v|t)^6}{r}\mathrm{Ei}\left[\tfrac{2|v|t}{r}\right]\right), \\ &C = mc^2, \quad \text{where } \dot{r} = \tfrac{x\dot{x}+y\dot{y}}{r}. \end{aligned}$$

Here $\dot{\xi}$, $\dot{r}$, and $\dot{\varphi}$ correspond to the derivatives of t, r, and φ with respect to the evolution parameter τ, respectively.

The parametrized Lagrangian $L(\Delta\xi,\tau) \equiv L(\tau)\dot{\xi}$ which enters, is a positive-homogeneous of first order function. It hence defines the arc-element for the space of linear elements, called also *Finsler norm.*

If we consider the dynamics of the system on a mass surface, the metric function is defined as

$$F^2(x^j,\ \ y^j) = A\frac{\dot{\xi}^3}{\dot{r}} + B\dot{\xi}^2 - C\frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{2c^2}. \tag{4}$$

where $x^j = (t,r,\phi)$ and $y^j = \dot{x}^j = (\dot{\xi},\dot{r},\dot{\phi})$

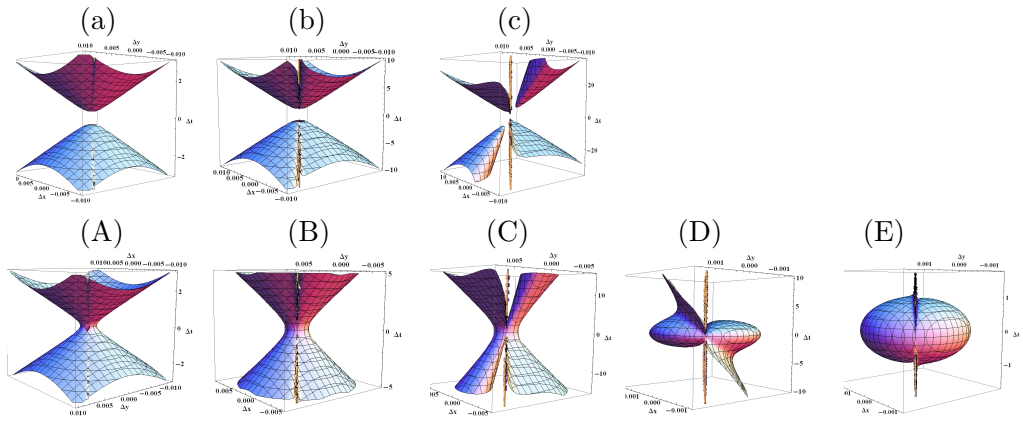


Figure 1: Indicatrixes of space F^2 at A equals to 0.00001 and different B: time-like, B equals to (a) 10, (b) 1, (c) 0.1; and space-like, B equals to (A) 10, (B) 1, (C) 0.1 (D) -0.1, (E) -1.

The metric F^2 is a pseudo-Finsler type metric and, when electrocapillary forces are absent ($v \to 0$, $A \to 0$), the space turns to a Minkowski space.

Let consider the unit sphere of the Finsler space (4) (indicatrix). In fig. 1 the time-like and space-like indicatrixes of space are shown. As one can see, the process of monolayer compression leads to distortion of space from the Minkowski type for 2D gas to the one, in which particle motion is significantly limited that corresponds to condensed monolayer state. Because of this, Finsler space can describe a phase transition in Langmuir monolayer. From this point of view, the study of indicatrix distortion can reveal the properties of phase transition in monolayer.

3 Distortion of monolayer space under the action of electrocapillary forces

According [11] the indicatrix of Finsler space $F(x, y)$ can be viewed as a color pattern on vector space T_xM, then Cartan torsion is the rate of tangential change of the color pattern at y, $y \in T_xM$. At that, F is Euclidean if and only if Cartan torsion is equal to zero.

Another quantities that distinguish Euclidean spaces from Finsler one is distortion and S-curvature. Distortion τ is defined as:

$$\tau := \ln \frac{\sqrt{det\,(g_{ij})}}{\sigma_F} \tag{5}$$

where

$$\sigma_F := \frac{Vol(B^n)}{Vol\{(y^i)\epsilon R^n \mid F(y^i b_i) < 1\}} \tag{6}$$

$Vol\{(y^i)\epsilon R^n \mid F(y^i b_i) < 1\}$ is a volume of indicatrix V:

$$V = \int_{-\infty}^{\infty} \int_{-\tilde{r}}^{\tilde{r}} \int_{\dot{\xi}_0}^{\dot{\xi}_{ind}} d\dot{\varphi} d\dot{r} d\dot{\xi} \tag{7}$$

where $\dot{\xi}_0$ and $\dot{\xi}_{ind}$ are solutions of equations

$$A\frac{\dot{\xi}_0^3}{\dot{r}} + B\dot{\xi}_0^2 - C(\dot{r}^2 + r^2\dot{\varphi}^2) = 0 \tag{8}$$

$$A\frac{\dot{\xi}_{ind}^3}{\dot{r}} + B\dot{\xi}_{ind}^2 - C(\dot{r}^2 + r^2\dot{\varphi}^2) = 1. \tag{9}$$

In the explicit form $\dot{\xi}_0$ and $\dot{\xi}_{ind}$ equal to

$$\dot{\xi}_{ind(0)} = \frac{1}{6A}\left(-2B\dot{r} + \frac{4B^2\dot{r}^2}{X_{ind(0)}} + X_{ind(0)}\right) \tag{10}$$

$$X_{ind} = 2^{2/3}\left[-2B^3\dot{r}^3 + 27A^2\dot{r}\left(1 + C\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\right)\right.$$
$$\left.+\sqrt{-4B^6\dot{r}^6 + \left(2B^3\dot{r}^3 - 27A^2\dot{r}\left(1 + C\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\right)\right)}\right]^{1/3} \tag{11}$$

$$X_0 = 2^{2/3}\left[-2B^3\dot{r}^3 + 27A^2\dot{r}C\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \sqrt{-4B^6\dot{r}^6 + \left(2B^3\dot{r}^3 - 27A^2\dot{r}C\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\right)}\right]^{1/3} \tag{12}$$

Then the volume integral can be divided into a sum of four integrals

$$V = \frac{1}{6A}\left[\int_{-\infty}^{\infty}\int_{-\tilde{r}}^{\tilde{r}}\frac{4B^2\dot{r}^2}{X_{ind}}d\dot{\varphi}d\dot{r} + \int_{-\infty}^{\infty}\int_{-\tilde{r}}^{\tilde{r}}X_{ind}d\dot{\varphi}d\dot{r} - \int_{-\infty}^{\infty}\int_{-\tilde{r}}^{\tilde{r}}\frac{4B^2\dot{r}^2}{X_0}d\dot{\varphi}d\dot{r} - \int_{-\infty}^{\infty}\int_{-\tilde{r}}^{\tilde{r}}X_0 d\dot{\varphi}d\dot{r}\right] \tag{13}$$

Let denote them as Y_1, Y_2, Y_3 and Y_4, respectively.

Using the following step-by-step transformations

$$r_1 = 1 + C(\dot{r}^2 + r^2\dot{\varphi}^2) \tag{14}$$

$$r_2 = \frac{27A^2Cr_1}{r1 - (1 + Cr^2\dot{\varphi}^2)} \tag{15}$$

$$r_3^2 = \frac{r_2}{r_2 - 4B^3}, \tag{16}$$

one can transform the integrals to the form that can be calculated analytically:

$$Y_1 = 6B^4\int_{-\infty}^{\infty}(1 + Cr^2\dot{\varphi}^2)d\dot{\varphi}\int_1^{\infty}\frac{r_3}{\left[(4B^2 - 27A^2C)r_3^2 + 27A^2C\right]^2}\left(\frac{r_3 - 1}{r_3 + 1}\right)^{1/3}dr_3 \tag{17}$$

$$Y_2 = 6B^4\int_{-\infty}^{\infty}(1 + Cr^2\dot{\varphi}^2)d\dot{\varphi}\int_1^{\infty}\frac{r_3}{\left[(4B^2 - 27A^2C)r_3^2 + 27A^2C\right]^2}\left(\frac{r_3 + 1}{r_3 - 1}\right)^{1/3}dr_3 \tag{18}$$

The unit ball B^n corresponds to the indicatrix of a Minkowski space.

In spite of the fact that the volume of indicatrix of Minkowski space and the volume of monolayer space are infinite, the ratio of these volumes is finite.

Example 1. Let consider the particular case for $A = \frac{2}{\sqrt{27}}$, $B = C = 1$ and $r = 0$. Then the metric function F has a form:

$$F^2 = \frac{2}{\sqrt{27}}\frac{\dot{\xi}^3}{\dot{r}} + \dot{\xi}^2 - \dot{r}^2 \tag{19}$$

Fig. 2 represents the indicatrixes and zero-distant curves for this Finsler space (19) and the Minkowski space. As one can see, cubic term in metric function turn to decrease of indicatrix volume due to its distortion.

It can be found that for this particular case

$$\sigma_F = \frac{\sqrt{3}}{2}\frac{\gamma(\frac{1}{3})\gamma(\frac{2}{3})}{2\gamma(\frac{2}{3})\gamma(\frac{4}{3}) + \gamma(\frac{1}{3})\gamma(\frac{5}{3})} \approx 1.29904 \tag{20}$$

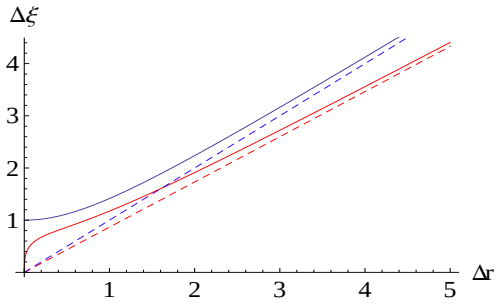


Figure 2: The indicatrixes (solid lines) and zero-distant curves (dashed lines) for Minkowski space (blue) and Finsler space (19)(red).

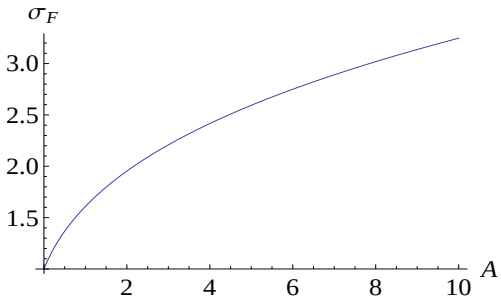


Figure 3: The dependence of σ_F on A.

Example 2. Let consider the particular case for $B = C = 1$ and $r = 0$. Then the metric function F has a form:

$$F^2 = A\frac{\dot{\xi}^3}{\dot{r}} + \dot{\xi}^2 - \dot{r}^2 \tag{21}$$

Denote $27A \equiv a$. Then

$$Y_1 = \int \frac{2^{1/3}\dot{r}^2}{\left[-2\dot{r}^3 + a\dot{r}\left(1+\dot{r}^2\right) + \sqrt{a\dot{r}^2\left(1+\dot{r}^2\right)\left(-4\dot{r}^3 + a\dot{r}\left(1+\dot{r}^2\right)\right)}\right]^{1/3}} d\dot{r} =$$
$$\int \frac{4az}{\left((4-a)z^2+a\right)^2}\left(\frac{z-1}{z+1}\right)^{1/3} dz = \frac{a}{2}\left[\frac{z^2-1}{(4-a)z^2+a}\left(\frac{z-1}{z+1}\right)^{1/3} - \right.$$
$$\frac{1}{\sqrt{a(a-4)}}\left(\left(\frac{\sqrt{a}+\sqrt{a-4}}{2}\right)^{2/3}\ln\left[\left(\frac{\sqrt{a}+\sqrt{a-4}}{2}\right)^{2/3} - \left(\frac{z-1}{z+1}\right)^{1/3}\right] - \right.$$
$$\left.\left.\left(\frac{\sqrt{a}-\sqrt{a-4}}{2}\right)^{2/3}\ln\left[\left(\frac{\sqrt{a}-\sqrt{a-4}}{2}\right)^{2/3} - \left(\frac{z-1}{z+1}\right)^{1/3}\right]\right)\right]_{z\to\infty} \tag{22}$$

And similar expressions can be obtained for Y_2, Y_3 and Y_4.

In fig. 3 the results on numerical calculation of σ_F in dependence on value of A are shown. One can see that increase of A leads to rise σ_F. As obvious, at $A \to 0$, σ_F turns to zero too.

However, in the general case the problem to find σ_F is too difficult. Because of this, instead of distortion τ one can consider the mean Cartan tensor I_i defined as

$$I_i = \tau_{y^i} = \frac{\partial}{\partial y^i}\left[\ln\sqrt{det\left(g_{jk}\left(y\right)\right)}\right] = \frac{1}{2}g^{jk}\frac{\partial g_{jk}}{\partial y^i} = g^{jk}C_{ijk} \tag{23}$$

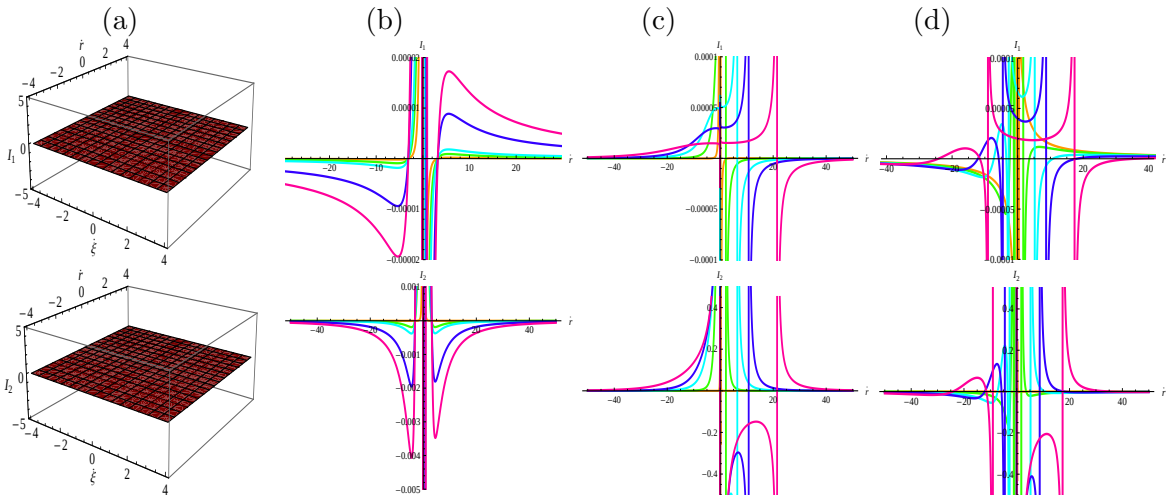


Figure 4: The dependencies of mean Cartan tensor time (I_1) and spatial (I_2) components on displacement of the particle in the monolayer obtained by the method of numerical modeling, for different monolayer space parameters: (a) A=0, (b) B=10^{-5} A=10^{-11} (orange), $5\cdot10^{-11}$(green), 10^{-10} (light blue), $5\cdot10^{-10}$ (blue), 10^{-9} (red), (c) B=10^{-5} A=10^{-11} $\dot{\xi}=10^3$ (orange), 10^4 (green), $3\cdot10^4$ (light blue), $5\cdot10^4$ (blue), 10^5 (red), (d) B=10^{-7} A=10^{-11} $\dot{\xi}=10^3$ (orange), 10^4 (green), $3\cdot10^4$ (light blue), $5\cdot10^4$ (blue), 10^5 (red)

The mean Cartan tensor takes a following form:

$$I_i = \begin{pmatrix} \frac{3A\left(2A\dot{\xi}^3+2B\dot{r}\dot{\xi}^2-C\dot{r}^3\right)}{3A^2\dot{\xi}^4+A\left(4B\dot{r}\dot{\xi}^3-6C\dot{r}^3\dot{\xi}\right)-2BC\dot{r}^4} \\ \frac{3A\dot{\xi}\left(-2A\dot{\xi}^3-2B\dot{r}\dot{\xi}^2+C\dot{r}^3\right)}{\dot{r}\left(3A^2\dot{\xi}^4+A\left(4B\dot{r}\dot{\xi}^3-6C\dot{r}^3\dot{\xi}\right)-2BC\dot{r}^4\right)} \\ 0 \end{pmatrix} \tag{24}$$

for Finsler space with norm

$$F^2 = A\frac{\dot{\xi}^3}{\dot{r}} + B\dot{\xi}^2 - C(\dot{r}^2 + r^2\dot{\varphi}^2). \tag{25}$$

In fig. 4 the dependencies of I_i components on displacement of the particle in the monolayer obtained by the method of numerical calculation are shown for different monolayer space parameters. One can see, that at very small compression speed, the particles of monolayer move in the Minkowski space-time with vanishing distortion that corresponds to white color. Accordingly performed simulations, there is a range of particle velocity values at that the time and space components of distortion are different from zero. This proves the emergence of space-time structures at Langmuir monolayer compression.

4 S-curvature of monolayer space under the action of electrocapillary forces

S-curvature is the rate of change of the distortion along geodesics:

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}\left(\ln \sigma_F(x)\right). \tag{26}$$

where G^i are spray components for function F^2 (4).

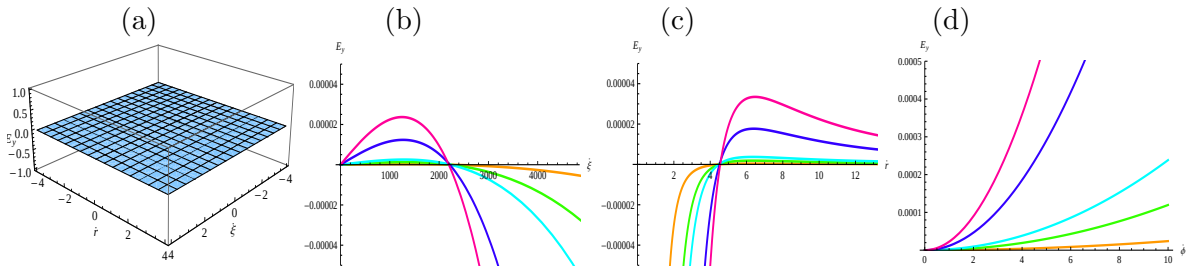


Figure 5: The dependencies of mean Berwald curvature on displacement of the particle in the monolayer obtained by the method of numerical modeling, for different monolayer space parameters: (a) A=0, (b-d) B=10^{-5} A=10^{-11} (orange), $5 \cdot 10^{-11}$(green), 10^{-10} (light blue), $5 \cdot 10^{-10}$ (blue), 10^{-9} (red), (b) $\dot{\varphi} = 1$ $\dot{r} = 10$ (c) $\dot{\varphi} = 1$ $\dot{\xi} = 1000$, (d) $\dot{\xi} = 1000$ $\dot{r} = 10$

Due to the problem of determination $\sigma_F(x)$, we can consider E-tensor:

$$E_{ij} := \frac{1}{2}\frac{\partial^2 S}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left[\frac{\partial G^m}{\partial y^m}\right], \tag{27}$$

which is associated with the S-curvature. Based on E-tensor one can defined E-curvature or the mean Berwald curvature:

$$E_y(u,v) := E_{ij}(x,y)u^i v^i \tag{28}$$

where $u, v \epsilon T_x M$

Let choose $u = (0, \dot{r}, 0)$ and $v = (0, 0, \dot{\varphi})$, then the mean Berwald curvature is equal to

$$E_y = \frac{3AC\dot{r}^2\dot{r}^2\dot{\xi}\dot{\phi}^2}{\left(3A^2\dot{\xi}^4 + A\left(4B\dot{r}\dot{\xi}^3 - 6C\dot{r}^3\dot{\xi}\right) - 2BC\dot{r}^4\right)^3}$$
$$\left(36A^4\dot{\xi}^8 + A^3\left(90B\dot{r}\dot{\xi}^7 + 99C\dot{r}^3\dot{\xi}^5\right) + 2A^2\dot{r}^2\dot{\xi}^2\left(44B^2\dot{\xi}^4 + 90BC\dot{r}^2\dot{\xi}^2 - 9C^2\dot{r}^4\right)\right.$$
$$\left. + 2AB\dot{r}^3\dot{\xi}\left(16B^2\dot{\xi}^4 + 58BC\dot{r}^2\dot{\xi}^2 - 9C^2\dot{r}^4\right) - 8B^2C\dot{r}^6\left(C\dot{r}^2 - 4B\dot{\xi}^2\right)\right) \tag{29}$$

The dependencies of mean Berwald curvature on space-time displacement of the particle in the monolayer at different values of A are shown in fig. 5. As follows from the results, the mean Berwald curvature in modulus increases with increase of A that corresponds rise of monolayer compression speed and, consequently, electrocapillary phenomenon contribution. At that, E_y changes a sign at some $\dot{r}$ and $\dot{\xi}$ independently on A and, respectively, on compression speed. In [10] we demonstrated that the Berwald curvature is proportional to compressibility of monolayer and negative values correspond to unstable region of isotherm for the phase transition process. As one can see from fig.5(b), the mean Berwald curvature is not zero for large displacements $\dot{\xi}$. We can assume that the Finsler corrections are not negligible when the phase transition occurs.

5 Conclusion

Accordingly performed simulations, there is a range of particle velocity values at that the time and space components of distortion are different from zero. This proves the emergence of space-time structures at Langmuir monolayer compression. It was shown also that the time component of mean Cartan tensor and the mean Berwald curvature is not zero at large values of displacements and, respectively, the Finsler corrections are significant, when the phase transition occurs.

Acknowledgments. The present work was developed under the auspices of the Grant No. 11962012-BRFFR-RAF14RA-006, within the cooperation framework between the Romanian Academy and the Belarussian Republican Foundation for Fundamental Research.

References

[1] S. Acharya, J.P. Hill, K. Ariga. Soft Langmuir–Blodgett Technique for Hard Nanomaterials, Adv. Mater. **21**, 2959 (2009).

[2] H. Motschmann, H. Möhwald, *Langmuir–Blodgett Films*, Handbook of Applied Surface and Colloid Chemistry (Ed. Krister Holmberg), Ch. **28** (John Wiley & Sons, 2001), 629-648.

[3] S.A. Kane, M. Compton, N. Wilder. Interactions determining the growth of chiral domains in phospholipid monolayers: experimental results and comparison with theory. Langmuir bf 16, 8447 (2000).

[4] H.V. Grushevskaya, G.G. Krylov, N.G. Krylova, I.V. Lipnevich. Modeling of the behavior and statistical analysis of compressibility in the process of Langmuir monolayer structurization. Journal of Physics: Conference Series **643**, 012015 (2015)

[5] D. Andelman, F. Brochard, C. Knobler, F. Rondelez, it Structures and phase transitions in Langmuir monolayers. "Micelles, Membranes, Microemulsions and Monolayers"., (Heidelberg, Springer, 1994). Pp.559 –602.

[6] D. Orsi, G. Baldi, P. Cicuta, L. Cristofolini. On the relation between hierarchical morphology and mechanical properties of a colloidal 2D gel system. Colloids and Surfaces A: Physicochemical and Engineering Aspects. **413**, 71-77 (2012).

[7] H.V. Grushevskaya, G. Krylov, V.V. Hrushevsky. Kinetic theory of phase foliation at formation of Langmuir-Blodgett monolayers. J. Nonlin. Phenom. in Complex Sys. **7**, 17–33 (2004).

[8] Grushevskaya H.V., Krylova N.G. Effects of Finsler geometry in physics surface phenomena: case of monolayer systems. Hypercomplex Numbers in Geom. and Phys. **8**, 128 (2011).

[9] M. Neagu, N.G. Krylova, H.V. Grushevskaya. Jet theoretical Yang-Mills energy in the geometric dynamics of two dimensional monolayer. J. Math. Phys. **54**, 031508 (2013).

[10] V. Balan, G. Grushevskaya, N. Krylova, M. Neagu, A. Oana. On the Berwald-Lagrange scalar curvature in the structuring process of the LB-monolayer. Applied Sciences. **15**, 30 (2013).

[11] S.S. Chern, Z. Shen, *Riemann-Finsler Geometry. In Nankai Tracts in Mathematics. Vol. 6*, (World Scientific, 2005).

Calculation of inflanton field

M.A. Knyazev
maknyazev@bntu.by; Belarussian National Technical University, Minsk, Belarus

Abstract

The field of inflanton in the inflation model with a single minimally coupled slowly-rolling scalar field is calculated in the lowest-order approximation by using the properties of the Korteweg-de Vries (KdV) equation. Under some conditions the cosmological Friedmann equation may be reduced to the KdV equation. This allows by using the one-soliton solution for the KdV equation to construct an expression for the Hubble parameter then find the scalar field and determine a time of inflation in the model.

1 Introduction

The theory of inflation represents one of the most perspective directions in a studying of the early universe [1]. By using it many problems of the cosmological standard model have been solved. In the inflationary model the early universe was in unstable false-vacuum state and possesses an energy for exponential expansion. Usually this energy is considered as the energy of a self-interactive scalar field which is called inflanton. There are also the models with two and more scalar fields. After expansion the false vacuum state decayed and the universe reheated. Now the main problem of this theory is a prediction of the initial configuration that leads to the observable distribution of CMB [2]. That is why now a determination of this filed is one of the actual problems in cosmology. It may be done by construction the new solutions for the Friedman equation. To do this not only the physical assumptions may be used. The using of mathematical links between the nonlinear equations are also very fruitful.

Recently a new approach to solve the cosmological equations was developed. In this approach a method based on the Korteweg-de Vries equation (KdV) is used [3]. This equation is an integrable one. It has the multisolitonic solutions and for every of such solution a corresponding conservation law can be constructed. For the travelling wave case the KdV equation may be reduced to the ordinary differential equation of the second order. There is an inflation cosmological model in which the Friedman equation may be presented in the same form. It allows to construct a solitonic solution for this equation and then calculate the field of inflanton.

It should be pointed out that now an alternative approach to the same problem based on the Schrödinger equation (SE) is also developed [4]. According to this approach a studying of the models of inflation with only one scalar field does not require the KdV equation. According to the opinion of the authors of paper [4] this equation may be useful for the models of inflation with multiple scalar fields.

2 Method

To present the method of calculation in a short let us follow to [3]. Consider (1+1)-dimensional the KdV equation in the form

$$u_t + u_{xxx} + \frac{3}{u_0} u u_x = 0, \tag{1}$$

where $u = u(x,t)$, $u_t = \frac{\partial u}{\partial t}$, $u_{xxx} = \frac{\partial^3 u}{\partial t^3}$ and u_0 is a constant. This equation has one solitonic solution of the form

$$u = u_0 \lambda^2 \operatorname{sech}^2[\frac{\lambda(x - \lambda^2 t)}{2}], \tag{2}$$

where λ is a constant corresponding to the wave number in the linear case.

For $\phi = x - \lambda^2 t$ Eq. (1) can be written as the nonlinear ordinary differential equation

$$-\lambda^2 u' + u''' + \frac{3}{u_0} u u' = 0, \tag{3}$$

where a prime denotes $\frac{d}{d\phi}$. For Eq. (3) there is an auto-Bäcklund transformation

$$u = u_0(\lambda^2 - 4y^2), \tag{4}$$

where y is a solution for the the Riccati equation

$$y' = \frac{u}{4u_0} = \frac{\lambda^2}{4} - y^2. \tag{5}$$

This solution also satisfies the equation

$$S[y(\phi)] = \frac{y'''}{y'} - \frac{3(y'')^2}{2(y')^2} = -\frac{\lambda^2}{2}. \tag{6}$$

The left-hand side of Eq. (6) is the Schwarzian derivation [5]. The properties of this operator allow conclude that if $\bar{y}$ is a particular solution for equation $S[y] = f(\phi)$, the general solution for this equation can be written in the form

$$y(\phi) = \frac{a\bar{y} + b}{c\bar{y} + d}, \tag{7}$$

where a, b, c and d are the constants and $ad - bc = 1$. If the general solution satisfies to Eq. (5) then one obtains a solution for the KdV equation.

3 Model

Consider the inflation model with single minimally coupled slowly-rolling scalar field ϕ [6]. Here and in the previous Section a same symbol is used for different variables but their physical meanings are clear. In this model a scale-invariant density perturbation spectral index generates to lowest-order in the slowly-roll approximation. For the standard Friedmann-Robertson-Walker universe the Friedmann equation in Hamilton-Jacobi form may be written as

$$3H^2 - 2(H')^2 = V(\phi), \tag{8}$$

$$\dot{\phi} = -2H', \tag{9}$$

where $H = H(\phi)$ is the Hubble parameter, $V(\phi)$ is the inflation potential. A prime and dot denote differentiation with respect to the inflation field and cosmic time. The inflation varies monotonically with cosmic time and $\dot{\phi} > 0$. The energy density $\rho(\phi) = 3H^2$.

4 Calculation of the field

By applying the method shortly described above in the Section 2 the Friedmann equation can be transformed to the KdV equation of the form [3]

$$-(1-n_s)H^{2\prime}+H^{2\prime\prime\prime}+\frac{3H^2H^{2\prime}}{H_0^2}=0, \tag{10}$$

where n_s is a spectral index and H_0 is a constant. For this equation under some conditions the next solution can be constructed

$$H^2(\phi)=H_0^2\lambda^2\,\mathrm{sech}^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p}), \tag{11}$$

where M_p is the Planck mass. If in Section 2 the parameter λ characterized the velocity of soliton, in the cosmological model under consideration it describes a deviation of the spectral index from the scale invariant Harrison-Zel'dovich value.

Consider the slowly-roll parameters [2]

$$\epsilon=\frac{M_p^2}{4\pi}[\frac{H'}{H}]^2, \tag{12}$$

$$\eta=\frac{M_p^2}{4\pi}\frac{H''}{H}, \tag{13}$$

$$\xi^2=\frac{M_p^4}{16\pi^2}\frac{H'H'''}{H^2}. \tag{14}$$

Now by using the solution determined by Eq. (11) these parameters may be calculated in explicit form

$$\epsilon=\frac{\lambda^2}{2}\tanh^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p}), \tag{15}$$

$$\eta=-\frac{\lambda^2}{2}[1-2\tanh^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p})], \tag{16}$$

$$\xi^2=\frac{\lambda^4}{4}\tanh^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p})[6\tanh^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p})-5]. \tag{17}$$

In [3] the deviation of the spectral index from the scale-invariant Harrison-Zel'dovich value $n_s=1$ follows from the properties of the differential equations and equals to $n_s=1-\lambda^2$. To the lowest-order in the slowly-roll approximation the spectral index takes the form $n_s=1-4\epsilon+2\eta$. Now the result $n_s=1-\lambda^2$ may be obtained by the direct calculation.

Let us take the inflation potential $V(\phi)$ in the form [2]

$$V(\phi)=\frac{(M_p)^2}{8\pi}H^2(3-\epsilon). \tag{18}$$

Then by substituting Eqs. (11) and (18) into the Friedmann equation after some algebra one obtains

$$\frac{8\pi}{M_p^2}\dot{\phi}^2 = H_0^2\lambda^4 \tanh^2(\frac{\lambda\sqrt{8\pi}\phi}{2M_p}), \tag{19}$$

where dot denotes a differentiation on time. By taking a positive square root from the both sides of Eq. (18) and using a new variable $y = \frac{\lambda\sqrt{8\pi}\phi}{2M_p}$ this equation may be transformed into equation of a form

$$\dot{y} = \frac{H_0\lambda^3}{2}\tanh^2(y). \tag{20}$$

After an integration of Eq. (20) [7] and return back to the initial variable ϕ one obtains for the inflanton field

$$\phi = \frac{2C_0M_p}{\lambda\sqrt{8\pi}}\operatorname{arcsinh}(\exp(H_0\lambda^3 t)), \tag{21}$$

where C_0 is a constant of integration. This result describes an expansion and in general agrees with a behavior of the scalar field in inflation theory. Nevertheless Eq. (21) describes a more slow process of expansion than exponential one. At the same time for this regime a transition to the next stage of evolution of the universe coming after the inflation is more simple. A possible way to determine the constant C_0 may be connected with a consideration of the initial configuration that leads to the modern distribution of CMB.

Now a time of inflation can be evaluated. In the lowest-order approximation the inflation ends when the condition $\epsilon = 1$ is fulfilled [2]. Let us substitute Eqs. (15) and (21) into this condition. One obtains

$$\frac{\lambda^2}{2}\tanh^2(C_0\operatorname{arcsinh}(\exp(H_0\lambda^3 t))) = 1. \tag{22}$$

Then from Eq. (22) the next equation for t follows

$$t = \frac{1}{H_0\lambda^3}\ln(\sinh(\frac{\operatorname{arctanh}\frac{\sqrt{2}}{\lambda}}{C_0})). \tag{23}$$

It is possible conclude from the last equation that in the considered approximation a time of inflation determines mainly by the deviation of the spectral index from the scale-invariant value. But the initial data should be accounted also. As a further step to calculate the time of inflation the next order of approximation should be considered.

References

[1] A. Linde. Particle Physics and Inflationary Cosmology. Harwood, Chur, Switzerland, 1990.

[2] A. Vallinotto, E.J. Copeland, E.W. Kolb, A.R. Liddle and D.A. Steer. Inflationary potentials yelding constant scalar spectral indices. http://xxx.lanl.gov(arXiv:astro-ph/0311005).

[3] J.E. Lidsey. Cosmology and the Korteweg-de Vries equation. http://xxx.lanl.gov(arXiv:astro-ph/1205.5641).

[4] A.V. Yaparova, A.V. Yurov, V.A. Yurov. The KdV in cosmology: a useful tool or a distraction? http://xxx.lanl.gov(arXiv:gr-qc/14056436).

[5] E. Hille. Ordinary Differential equations in the Complex Domain. Wiley, New York, 1976.

[6] D.H. Lyth and A.R. Liddle. The Primordial Density Perturbation. Cambridge University Press, 2009.

[7] A.D. Polyanin, V.F. Zaitsev. Handbook of Exact Solutions for Ordinary Differential Equations. CRC Press, Boca Raton, New York, 1995.

Eigenfunction expansions in the imaginary Lobachevsky space

Yu.A. Kurochkin
yukuroch@bas-net.by; B.I. Stepanov Institute of Physics
V.S. Otchik
B.I. Stepanov Institute of Physics
D. Petrosyan
Joint Institute for Nuclear Research, Dubna, Russia
G.S. Pogosyan
pogosyan@ysu.am; Joint Institute for Nuclear Research, Dubna, Russia
and Yerevan State University, Armenia

Abstract

Eigenfunctions of the Schrödinger equation with the Coulomb potential in the imaginary Lobachevsky space are studied in two coordinate systems admitting solutions in terms of hypergeometric functions. Normalization and coefficients of mutual expansions for some sets of solutions are found.

Geometry of spaces of constant curvature with (pseudo)-orthogonal group of motion provides a natural framework for treatment of numerous physical problems. For example, the velocity space in relativity is the Lobachevsky space, and the imaginary Lobachevsky space corresponds to an unphysical region of momentum variables, which is also of interest in study of scattering processes. Metrics of spaces of constant curvature appear as simplest solutions of Einstein equations and are used as a background in models aimed for study of various effects of curvature in quantum theory. Derivation of eigenfunction expansions is an essential element of such study.

Eigenfunction expansions associated with invariant operators on various hyperboloids in arbitrary dimensions were derived in [1] and in special case of four-dimensional pseudo-Euclidean space in [2]. Authors of these works used the canonical subgroup reduction and separation of variables associated with it. Harmonic analysis in homogeneous spaces based on the methods of integral geometry was developed in [3]. These methods have been applied in [4] to obtain eigenfunction expansions in several coordinate systems in the Lobachevsky space (the upper sheet of a double-sheeted hyperboloid), and also in [5], where separable bases on the one and two sheeted hyperboloids were studied. (In [3, 5] the imaginary Lobachevsky space is defined as the single sheeted hyperboloid with geometrically opposed points identified).

Study of quantum mechanical problems in spaces of constant curvature was initiated by Schrödinger [6], who solved the Kepler–Coulomb problem on a three-dimensional sphere. In the Lobachevsky space, this problem was solved by Infeld and Schild [7]. Since then, a considerable number of works has appeared concerning the Kepler–Coulomb problem in these two spaces (see e.g. [8]–[13] and references therein).

On the other hand, C. Grosche [14] in his development of path integral in the imaginary Lobachevsky space incorporated in it an analog of the Coulomb potential and presented solution of thus formulated Kepler–Coulomb problem in this space. For all three spaces of constant curvature, symmetry of this problem is essentially common [15]. However, much less is known

about eigenfunctions of the problem in the imaginary Lobachevsky space than in two other ones. In this paper we consider the Kepler–Coulomb problem in two coordinate systems of imaginary Lobachevsky space admitting separation of variables and solutions in terms of hypergeometric functions.

In terms of ambient pseudo-Euclidean space, the imaginary Lobachevsky space is the single-sheet hyperboloid

$$x_\mu x_\mu = \mathbf{x}^2 + x_4^2 = \mathbf{x}^2 - x_0^2 = R^2.$$

The Schrödinger equation for the Kepler problem in this space we write as

$$H\Psi = E\Psi, \quad H = -\frac{1}{2}\Delta + V, \tag{1}$$

where Δ is the Laplace operator on the hyperboloid, and the Coulomb potential V is

$$V = -\frac{\alpha}{R}\frac{x_0}{|\mathbf{x}|} \qquad (\alpha > 0).$$

The spherical coordinate system in the imaginary Lobachevsky space can be introduced by the relations

$$\begin{gathered} x_0 = R\sinh\tau,\ x_1 = R\cosh\tau\sin\theta\cos\phi,\ x_2 = R\cosh\tau\sin\theta\sin\phi,\ x_3 = R\cosh\tau\cos\theta, \\ -\infty < \tau < \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi. \end{gathered} \tag{2}$$

In the coordinates τ, θ, ϕ, the line element is $ds^2 = R^2[d\tau^2 - \cosh^2\tau(d\theta^2 + \sin^2\theta d\phi^2)]$, and the Hamiltonian (1) is

$$H = -\frac{1}{2R^2}\left[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} - \frac{1}{\cosh^2\tau}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right] - \frac{\alpha}{R}\tanh\tau. \tag{3}$$

Writing the wave function in the form

$$\Psi = S(\tau)Y_l^m(\theta,\phi), \tag{4}$$

where Y_l^m are spherical harmonics [16], we obtain an equation for the radial function S

$$\frac{d^2S}{d\tau^2} + 2\tanh\tau\frac{dS}{d\tau} + \left[\frac{l(l+1)}{\cosh\tau^2} + 2\alpha R\tanh\tau + 2ER^2\right]S = 0. \tag{5}$$

Denoting $z = (1+\tanh\tau)/2 = (1+e^{-2\tau})^{-1}$, we get from (5) equation

$$\frac{d^2S}{dz^2} + \frac{1}{z(z-1)}\left[-l(l+1) + \frac{ER^2+\alpha R}{2(z-1)} - \frac{ER^2-\alpha R}{2z}\right]S = 0, \tag{6}$$

which can be solved in terms of hypergeometric functions. Setting further

$$S(\tau) = (\cosh\tau)^{-1}f(\tau), \tag{7}$$

we arrive at equation

$$\frac{d^2f}{d\tau^2} + \left[\frac{l(l+1)}{\cosh\tau^2} + 2\alpha R(\tanh\tau - 1) + \Lambda\right]f = 0, \tag{8}$$

where $\Lambda = 2ER^2 + 2\alpha R - 1$. Equation (8) has the form of the one-dimensional Schrödinger equation with the Rosen–Morse potential [17]. For the discrete energy states, normalization of wave functions for this potential was accomplised in the work [18]. Here we use the method

of Titchmarsh [19], which allows one to obtain normalization constants for both discrete and continuous states. From now on we consider Λ as a complex variable. Let $f_1(\tau)$ and $f_2(\tau)$ be two linearly independent solutions of equation (8) such that for $\mathrm{Im}\Lambda > 0$

$$f_1(\tau) \in L^2(-\infty, 0), \quad f_2(\tau) \in L^2(0, +\infty), \tag{9}$$

where $L^2(a, b)$ denotes the space of square integrable functions on the interval (a, b). Consider now a function

$$\Phi(\tau, \Lambda) = \frac{f_2(\tau)}{W(f_1, f_2)} \int_{-\infty}^{\tau} f_1(\tau')\varphi(\tau')d\tau' + \frac{f_1(\tau)}{W(f_1, f_2)} \int_{\tau}^{\infty} f_2(\tau')\varphi(\tau')d\tau'. \tag{10}$$

Here $\varphi(\tau)$ is an arbitrary function square integrable on $(-\infty, +\infty)$ and $W(f_1, f_2)$ is the Wronskian

$$W(f_1, f_2) = f_1 \frac{df_2}{d\tau} - f_2 \frac{df_1}{d\tau}.$$

Then the eigenfunction expansion of $\varphi(\tau)$ associated with equation (8) is

$$\varphi(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} [-\mathrm{Im}\Phi(\tau, \Lambda)]d\Lambda + \sum_n \mathrm{Res}\, \Phi(\tau, \Lambda_n), \tag{11}$$

where $\mathrm{Res}\, \Phi(\tau, \Lambda_n)$ are residues of $\Phi(\tau, \Lambda)$ in the complex Λ plane.

First we consider the case of free motion $\alpha = 0$. In this case, the equation (8) reduces to

$$\frac{d^2 f}{d\tau^2} + \left[\frac{l(l+1)}{\cosh \tau^2} + \Lambda\right] f = 0, \tag{12}$$

with $\Lambda = 2ER^2 - 1$. Consider two solutions of (12) satisfying conditions (9)

$$f_1(\tau) = (2\cosh\tau)^{ip} {}_2F_1\left(l + 1 - ip, -l - ip; 1 - ip; (1 + e^{-2\tau})^{-1}\right), \tag{13}$$

$$f_2(\tau) = (2\cosh\tau)^{ip} {}_2F_1\left(l + 1 - ip, -l - ip; 1 - ip; (1 + e^{2\tau})^{-1}\right), \tag{14}$$

where $p = \sqrt{\Lambda}$. The Wronskian of solutions (13) and (14) is

$$W(f_1, f_2) = -\frac{2\left[\Gamma(1 - ip)\right]^2}{\Gamma(l + 1 - ip)\,\Gamma(-l - ip)}. \tag{15}$$

By using properties of hypergeometric functions [20] one can easily check that

$$f_1^*(\tau) = f_1(\tau), \quad f_2^*(\tau) = f_2(\tau) \quad \text{for } \Lambda < 0; \tag{16}$$

$$\begin{aligned} f_1^*(\tau) &= \frac{\Gamma(l + 1 - ip)\,\Gamma(-l - ip)}{\Gamma(1 - ip)\,\Gamma(-ip)} f_2(\tau), \\ f_2^*(\tau) &= \frac{\Gamma(l + 1 - ip)\,\Gamma(-l - ip)}{\Gamma(1 - ip)\,\Gamma(-ip)} f_1(\tau) \end{aligned} \quad \text{for } \Lambda > 0. \tag{17}$$

Using equations (15) – (17) we find the discontinuity of the function $\Phi(\tau, \Lambda)$ (10) on the real axis of complex Λ plane:

$$\mathrm{Im}\Phi(\tau, \Lambda) = 0 \quad \text{for } \Lambda < 0; \tag{18}$$

$$\mathrm{Im}\Phi(\tau, \Lambda) = -\frac{1}{4p}\left[f_1^*(\tau) \int_{-\infty}^{\infty} f_1(\tau')\varphi(\tau')d\tau' + f_2^*(\tau) \int_{-\infty}^{\infty} f_2(\tau')\varphi(\tau')d\tau'\right] \quad \text{for } \Lambda > 0. \tag{19}$$

The function $\Phi(\tau, \Lambda)$ has poles at the points $-i\sqrt{\Lambda} = n$, where $n = 1, 2, \ldots, l$. Since $f_1(\tau) = (-1)^{l-n} f_2(\tau)$ for these values of Λ, the residues of $\Phi(\tau, \Lambda)$ are found to be

$$\mathrm{Res}\, \Phi(\tau, \Lambda_n) = \frac{(l + n)!\, n}{(l - n)!(n!)^2} f_{1n}(\tau) \int_{-\infty}^{\infty} f_{1n}(\tau')\varphi(\tau')d\tau', \tag{20}$$

where

$$f_{1n}(\tau) = (2\cosh\tau)^{-n}\,{}_2F_1\left(n+l+1, n-l; n+1;\, (1+e^{-2\tau})^{-1}\right). \tag{21}$$

Thus eigenfunction expansion (11) of an arbitrary function φ takes in case of equation (12) the form

$$\begin{aligned}\varphi(\tau) = \frac{1}{4\pi}\int_0^\infty \frac{d\Lambda}{\sqrt{\Lambda}}\left[f_1^*(\tau)\int_{-\infty}^{\infty} f_1(\tau')\varphi(\tau')d\tau' + f_2^*(\tau)\int_{-\infty}^{\infty} f_2(\tau')\varphi(\tau')d\tau'\right] \\ +\sum_{n=1}^{l}\frac{(l+n)!n}{(l-n)!(n!)^2} f_{1n}(\tau)\int_{-\infty}^{\infty} f_{1n}(\tau')\varphi(\tau')d\tau'.\end{aligned} \tag{22}$$

Expressions for normalized eigenfunctions of equation (12) can be easily deduced from the expansion (22).

Now we return to the equation (8) with $\alpha > 0$. Introducing notations

$$p = \sqrt{\Lambda}, \quad q = \sqrt{\Lambda - 4\alpha R}, \quad \nu = -\frac{i}{2}(p+q), \tag{23}$$

we can write solutions of (8) satisfying conditions (9) as follows:

$$f_1(\tau) = (1+e^{-2\tau})^{iq/2}(1+e^{2\tau})^{ip/2}\,{}_2F_1\left(\nu+l+1, \nu-l; 1-iq;\, (1+e^{-2\tau})^{-1}\right), \tag{24}$$

$$f_2(\tau) = (1+e^{-2\tau})^{iq/2}(1+e^{2\tau})^{ip/2}\,{}_2F_1\left(\nu+l+1, \nu-l; 1-ip;\, (1+e^{2\tau})^{-1}\right). \tag{25}$$

Wronskian of the solutions (24) and (25) is

$$W(f_1, f_2) = -\frac{2\Gamma(1-ip)\,\Gamma(1-iq)}{\Gamma(\nu+l+1)\,\Gamma(\nu-l)}. \tag{26}$$

Denoting $\sqrt{4\alpha R - \Lambda} = r$ for $\Lambda < 4\alpha R$ and using relations between solutions of hypergeometric equation we find

$$f_1^*(\tau) = f_1(\tau), \quad f_2^*(\tau) = f_2(\tau) \quad \text{for } \Lambda < 0; \tag{27}$$

$$\begin{aligned}f_1^*(\tau) = f_1(\tau) = \frac{\Gamma(ip)\,\Gamma(1+r)\,f_2(\tau)}{\Gamma\left(l+1+\frac{ip}{2}+\frac{r}{2}\right)\Gamma\left(-l+\frac{ip}{2}+\frac{r}{2}\right)} \\ +\frac{\Gamma(-ip)\,\Gamma(1+r)\,f_2^*(\tau)}{\Gamma\left(l+1-\frac{ip}{2}+\frac{r}{2}\right)\Gamma\left(-l-\frac{ip}{2}+\frac{r}{2}\right)} \quad \text{for } 0 < \Lambda < 4\alpha R;\end{aligned} \tag{28}$$

$$\begin{aligned}f_1(\tau) = \frac{\Gamma(ip)\,\Gamma(1-iq)\,f_2(\tau)}{\Gamma\left(l+1+\frac{ip}{2}-\frac{iq}{2}\right)\Gamma\left(-l+\frac{ip}{2}-\frac{iq}{2}\right)} + \frac{\Gamma(-ip)\,\Gamma(1-iq)\,f_2^*(\tau)}{\Gamma\left(l+1-\frac{ip}{2}-\frac{iq}{2}\right)\Gamma\left(-l-\frac{ip}{2}-\frac{iq}{2}\right)}, \\ f_2(\tau) = \frac{\Gamma(1-ip)\,\Gamma(iq)\,f_1(\tau)}{\Gamma\left(l+1-\frac{ip}{2}+\frac{iq}{2}\right)\Gamma\left(-l-\frac{ip}{2}+\frac{iq}{2}\right)} + \frac{\Gamma(1-ip)\,\Gamma(-iq)\,f_1^*(\tau)}{\Gamma\left(l+1-\frac{ip}{2}-\frac{iq}{2}\right)\Gamma\left(-l-\frac{ip}{2}-\frac{iq}{2}\right)} \\ \text{for } \Lambda > 4\alpha R.\end{aligned} \tag{29}$$

By use of equations (26) – (29) we obtain:

$$\mathrm{Im}\Phi(\tau, \Lambda) = 0 \quad \text{for } \Lambda < 0; \tag{30}$$

$$\begin{aligned}\mathrm{Im}\Phi(\tau, \Lambda) = -\frac{\sinh\pi p\,\left|l+1-\frac{ip}{2}+\frac{r}{2}\right|^2\left|-l-\frac{ip}{2}+\frac{r}{2}\right|^2}{4\pi\left[\Gamma\left(1+\frac{r}{2}\right)\right]^2} f_1(\tau)\int_{-\infty}^{\infty} f_1(\tau')\varphi(\tau')d\tau' \\ \text{for } 0 < \Lambda < 4\alpha R;\end{aligned} \tag{31}$$

$$\mathrm{Im}\Phi(\tau,\Lambda)=-\frac{\sinh\pi p\,\sinh\pi q}{4\sinh^2\frac{\pi}{2}(p+q)}\left[\frac{1}{q}f_1^*(\tau)\int_{-\infty}^{\infty}f_1(\tau')\varphi(\tau')d\tau'+\frac{1}{p}f_2^*(\tau)\int_{-\infty}^{\infty}f_2(\tau')\varphi(\tau')d\tau'\right] \quad (32)$$
$$\text{for}\,\Lambda>4\alpha R.$$

The function $\Phi(\tau,\Lambda)$ (10) has poles at the points

$$\nu=-i\left(\sqrt{\Lambda}+\sqrt{\Lambda-4\alpha R}\right)/2=n, \quad (33)$$

where $n=\delta,\,\delta+1,\ldots,l$, and $\delta=\max\{[\sqrt{\alpha R}],\,1\}$. For these values of ν we have

$$f_2(\tau)=(-1)^{n-l}\frac{\Gamma(n-\sigma+1)\Gamma(l+\sigma+1)}{\Gamma(n+\sigma+1)\Gamma(l-\sigma+1)}f_1(\tau),\quad \sigma=\frac{\alpha R}{n}.$$

Thus we find

$$\mathrm{Res}\Phi(\tau,\Lambda_n)=\frac{(l+n)!\Gamma(l+\sigma+1)(n^2-\sigma^2)}{(l-n)!\Gamma(l-\sigma+1)[\Gamma(n+\sigma+1)]^2 n}f_{1n}(\tau)\int_{-\infty}^{\infty}f_{1n}(\tau')\varphi(\tau')d\tau', \quad (34)$$

where

$$f_{1n}(\tau)=e^{\sigma\tau}(2\cosh\tau)^{-n}{}_2F_1\left(n+l+1,n-l;n+\sigma+1;\,(1+e^{-2\tau})^{-1}\right). \quad (35)$$

From equation (33) follows expression for discrete energy levels

$$E_n=-\frac{\alpha^2}{2n^2}-\frac{n^2-1}{2R^2}. \quad (36)$$

Now we can write the eigenfunction expansion associated with the equation (8) for an arbitrary function $\varphi(\tau)$

$$\begin{aligned}\varphi(\tau)=&\int_0^{4\alpha R}d\Lambda\frac{\sinh\pi p\,\left|l+1-\frac{ip}{2}+\frac{r}{2}\right|^2\left|-l-\frac{ip}{2}+\frac{r}{2}\right|^2}{4\pi^2\left[\Gamma\left(1+\frac{r}{2}\right)\right]^2}f_1(\tau)\int_{-\infty}^{\infty}f_1(\tau')\varphi(\tau')d\tau'\\
&+\frac{1}{4\pi}\int_{4\alpha R}^{\infty}d\Lambda\frac{\sinh\pi p\,\sinh\pi q}{\sinh^2\frac{\pi}{2}(p+q)}\left[\frac{1}{q}f_1^*(\tau)\int_{-\infty}^{\infty}f_1(\tau')\varphi(\tau')d\tau'+\frac{1}{p}f_2^*(\tau)\int_{-\infty}^{\infty}f_2(\tau')\varphi(\tau')d\tau'\right] \quad (37)\\
&+\sum_{n=\delta}^{l}\frac{(l+n)!\Gamma(l+\sigma+1)(n^2-\sigma^2)}{(l-n)!\Gamma(l-\sigma+1)[\Gamma(n+\sigma+1)]^2 n}f_{1n}(\tau)\int_{-\infty}^{\infty}f_{1n}(\tau')\varphi(\tau')d\tau'.\end{aligned}$$

Note that only $f_1(\tau)$ enters the expansion for the interval $0<\Lambda<4\alpha R$ of the continuous spectrum. We can see from (24) and (37) that the normalized eigenfunction for this interval can be written as

$${}^0f_\Lambda(\tau)={}^0N_\Lambda(1+e^{-2\tau})^{-r/2}(1+e^{2\tau})^{ip/2}{}_2F_1\left(\nu+l+1,\nu-l;1+r;\,(1+e^{-2\tau})^{-1}\right), \quad (38)$$

where $\nu=(r-ip)/2$, and

$${}^0N_\Lambda=\frac{\sqrt{\sinh\pi p}\left|l+1-\frac{ip}{2}+\frac{r}{2}\right|\left|-l-\frac{ip}{2}+\frac{r}{2}\right|}{2\pi\Gamma\left(1+\frac{r}{2}\right)}.$$

For $\Lambda>4\alpha R$, two eigenfunctions enter the expansion (37). The normalized functions are

$${}^1f_\Lambda(\tau)={}^1N_\Lambda(1+e^{-2\tau})^{iq/2}(1+e^{2\tau})^{ip/2}{}_2F_1\left(\nu+l+1,\nu-l;1-iq;\,(1+e^{-2\tau})^{-1}\right), \quad (39)$$

$${}^2f_\Lambda(\tau)={}^2N_\Lambda(1+e^{-2\tau})^{iq/2}(1+e^{2\tau})^{ip/2}{}_2F_1\left(\nu+l+1,\nu-l;1-ip;\,(1+e^{2\tau})^{-1}\right), \quad (40)$$

where

$$ {}^1N_\Lambda = \frac{\sqrt{\sinh \pi p \, \sinh \pi q}}{2\sqrt{\pi q}\, \sinh \frac{\pi}{2}(p+q)}, \quad {}^2N_\Lambda = \frac{\sqrt{\sinh \pi p \, \sinh \pi q}}{2\sqrt{\pi p}\, \sinh \frac{\pi}{2}(p+q)}. $$

Normalized eigenfunctions for discrete states are

$$ f_{nl}(\tau) = N_{nl} e^{\sigma\tau} (2\cosh\tau)^{-n} {}_2F_1\left(n+l+1, n-l; n+\sigma+1; (1+e^{-2\tau})^{-1}\right), \tag{41} $$

where

$$ N_{nl} = \frac{1}{\Gamma(n+\sigma+1)} \sqrt{\frac{(n^2-\sigma^2)(n+l)!\Gamma(l+\sigma+1)}{n(l-n)!\Gamma(l-\sigma+1)}}. $$

Functions (38) – (41) satisfy orthogonality relations

$$ \int_{-\infty}^{\infty} {}^0f_\Lambda(\tau)\, {}^0f_{\Lambda'}(\tau) d\tau = \delta(\Lambda - \Lambda'), \quad 0 < \Lambda < 4\alpha R; \tag{42} $$

$$ \int_{-\infty}^{\infty} {}^if^*_\Lambda(\tau)\, {}^jf_{\Lambda'}(\tau) d\tau = \delta(\Lambda - \Lambda')\delta_{ij}, \quad i,j = 1,2;\ \Lambda > 4\alpha R; \tag{43} $$

$$ \int_{-\infty}^{\infty} f_{nl}(\tau) f_{n'l}(\tau) d\tau = \delta_{nn'}. \tag{44} $$

Taking into account expressions (4) and (7) for wave functions $\Psi(\tau, \theta, \phi)$, we can now easily obtain solutions of the Schrödinger equation (1) normalized with respect to the scalar product

$$ (\Psi_1,\, \Psi_2) = \iiint\limits_V \Psi_1^* \Psi_2 dV = \iiint\limits_V \Psi_1^* \Psi_2 R^3 \cosh^2\tau \sin\theta d\tau d\theta d\phi, \tag{45} $$

where integration is taken over the whole hyperboloid. In particular, the wave functions of discrete states $\Psi_{nlm}(\tau,\, \theta,\, \phi)$ normalized by the condition $(\Psi_{nlm},\, \Psi_{nlm}) = 1$, have the form

$$ \Psi_{nlm}(\tau,\, \theta,\, \phi) = N_{nl} R^{-3/2} (\cosh\tau)^{-1} f_{nl}(\tau) Y_{lm}(\theta, \phi). \tag{46} $$

In the imaginary as well as in real Lobachevsky space there exists, besides spherical coordinates, one more separable coordinate system, in which solutions of Schrödinger equation with the Coulomb potential can be expressed through hypergeometric functions. In the case of Lobachevsky space, such solutions were studied in [10]. The corresponding coordinate system from Olevskiĭ's list [21] was named in [12] elliptic-parabolic. Solutions of the Kepler–Coulomb problem in a similar system in the imaginary Lobachevsky space were considered in [22]. (Eigenfunctions of the Laplace operator in the imaginary Lobachevsky space in a variety of coordinate systems were found in [5]). Now we proceed to find normalization coefficients for some sets of eigenfunctions. We define an analog of elliptic-parabolic coordinate system with the help of relations

$$ t_1 = \frac{|\mathbf{x}| - x_3}{|\mathbf{x}| + x_0},\ t_2 = \frac{|\mathbf{x}| + x_3}{|\mathbf{x}| - x_0}. \tag{47} $$

Then for $x_0 + x_3 > 0$ we have $0 \le t_1 < 1$, $t_2 > 1$, and the coordinates of the pseudoeuclidean space are expressed as

$$ x_0 = \frac{R(t_1+t_2-2)}{2\sqrt{(1-t_1)(t_2-1)}},\ x_1 = R\sqrt{t_1 t_2}\cos\phi,\ x_2 = R\sqrt{t_1 t_2}\sin\phi,\ x_3 = \frac{R(t_1+t_2-2t_1t_2)}{2\sqrt{(1-t_1)(t_2-1)}}. \tag{48} $$

It should be noted that coordinates t_1, t_2 only cover one half of the one-sheeted hyperboloid. To cover the other half, one can e.g. change signs of x_0 and x_3 in (48).

In the coordinates t_1, t_2, ϕ, Hamiltonian (1) takes the form

$$ H = \frac{2}{R^2}\left[\frac{1-t_1}{t_2-t_1}\frac{\partial}{\partial t_1} t_1(1-t_1)\frac{\partial}{\partial t_1} + \frac{1-t_2}{t_1-t_2}\frac{\partial}{\partial t_2} t_2(1-t_2)\frac{\partial}{\partial t_2} + \frac{1}{4t_1t_2}\frac{\partial^2}{\partial\phi^2}\right] - \frac{\alpha}{R}\frac{t_1+t_2-2}{t_2-t_1}. \tag{49} $$

Substitution $\Psi = S_1(t_1)S_2(t_2)e^{im\phi}$ separates the variables in the Schrödinger equation and we obtain the equations for S_1 and S_2

$$\frac{d}{dt_1}(t_1-1)t_1\frac{dS_1}{dt_1}+\left[\frac{ER^2-\alpha R}{2}-\frac{m^2}{4t_1}+\frac{A}{4(t_1-1)}\right]S_1=0, \tag{50}$$

$$\frac{d}{dt_2}(t_2-1)t_2\frac{dS_2}{dt_2}+\left[\frac{ER^2+\alpha R}{2}-\frac{m^2}{4t_2}+\frac{A}{4(t_2-1)}\right]S_2=0. \tag{51}$$

Solutions of these equations can be expressed in terms of hypergeometric functions. Spectra of energy E and of the separation constant A should be found by imposing appropriate boundary conditions on these solutions. But this task is complicated by the fact that both E and A enter each of the equations. In the case of free motion, when $\alpha = 0$, one finds that solutions of the equations (50) and (51) cannot be simultaneously finite for any values of E and A. On the other hand, the existence of the discrete energy spectrum (see (36)) was demonstrated by solving the problem in the spherical coordinates. This means that for the discrete values of energy, the separation constant A may have continuous spectrum. Now we will find corresponding wave functions and their normalization coefficients.

Let us denote by $S_{EA}(t)$ a solution of any of the equations (50) and (51) with given values E and A. Then we have

$$\begin{aligned}&\int_a^b\left[2R^2(E-E')+(A-A')(t-1)^{-1}\right]S_{EA}S_{E'A'}dt\\&\quad=-4\left[t(t-1)\left(S_{E'A'}\frac{dS_{EA}}{dt}-S_{EA}\frac{dS_{E'A'}}{dt}\right)\right]\Big|_a^b.\end{aligned} \tag{52}$$

Let $S_{1nk}(t_1)$ be a solution of (50) finite at $t_1 = 0$, and $S_{2nk}(t_2)$ a solution of (51) vanishing for $t_2 \to \infty$. Then, up to normalization constants, we have

$$\begin{aligned}&S_{1nk}(t_1)=t_1^{|m|/2}(1-t_1)^{-ik/2}\\&\times\ {}_2F_1\left((1+n+\sigma+|m|-ik)/2,(1-n-\sigma+|m|-ik)/2;|m|+1;t_1\right),\end{aligned} \tag{53}$$

$$\begin{aligned}&S_{2nk}(t_2)=t_2^{(ik-n+\sigma-1)/2}(t_2-1)^{-ik/2}\\&\times\ {}_2F_1\left((1+n-\sigma+|m|-ik)/2,(1+n-\sigma-|m|-ik)/2;n-\sigma+1;1/t_2\right),\end{aligned} \tag{54}$$

where $k=\sqrt{A}$ and $\sigma=\alpha R/n$. Both S_{1nk} and S_{2nk} are real functions. By using equation (52) we can evaluate the following integrals:

$$\int_0^1 S_{1nk}^2(t)dt=\frac{2\pi(|m|!)^2\,\mathrm{Im}\left[\psi\left(\frac{|m|+1-n-\sigma+ik}{2}\right)-\psi\left(\frac{|m|+1+n+\sigma+ik}{2}\right)\right]}{(n+\sigma)\sinh\pi k\left|\Gamma\left(\frac{|m|+1+n+\sigma+ik}{2}\right)\right|^2\left|\Gamma\left(\frac{|m|+1-n-\sigma+ik}{2}\right)\right|^2}, \tag{55}$$

$$\int_1^\infty S_{2nk}^2(t)dt=\frac{2\pi(n-\sigma)[\Gamma(n-\sigma)]^2\,\mathrm{Im}\left[\psi\left(\frac{|m|+1+n-\sigma+ik}{2}\right)+\psi\left(\frac{-|m|+1+n-\sigma+ik}{2}\right)\right]}{\sinh\pi k\left|\Gamma\left(\frac{|m|+1+n-\sigma+ik}{2}\right)\right|^2\left|\Gamma\left(\frac{-|m|+1+n-\sigma+ik}{2}\right)\right|^2}, \tag{56}$$

$$\int_0^1 S_{1nk}(t)S_{1nk'}(t)\frac{dt}{1-t}=\frac{4\pi^2(|m|!)^2\delta(k-k')}{k\sinh\pi k\left|\Gamma\left(\frac{|m|+1+n+\sigma+ik}{2}\right)\right|^2\left|\Gamma\left(\frac{|m|+1-n-\sigma+ik}{2}\right)\right|^2}, \tag{57}$$

$$\int_1^\infty S_{2nk}(t)S_{2nk'}(t)\frac{dt}{t-1}=\frac{4\pi^2[\Gamma(n-\sigma+1)]^2\delta(k-k')}{k\sinh\pi k\left|\Gamma\left(\frac{|m|+1+n-\sigma+ik}{2}\right)\right|^2\left|\Gamma\left(\frac{-|m|+1+n-\sigma+ik}{2}\right)\right|^2}. \tag{58}$$

Here $\psi(z) = d\ln\Gamma(z)/dz$. Now, denoting $\Psi_{nkm}(t_1,t_2,\varphi) = N^n_{km} S_{1nk}(t_1) S_{2nk}(t_2) e^{im\phi}$ and using integrals (55) – (58), we obtain the normalization relation

$$\int\limits_{x_0+x_3>0}\int\int\int \Psi^*_{nkm}(t_1,t_2,\phi)\Psi_{nk'm}(t_1,t_2,\phi)dV = \pi\delta(k-k'), \tag{59}$$

where dV is the volume element in coordinates t_1, t_2, ϕ,

$$dV = \frac{R^3}{4}\left(\frac{1}{1-t_1}+\frac{1}{t_2-1}\right)dt_1 dt_2 d\phi,$$

and the normalization coefficient is

$$N^n_{km} = \frac{\sqrt{k}\sinh\pi k\sqrt{n^2-\sigma^2}}{2\sqrt{\pi}R^{3/2}\Gamma(n-\sigma+1)|m|!\sqrt{I(n,\sigma)}}$$
$$\times\left|\Gamma\left(\tfrac{|m|+1+n+\sigma+ik}{2}\right)\right|\left|\Gamma\left(\tfrac{|m|+1-n-\sigma+ik}{2}\right)\right|\left|\Gamma\left(\tfrac{|m|+1+n-\sigma+ik}{2}\right)\right|\left|\Gamma\left(\tfrac{-|m|+1+n-\sigma+ik}{2}\right)\right|.$$

Here

$$I(n,\sigma) = \mathrm{Im}\left\{(n-\sigma)\left[\psi\left(\tfrac{|m|+1-n-\sigma+ik}{2}\right)-\psi\left(\tfrac{|m|+1+n+\sigma+ik}{2}\right)\right]\right.$$
$$\left.+(n+\sigma)\left[\psi\left(\tfrac{|m|+1+n-\sigma+ik}{2}\right)+\psi\left(\tfrac{-|m|+1+n-\sigma+ik}{2}\right)\right]\right\}.$$

Normalization simplifies greatly for $\alpha = 0$. In this case,

$$N^n_{km} = \frac{\sqrt{nk\sinh\pi k}\left|\Gamma\left(\frac{|m|+n-ik+1}{2}\right)\right|^2}{\pi R^{3/2} n!}.$$

For $E = E_n$ the separation constant A may also take discrete values. This is only possible if $n-\sigma < |m| < n+\sigma$. Then we may write $A = -[\sigma-(n_1+n_2+1)]^2$, where $n_1 = 0, 1, 2, ..., \frac{1}{2}[n-\sigma-|m|-1)]$, $n_2 = 0, 1, 2, ..., \frac{1}{2}[|m|-n+\sigma-1)]$, and $n = |m|+n_1-n_2$. In this case the wave functions $\Psi_{n_1n_2m}(t_1,t_2,\phi)$ normalized by the condition

$$\int\limits_{x_0+x_3>0}\int\int\int \Psi^*_{n_1n_2m}(t_1,t_2,\phi)\Psi_{n_1n_2m}(t_1,t_2,\phi)dV = \frac{1}{2}, \tag{60}$$

take the form

$$\Psi_{n_1n_2m}(t_1,t_2,\phi) = N^n_{n_1n_2}\, S^n_{1n_1n_2}(t_1)\, S^n_{2n_1n_2}(t_2)\,\frac{e^{im\phi}}{\sqrt{2\pi}}, \tag{61}$$

where

$$S^n_{1n_1n_2} = t_1^{|m|/2}(1-t_1)^{-n_1+(n+\sigma-|m|-1)/2}\,{}_2F_1(-n_1,\, n+\sigma-n_1;\, |m|+1;\, t_1),$$
$$S^n_{2n_1n_2} = t_2^{n_2-|m|/2}(t_2-1)^{-n_2+(|m|-n+\sigma-1)/2}\,{}_2F_1(-n_2,\, |m|-n_2;\, n-\sigma+1;\, 1/t_2),$$

and

$$N^n_{n_1n_2} = \frac{\sqrt{(N^2-\sigma^2)(\sigma-n_1-n_2-1)}}{|m|!\Gamma(N-\sigma+1)}\sqrt{\frac{(|m|+n_1)!(|m|-n_2-1)!}{NR^3 n_1! n_2!}}$$
$$\times\sqrt{\frac{\Gamma(|m|+\sigma-n_2)\Gamma(|m|+n_1-\sigma+1)}{\Gamma(\sigma-n_1)\Gamma(\sigma-n_2)}}.$$

For fixed values of energy, the wave functions (61) can be expanded in terms of eigenfunctions $\Psi_{nlm}(\tau,\theta,\phi)$ (46) as

$$\Psi_{n_1n_2m}(t_1,\, t_2,\phi) = \sum_{l=n}^{\infty} W^{nlm}_{n_1n_2}\,\Psi_{nlm}(\tau,\theta,\phi). \tag{62}$$

To find expansion coefficients $W^{nlm}_{n_1n_2}$, we express coordinates (t_1, t_2) on the left side of (62) through spherical coordinates: $t_1 = (1+\cos\theta)/(1+\tanh\tau)$, $t_2 = (1-\cos\theta)/(1-\tanh\tau)$. Then, taking the limit $\tau \to \infty$ and using the orthogonality relations for spherical function $Y_{lm}(\theta,\phi)$, we obtain the following expression:

$$\begin{aligned} W^{Nlm}_{n_1n_2m} &= \frac{(-1)^{n+l+\frac{m-|m|}{2}}}{\Gamma(n_2-n_1+1)}\sqrt{\frac{(2l+1)(l+|m|)!n_2!(|m|-n_2-1)!(|m|+n_1)!(l-n)!}{2(l-|m|)!n_1!(l+n)!}} \\ &\times \frac{\sqrt{\Gamma(|m|+n_1-\sigma+1)\Gamma(|m|-n_2+\sigma)}}{\Gamma(\sigma+|m|+1)}\sqrt{\frac{\Gamma(\sigma-n_1)\Gamma(l+\sigma+1)}{\Gamma(\sigma-n_2)\Gamma(l-\sigma+1)}} \\ &\times \sqrt{(\sigma-n_1-n_2-1)}\ {}_4F_3\left[\begin{array}{c} |m|+l+1, |m|-l, n_2+1, \sigma-n_1; 1 \\ |m|+1, n_2-n_1+1, \sigma+|m|+1 \end{array}\right]. \end{aligned}$$

Here ${}_4F_3$ denotes the generalized hypergeometric function [16, 23]. Using now the relation [23]

$${}_4F_3\left[\begin{array}{c} -n, b, c, d, 1 \\ e, f, g \end{array}\right] = \frac{(f-b)_n(g-b)_n}{(f)_n(g)_n}{}_4F_3\left[\begin{array}{c} -n, b, e-c, e-d, 1 \\ e, b-f-n+1, b-g-n+1 \end{array}\right],$$

we can rewrite $W^{nlm}_{n_1n_2}$ in the form

$$\begin{aligned} W^{nlm}_{n_1n_2} &= \frac{(-1)^{n+l+\frac{m-|m|}{2}}\Gamma(l+n_2+1-|m|-\sigma)\Gamma(l+n_1+1)}{\Gamma(n_2+1+\sigma)}\sqrt{\frac{(2l+1)(l+|m|)!}{2(l-|m|)!}} \\ &\times \sqrt{\frac{n_2!(|m|-n_2-1)!}{n_1!(l+n)!(l-n)!(|m|+n_1)!}}\sqrt{\frac{\Gamma(|m|+n_1-\sigma+1)\Gamma(|m|-n_2+\sigma)}{\Gamma(l+\sigma+1)\Gamma(l-\sigma+1)}} \\ &\times \sqrt{\frac{\Gamma(\sigma-n_1)(\sigma-n_1-n_2-1)}{\Gamma(\sigma-n_2)}}{}_4F_3\left[\begin{array}{c} |m|-l, \sigma-n_1, |m|-n_2, -l; 1 \\ |m|+l, \sigma-n_2+|m|-l, -n_1-l \end{array}\right]. \end{aligned}$$

The expansion coefficients $W^{nlm}_{n_1n_2}$ can also be expressed through the Racah coefficients of the $SU(2)$ group. This will be discussed elsewhere.

The work of Yu.A.K., V.S.O., and G.S.P. was supported in part by the Armenian-Belarusian grants Nos. 13RB-035 and Ph14ARM-029 from SCS and FFR.

References

[1] N. Limić, J. Niederle, and R. Rączka. Eigenfunction expansions associated with the second order invariant operator on hyperboloids and cones. // J. Math. Phys. 1967. Vol. 8, no. 3. P. 1079–1093.

[2] J.S. Zmuidzinas. Unitary representations of the Lorentz group on 4-vector manifolds. // J. Math. Phys. 1966. Vol. 7, no. 4. P. 764–780.

[3] I.M. Gelfand, M.I. Graev and N.Ya. Vilenkin. Generalized functions. Vol. 5: Integral geometry and representation theory. – AP: N-Y, 1966. 449 p.

[4] N.Ya. Vilenkin and Ya.A. Smorodinskiĭ. Invariant expansions of relativistic amplitudes. // Sov. Phys. JETP. 1964. Vol. 19, no. 5. P. 1209–1218.

[5] E.G. Kalnins and W. Miller Jr. Lie theory and the wave equation in space-time. I. The Lorentz group. // J. Math. Phys. 1977. Vol. 18, no. 1. P. 1–16.

[6] E. Schrödinger. A method of determining quantum-mechanical eigenvalues and eigenfunctions. // Proc. Roy. Irish Acad. A. 1940. Vol. 46. P. 9–16. 1941. P. 183–206. Vol. 47. P. 53–54.

[7] L. Infeld and A. Schild. A note on the Kepler problem in a space of constant negative curvature. // Phys. Rev. 1945. Vol. 67, no. 2. P. 121–122.

[8] P.W. Higgs. Dynamical symmetries in a spherical geometry I. // J. Phys. A. 1979. Vol. 12, no 3. P. 309–323.

[9] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik. On the quantum-mechanical Kepler problem in a three-dimensional Lobachevsky space. // Dokl. AN BSSR. 1980. Vol. 24, no. 1. P. 19–22.

[10] A.A. Bogush, V.S. Otchik, V.M. Red'kov. Separation of variables in Schrödinger equation and normalized state functions for Kepler problem in three-dimensional spaces of constant curvature. // Izv. AN BSSR, ser. fiz.-mat. nauk. 1983. No. 3. P. 56–62.

[11] A.O. Barut, A. Inomata, G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. // J. Phys. A. 1987. Vol. 20, no. P. 6271–6280.

[12] C. Grosche, G.S. Pogosyan and A.N. Sissakian. Path integral discussion for Smorodinsky-Winternitz potentials: II. The two- and three-dimensional sphere // Fortschr. Phys. 1995. Vol. 43, no. 6. P. 523–563; Path integral appraoach for superintegrable potentials on the three-dimensional hyperboloid. // Phys. Element. Part. At. Nucl. 1997. Vol. 28, no. 5. P. 1229–1294.

[13] A. Nersessian and G. Pogosyan. Relation of the oscillator and Coulomb systems on spheres and pseudospheres. // Phys. Rev. A. 2001. Vol. 63, no. 2. 020103(R).

[14] C. Grosche. On the Path Integral in Imaginary Lobachevsky Space. // J. Phys. A. 1994. V. 27, no. 10. P. 3475–3490.

[15] A.A. Bogush, Yu.A. Kurochkin and V.S. Otchik. Algebra of conserved operators for the Kepler–Coulomb problem in spaces of constant curvature. // Phys. At. Nucl. 1998. Vol. 61, no. 10. P. 1889–1892.

[16] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii. Quantum Theory of Angular Momentum. – World Scientific: Singapore, 1988. 524 p.

[17] N. Rosen and P.M. Morse. On the vibrations of polyatomic molecules. // Phys. Rev. 1932. Vol. 42, no. 2. P. 210–217.

[18] M. Nieto. Exact wave-function normalization constants for the $B_0 \tanh z - U_0 \cosh^{-2} z$ and Pöschl-Teller potentials. // Phys. Rev. A. 1978. Vol. 17, no. 4. P. 1273–1283.

[19] E.C. Titchmarsh. Eigenfunction expansions associated with second-order differential equations. – Oxford Univ. Press, 1953. Part I. 203 p.

[20] A. Erdelyi et al. Higher transcendental functions. – McGraw-Hill: N-Y, 1953. Vol. 1. 302 p.

[21] M.N. Olevskiĭ. Triorthogonal systems in spaces of constant curvature in which the equation $\Delta_2 u + \lambda u = 0$ allows the complete separation of variables. // Matem. Sbornik. 1950. Vol. 27, no. 3. P. 379–426 (in Russian).

[22] Yu.A. Kurochkin, V.S. Otchik, E.M. Ovsiyuk, Dz.V. Shoukavy. On some integrable systems in the extended Lobachevsky space. // Phys. At. Nucl. 2011. Vol. 74, no. 6. P. 944–948.

[23] W.N. Bailey. Generalized hypergeometric series. – Cambridge Univ. Press, 1935. 108 p.

Two Coulomb centers problem in the Lobachevsky space

V.S. Otchik
votchyk@tut.by; B.I. Stepanov Institute of Physics

Abstract

The quantum mechanical problem of the motion of a charged particle in the field of two Coulomb centers in the three-dimensional Lobachevsky space is considered. Variables in the Schrödinger equation are separated and solutions of the obtained ordinary differential equations are found in the form of series of hypergeometric functions. Expressions for the energy levels of the system in the limit of small intercenter separation are presented.

Models based on the use of solutions of the quantum mechanical Kepler – Coulomb problem in spaces of constant curvature have been applied for phenomenological description of quarconium spectra [1] and excitons in quantum dots [2]. This problem was first treated by Schrödinger [3] for the case of positive curvature and by Infeld and Schild [4] for negative curvature. After these first works, a considerable number of papers have appeared which are devoted to different aspects of this problem including symmetry and separation of variables (see e.g. [5]–[14]).

High symmetry related to the existence of conserved analog of the Runge – Lenz vector also enables the separation of variables in the problem of two Coulomb centers in spaces of constant curvature [15].

In this note we obtain solutions of the Schrödinger equation with the two-centers Coulomb potential in the Lobachevsky space and use these solutions to derive asymptotic expressions for energy levels of the system in the case of small intercenter distances.

We use the embedding of the Lobachevsky space ${}^1S_3$ in a four-dimensional pseudo-Euclidean space with coordinates x_μ, $\mu = 0, 1, 2, 3$. Then for points of the Lobachevsky space we have $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0^2 - \mathbf{x}^2 = \rho^2$, where $x = (x_0, x_1, x_2, x_3)$, $\mathbf{x} = (x_1, x_2, x_3)$, and constant ρ denotes the curvature radius.

The Schrödinger equation for a particle moving in the potential field U in the Lobachevsky space we write in the form

$$H\Psi = E\Psi, \quad H = -\frac{1}{2}\Delta + U, \tag{1}$$

where Δ is the Laplacian operator in ${}^1S_3$. We use units such that $\hbar = m = 1$, where m is the mass of the particle.

The Coulomb potential of a charge Z located at the point ${}^0x = (\rho, 0, 0, 0)$, we take in the form $U = -Zx_0/|\mathbf{x}|$. Such a choice leads to a similarity between some formulae relating to spaces of positive and negative curvature. In particular, energy spectrum is given by the formula

$$E_n = -\frac{Z^2}{2n^2} - \frac{n^2-1}{2\rho^2}, \quad n = 1, 2, \ldots,$$

which can be obtained from the corresponding formula for the three-dimensional sphere [3] by a substitution $R^2 \to -\rho^2$, where R denotes the radius of the sphere. Absence of dependence on the angular momentum in this expression is due to the existence of conserved analog of Runge–Lenz vector [8], which also enables separation of variables in the corresponding two-center problem.

If an electric charge Z_1 is located at the point ${}^1x = ({}^1x_0,\ {}^1x_1,\ {}^1x_2,\ {}^1x_3)$ in the Lobachevsky space, then its potential is

$$U_1 = -\frac{Z_1}{\rho}\frac{({}^1xx)}{\sqrt{\rho^4 - ({}^1xx)^2}},$$

where $({}^1xx) = {}^1x_0x_0 - {}^1x_1x_1 - {}^1x_2x_2 - {}^1x_3x_3$.

Now we introduce coordinates s, t, ϕ, related to the Cartesian coordinates x_μ as follows:

$$x_0 = \frac{\rho\gamma\sqrt{\gamma^2-1}}{\sqrt{(\gamma^2-t^2)(\gamma^2-s^2)}},\ x_1 = \rho\gamma\sqrt{\frac{(t^2-1)(1-s^2)}{(\gamma^2-t^2)(\gamma^2-s^2)}}\cos\phi,$$

$$x_2 = \rho\gamma\sqrt{\frac{(t^2-1)(1-s^2)}{(\gamma^2-t^2)(\gamma^2-s^2)}}\sin\phi,\ x_3 = \frac{\rho st\sqrt{\gamma^2-1}}{\sqrt{(\gamma^2-t^2)(\gamma^2-s^2)}},$$

$$1 \le t < \gamma,\ -1 \le s \le 1,\ 0 \le \phi < 2\pi,$$

where γ is some constant.

The metrics of the Lobachevsky space in these coordinates takes the form

$$dl^2 = \frac{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)dt^2}{(t^2-1)(\gamma^2-s^2)(\gamma^2-t^2)^2} + \frac{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)ds^2}{(1-s^2)(\gamma^2-s^2)^2(\gamma^2-t^2)} + \frac{\rho^2\gamma^2(t^2-1)(1-s^2)d\phi^2}{(\gamma^2-s^2)(\gamma^2-t^2)},$$

and the Laplacian operator is

$$\Delta = \frac{(\gamma^2-s^2)(\gamma^2-t^2)}{\rho^2\gamma^2(\gamma^2-1)(t^2-s^2)}\left[(\gamma^2-t^2)\frac{\partial}{\partial t}(t^2-1)\frac{\partial}{\partial t}\right.$$

$$\left.+(\gamma^2-s^2)\frac{\partial}{\partial s}(1-s^2)\frac{\partial}{\partial s}\right] + \frac{(\gamma^2-s^2)(\gamma^2-t^2)}{\rho^2\gamma^2(t^2-1)(1-s^2)}\frac{\partial^2}{\partial\phi^2}.$$

If the charges Z_1 and Z_2 are located at the points with coordinates $t = 1$, $s = -1$ and $t = 1$, $s = 1$, respectively, then their potential is

$$U = -\frac{Z_1(\gamma^2+st)}{\rho\gamma(s+t)} - \frac{Z_2(\gamma^2-st)}{\rho\gamma(t-s)}. \tag{2}$$

Separating the Schrödinger equation (1) with the potential (2) by the substitution $\Psi = u(t)v(s)e^{\pm m\phi}$, $m = 0,\ 1,\ 2, ...$, one obtains ordinary differential equations

$$(\gamma^2-s^2)\frac{d}{ds}(1-s^2)\frac{dv}{ds} + \left[\frac{2Z_-\gamma\rho(\gamma^2-1)s}{\gamma^2-s^2} + \frac{2E\gamma^2\rho^2(1-s^2)}{\gamma^2-s^2} - \frac{m^2(\gamma^2-s^2)}{1-s^2} + \Lambda\right]v = 0, \tag{3}$$

$$(\gamma^2-t^2)\frac{d}{dt}(t^2-1)\frac{du}{dt} + \left[\frac{2Z_+\gamma\rho(\gamma^2-1)t}{\gamma^2-t^2} + \frac{2E\gamma^2\rho^2(t^2-1)}{\gamma^2-t^2} - \frac{m^2(\gamma^2-t^2)}{t^2-1} - \Lambda\right]u = 0, \tag{4}$$

where $Z_\pm = Z_2 \pm Z_1$.

In the limit of vanishing curvature we have $\rho \to \infty$ and $2\rho/\gamma \to d$, where d is the distance between charges in flat space. Assuming that in this limit $\Lambda/\gamma^2 \to \lambda$, we obtain the limiting form of equations (3) and (4):

$$\frac{d}{ds}(1-s^2)\frac{dv}{ds} + \left[bs - p^2(1-s^2) - \frac{m^2}{1-s^2} + \lambda\right]v = 0, \tag{5}$$

$$\frac{d}{dt}(t^2-1)\frac{du}{dt} + \left[at - p^2(t^2-1) - \frac{m^2}{t^2-1} - \lambda\right]u = 0, \tag{6}$$

where $p^2 = -Ed^2/2$, $a = Z_+ d$, $b = Z_- d$.

Each of the equations (3) and (4) has four regular singularities. Change of variables

$$x = \frac{(\gamma-1)(1-s)}{2(\gamma+s)}, \quad z = \frac{(1-\gamma)(t-1)}{2(\gamma+t)} \tag{7}$$

and substitutions

$$v = x^{m/2}(1-x)^{m/2}(x-a)^{\alpha_+} f, \quad u = (-z)^{m/2}(1-z)^{m/2}(z-a)^{\beta_-} g, \tag{8}$$

where

$$\alpha_\pm = \frac{1}{2}\left(1 + \sqrt{-2E\rho^2 \pm 2Z_-\rho + 1}\right), \quad \beta_\pm = \frac{1}{2}\left(1 + \sqrt{-2E\rho^2 \pm 2Z_+\rho + 1}\right),$$

result in the Heun equation [16] for each of the functions f and g:

$$\frac{d^2 f}{dx^2} + \left(\frac{m+1}{x} + \frac{m+1}{x-1} + \frac{2\alpha_+}{x-a}\right)\frac{df}{dx} + \frac{(Ax-p)f}{x(x-1)(x-a)} = 0, \tag{9}$$

$$\frac{d^2 g}{dz^2} + \left(\frac{m+1}{z} + \frac{m+1}{z-1} + \frac{2\beta_-}{z-a}\right)\frac{dg}{dx} + \frac{(Bz-q)g}{z(z-1)(z-a)} = 0, \tag{10}$$

where

$$a = -\frac{(\gamma-1)^2}{4\gamma}, \; p = (\alpha_+ + am)(m+1) + \frac{\rho Z_-}{2} + \frac{\Lambda}{4\gamma}, \; q = (\beta_- + am)(m+1) - \frac{\rho Z_+}{2} + \frac{\Lambda}{4\gamma},$$

$$A = (\alpha_+ + \alpha_- + m)(\alpha_+ - \alpha_- + m + 1), \quad B = (\beta_+ + \beta_- + m)(\beta_- - \beta_+ + m + 1).$$

When $\gamma \to \infty$, $\rho = \text{const}$, that is when the two charges are located at one point, the equations (9) and (10) turn into hypergeometric equations. Therefore we will look for solutions of the equations (9) and (10) at small intercenter separations in the form of series of hypergeometric functions. Thus, in order to obtain the solution of equation (3) bounded for $-1 \le s \le 1$, we write the solution f of the equation (9) as a series

$$f = \sum_{k=m-l}^{\infty} c_k \, {}_2F_1(m-l-k, m+l+k+1; m+1; x), \tag{11}$$

with integer $l \ge m$. Substituting (11) in (9), we obtain recurrence relations for c_k

$$\alpha_k c_{k+1} + \beta_k c_k + \gamma_k c_{k-1} = 0, \tag{12}$$

where

$$\alpha_k = \frac{\varepsilon(l+k-m+1)(l+k+\alpha_- - \alpha_+ + 1)(l+k-\alpha_- - \alpha_+ + 2)}{2l+2k+3},$$

$$\beta_k = \lambda - (l+k)(l+k+1), \quad \gamma_k = \frac{\varepsilon(l+k+m)(l+k-\alpha_- + \alpha_+)(l+k+\alpha_- + \alpha_+ - 1)}{2l+2k-1}$$

and

$$\lambda = \frac{\Lambda}{\gamma^2+1}, \quad \varepsilon = \frac{2\gamma}{(\gamma^2+1)}.$$

Introducing the continued fractions

$$R_k = \frac{c_k}{c_{k-1}} = \frac{-\gamma_k}{\beta_k + \alpha_k R_{k+1}}, \quad L_k = \frac{c_k}{c_{k+1}} = \frac{-\alpha_k}{\beta_k + \gamma_k L_{k-1}},$$

one gets a transcendental equation for λ

$$R_m L_{m-1} = 1. \tag{13}$$

For small distances between Coulomb centers, when ε is small, we are able to obtain for λ an expansion in powers of ε

$$\lambda_l = l(l+1) - \frac{\varepsilon^2}{2l(l+1)(2l-1)(2l+3)} \Big[2E\rho^2 l(l+1)(m^2+l^2+l-1)$$

$$+Z_-^2\rho^2\Big(3m^2 - l(l+1)\Big) + l^2(l+1)^2\Big(3l(l+1) - m^2 - 2\Big)\Big] + O(\varepsilon^4), \quad m \geq 1,$$

$$\lambda_0 = -\frac{1}{6}(2E\rho^2 + Z_-\rho^2)\varepsilon^2 + O(\varepsilon^4), \quad m = 0.$$

If λ is found from equation (13), then series (11) is converging inside the ellipse on the plane of complex x with foci at the singularities $x = 0$ and $x = 1$ and passing through the singularity $x = a$.

Now we turn to solutions of the equation (4). A solution of this equation bounded at $t = 1$ we write in the form

$$u_1^\nu = (-z)^{m/2}(1-z)^{m/2}(z-a)^{\beta_-} \sum_{k=-\infty}^{\infty} d_k^\nu P_k^\nu, \tag{14}$$

where

$$P_k^\nu = \Gamma(m+\nu+k+1)\Gamma(m-\nu-k)_2F_1(m-\nu-k, m+\nu+k+1; m+1; z). \tag{15}$$

Coefficients d_k^ν obey the recurrence relations

$$\alpha_k^\nu d_{k+1}^\nu + \beta_k^\nu d_k^\nu + \gamma_k^\nu d_{k-1}^\nu = 0, \tag{16}$$

where

$$\alpha_k^\nu = -\frac{\varepsilon(\nu+k+m+1)(\nu+k-\beta_-+\beta_++1)(\nu+k-\beta_--\beta_++2)}{2\nu+2k+3},$$

$$\beta_k^\nu = \lambda - (\nu+k)(\nu+k+1), \quad \gamma_k^\nu = -\frac{\varepsilon(\nu+k-m)(\nu+k+\beta_--\beta_+)(\nu+k+\beta_-+\beta_+-1)}{2\nu+2k-1}.$$

The auxiliary parameter ν is determined by the condition of convergence of series (14)

$$R_0^\nu L_{-1}^\nu = 1, \tag{17}$$

where

$$R_k^\nu = \frac{d_k^\nu}{d_{k-1}^\nu} = \frac{-\gamma_k^\nu}{\beta_k^\nu + \alpha_k^\nu R_{k+1}^\nu}, \quad L_k^\nu = \frac{d_k^\nu}{d_{k+1}^\nu} = \frac{-\alpha_k^\nu}{\beta_k^\nu + \gamma_k^\nu L_{k-1}^\nu}.$$

For small ε we have approximate expressions

$$\nu = l - \frac{2(l^2+l-3m^2)Z_1Z_2\rho^2\varepsilon^2}{l(l+1)(2l-1)(2l+1)(2l+3)} + O(\varepsilon^4), \quad l \geq 1;$$

$$\nu = \frac{2}{3}Z_1Z_2\rho^2\varepsilon^2 + O(\varepsilon^4), \quad l = 0.$$

If ν is found from equation (17), then coefficients d_k^ν can be obtained from relations

$$d_k^\nu = d_0^\nu R_1^\nu R_2^\nu \dots R_k^\nu, \quad d_{-k}^\nu = d_0^\nu L_{-1}^\nu L_{-2}^\nu \dots L_{-k}^\nu.$$

For instance, for $l=0$, if we take $d_0^\nu=1$, then

$$d_{-1}^\nu=-\frac{1}{2}\,\varepsilon\rho Z_+ +O(\varepsilon^3),\;\; d_1^\nu=\frac{1}{2}\,\varepsilon(\rho Z_+ -2\beta_-)+O(\varepsilon^3),\;\; d_{-2}^\nu=-\frac{1}{4}\,\varepsilon^2\rho Z_+(\rho Z_+ -2\beta_-)+O(\varepsilon^4),$$
$$d_2^\nu=\frac{1}{18}\,\varepsilon^2[\rho^2Z_+^2-4E\rho^2-6\rho Z_+ +6\beta_-(2-\rho Z_+)]+O(\varepsilon^4),\;\; d_{-3}^\nu=O(\varepsilon^3),\;\; d_3^\nu=O(\varepsilon^3).$$

If the coefficient d_0^ν is chosen in such a way that relation $d_0^{-\nu}=d_0^\nu$ is valid, then also relation $d_{-k}^{-\nu}=d_k^\nu$ holds.

The point $t=\gamma$ lies outside the region of convergence of the series (14). In order to obtain a solution of equation (4) bounded in the vicinity of this point, we use the variable $y=a/z$ and consider the series convergent inside the ellipse with foci at $y=0$ and $y=1$, i.e. at $t=-\gamma$ and $t=\gamma$:

$$\tilde u_1^\nu=y^{\beta_+}(y-1)^{\beta_-}(y-a)^{m/2}\sum_{k=-\infty}^{\infty}\tilde d_k^\nu\,\tilde P_k^\nu, \tag{18}$$

where

$$\tilde P_k^\nu=\Gamma(\beta_-+\beta_++\nu+k)\Gamma(\beta_-+\beta_+-\nu-k-1)\Gamma(1+\beta_--\beta_++\nu+k)\Gamma(\beta_--\beta_+-\nu-k)$$
$$\times{}_2F_1(\beta_-+\beta_++\nu+k,\beta_-+\beta_+-\nu-k-1;2\beta_-;1-y).$$

Coefficients $\tilde d_k^\nu$ obey the recurrence relations

$$\tilde\alpha_k^\nu\tilde d_{k+1}^\nu+\tilde\beta_k^\nu\tilde d_k^\nu+\tilde\gamma_k^\nu\tilde d_{k-1}^\nu=0, \tag{19}$$

where

$$\tilde\alpha_k^\nu=-\frac{\varepsilon(\nu+k-m+1)(\nu+k+\beta_-+\beta_+)(\nu+k+\beta_--\beta_++1)}{2\nu+2k+3},$$
$$\tilde\beta_k^\nu=\lambda-(\nu+k)(\nu+k+1),\quad \tilde\gamma_k^\nu=-\frac{\varepsilon(\nu+k+m)(\nu+k-\beta_-+\beta_+)(\nu+k+\beta_--\beta_++1)}{2\nu+2k-1}.$$

Recurrence relations (16) and (19) hold simultaneously if $\tilde d_k^\nu$ and d_k^ν are related by

$$\tilde d_k^\nu=(-1)^k\frac{(m+\nu+1)_k(2-\beta_--\beta_++\nu)_k(\beta_--\beta_+-\nu-k)_k}{(m-\nu-k)_k(1+\beta_--\beta_++\nu)_k(1-\beta_--\beta_+-\nu-k)_k}d_k^\nu,$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol.

To find connection between series solutions with different regions of convergence we consider the following solutions of the equation (4):

$$u_2^\nu=(-z)^{m/2}(1-z)^{m/2}(z-a)^{\beta_-}\sum_{k=-\infty}^{\infty}d_k^\nu Q_k^\nu, \tag{20}$$

where

$$Q_k^\nu=\frac{\Gamma(-2\nu-2k-1)\Gamma(m+\nu+k+1)}{\Gamma(-\nu-k)}$$
$$\times(-z)^{-(m+\nu+k+1)}{}_2F_1(m+\nu+k+1,\nu+k+1;2(\nu+k+1);1/z),$$

and

$$\tilde u_2^\nu=y^{\beta_+}(y-1)^{\beta_-}(y-a)^{m/2}\sum_{k=-\infty}^{\infty}\tilde d_k^\nu\tilde Q_k^\nu, \tag{21}$$

where

$$\tilde{Q}_k^\nu = \frac{\Gamma(\nu+k+\beta_-+\beta_+)\Gamma(\nu+k+\beta_--\beta_++1)}{\Gamma(2\nu+2k+2)}$$

$$\times y^{-(\nu+k+\beta_-+\beta_+)}{}_2F_1(\nu+k+\beta_-+\beta_+,\nu+k+\beta_--\beta_++1;2(\nu+k+1);1/y).$$

Using properties of hypergeometric functions [17], one can find relations between solutions (14) and (20), as well as between solutions (18) and (21)

$$u_1^\nu = \Gamma(m+1)(u_2^\nu + u_2^{-\nu-1}), \quad \tilde{u}_1^\nu = \frac{\pi\Gamma(2\beta_-)}{\sin 2\pi\nu}(\tilde{u}_2^\nu - \tilde{u}_2^{-\nu-1}). \tag{22}$$

Further, can easily check that when a closed circuit in the complex t plane is described in positive direction around the points $t=1$ and $t=-1$, the solutions u_2^ν and $\tilde{u}_2^{-\nu-1}$ are multiplied by the same factor $e^{2\pi\nu i}$. Therefore, they can differ only by a constant factor:

$$u_2^\nu = K_\nu \tilde{u}_2^{-\nu-1}. \tag{23}$$

Factor K_ν can be found by expanding u_2^ν and $\tilde{u}_2^{-\nu-1}$ into Laurent series and comparing like terms. Thus we find

$$K_\nu = -\frac{(-a)^{\beta_--\nu-m/2-1}\Gamma(\nu+1)\Gamma(\nu+m+1)}{2\cos\pi\nu\Gamma(-\nu-1+\beta_++\beta_-)\Gamma(-\nu-\beta_++\beta_-)}$$

$$\times\sum_{k=0}^{\infty}\frac{d_{-k}^\nu}{k!\Gamma(2\nu-k+2)}\left(\sum_{k=0}^{\infty}\frac{\tilde{d}_k^\nu}{k!\Gamma(-2\nu-k)}\right)^{-1}.$$

For bound states, the solution (14) finite at $t=\gamma$ coincides, up to a constant, with the solution (18) finite at $t=1$, i.e. $\tilde{u}_1^\nu = cu_1^\nu$. This implies, with account of (22) and (23), the equation

$$K_\nu + K_{-\nu-1} = 0,$$

from which the energy levels can be found. For small ε one obtains

$$E_n = E_n^0 - \frac{2Z_1Z_2(Z_+^2\rho^2-n^4)(l(l+1)-3m^2)}{n^3l(l+1)(2l-1)(2l+1)(2l+3)}\,\varepsilon^2 + O(\varepsilon^4), \quad l\geq 1,$$

$$E_n = E_n^0 + \frac{2Z_1Z_2(Z_+^2\rho^2-n^4)}{3n^3}\,\varepsilon^2(1-Z_+\rho\varepsilon) + O(\varepsilon^4), \quad l=0.$$

Here E_n^0 is the n-th energy level of the particle in the field of the point charge $Z_+ = Z_1+Z_2$.

References

[1] A.A. Izmest'ev. Exactly solvable potential model for quarconia // Sov. J. Nucl. Phys. 1990. Vol. 52, no. 6. P. 1068–1076.

[2] V.V. Gritsev and Yu.A. Kurochkin. Model of excitations in quantum dots based on quantum mechanics in spaces of constant curvature. // Phys. Rev. B. 2002. Vol. 64, no. 3. 035308.

[3] E. Schrödinger A method of determining quantum-mechanical eigenvalues and eigenfunctions // Proc. Roy. Irish Acad. A. 1940. Vol. 46. P. 9–16. 1941. Vol. 46. P. 183–206. Vol. 47. P. 53–54.

[4] L. Infeld and A. Schild. A note on the Kepler problem in a space of constant negative curvature. // Phys. Rev. 1945. Vol. 67, no. 2. P. 121–122.

[5] P.W. Higgs. Dynamical symmetries in a spherical geometry I. // J. Phys. A: Math. and Gen. 1979. Vol. 12, no 3. P. 309–323.

[6] H.I. Leemon. Dynamical symmetries in a spherical geometry II. // J. Phys. A: Math. and Gen. 1979. Vol. 12, no. 4. P. 489–501.

[7] Yu.A. Kurochkin, V.S. Otchik. Analog of the Runge–Lenz vector and energy spectrum in the Kepler problem on a three-dimensional sphere. // Dokl. AN BSSR. 1979. Vol. 23, no. 11. P. 987–990.

[8] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik. On the quantum-mechanical Kepler problem in a three-dimensional Lobachevsky space. // Dokl. AN BSSR. 1980. Vol. 24, no. 1. P. 19–22.

[9] A.A. Bogush, V.S. Otchik, V.M. Red'kov. Separation of variables in Schrödinger equation and normalized state functions for Kepler problem in three-dimensional spaces of constant curvature. // Izv. AN BSSR, ser. fiz.-mat. nauk. 1983. No. 3. P. 56–62.

[10] A.O. Barut, A. Inomata, G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. // J. Phys. A: Math. Gen. 1987. Vol. 20, no. P. 6271–6280.

[11] A.O. Barut, A. Inomata, G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature: II. Hyperbolic space. // J. Phys. A: Math. Gen. 1990. Vol. 23, no. P. 1179–1190.

[12] Ya.I. Granovskii, A.S. Zhedanov, I.M. Lutsenko. Quadratic algebras and dynamics in curved space: II. Kepler problem. // Theor. Math. Phys. 1992. Vol. 91, no. 3, P. 604-612.

[13] C. Grosche, G.S. Pogosyan and A.N. Sissakian. Path integral discussion for Smorodinsky-Winternitz potentials: II. The two- and three-dimensional sphere // Fortschr. Phys. 1995. Vol. 43, no. 6. P. 523–563.

[14] C. Grosche, G.S. Pogosyan and A.N. Sissakian. Path integral appraoach for superintegrable potentials on the three-dimensional hyperboloid. // Phys. Element. Part. At. Nucl. 1997. Vol. 28, no. 5. P. 1229–1294.

[15] V.S. Otchik Symmetry and separation of variables in the problem of two Coulomb centers in the three-dimensional spaces of constant curvature. // Doklady AN BSSR. 1991. Vol. 35, no. 5. P. 420–424.

[16] A. Erdelyi et al. Higher Transcendental Functions. – McGraw-Hill: N-Y, 1953. Vol. 3. 26+302 p.

[17] A. Erdelyi et al. Higher Transcendental Functions. – McGraw-Hill: N-Y, 1953. Vol. 1. 17+292 p.

Kepler motion on single-sheet hyperboloid

Yu.A. Kurochkin
yukuroch@bas-net.by; B.I. Stepanov Institute of Physics
L.G. Mardoyan
Joint Institute for Nuclear Research, Dubna, Russia
V.S. Otchik
B.I. Stepanov Institute of Physics
D.R. Petrosyan
Joint Institute for Nuclear Research, Dubna, Russia
G.S. Pogosyan
pogosyan@ysu.am; Departamento de Matematicas, CUCEI,
Universidad de Guadalajara, Guadalajara, Mexico
and Yerevan State University, Armenia

Abstract

The classical Kepler-Coulomb problem on the single-sheeted hyperboloid H_1^3 is solved in the framework of the Hamilton–Jacobi equation. We have proven that all the bounded orbits are closed and periodic. The paths are ellipses or circles for finite motion.

Classical and quantum mechanical systems in the spaces of constant curvature (positive and negative) have always drawn a lot of attention due to connection with the relativistic physics and gravity. The 2D and 3D (pseudo)-Riemannian spaces of constant curvature are the models of the relativistic space-times (including de Sitter and anti de Sitter spaces) and find application in many branches of physics, from the quantum gravity and cosmology [1, 2] to the quantum Hall effect [3] or coherent state quantization [4].

The problem of the motion of a classical particle in the field of gravity and of a charged particle in the Coulomb field in the spaces of constant curvature as in Euclidean space has a rich history. The introduction of hyperbolic geometry into the law of gravity can be found already in the work of Lobachevsky, who was the first to define the Kepler potential and found the trajectory of the classical motion [5] (see also the more recent articles devoted to the Kepler's problem on the three-dimensional sphere and pseudosphere, i.e. Lobachevsky space [6, 7, 8, 9]). The investigation of Kepler–Coulomb problem in quantum mechanics was motivated by wish to compare the properties of the Coulomb potential in the "open hyperbolic" or "closed" universe to that of an "open but flat" universe. Schrödinger [10] was the first who discussed this problem and discovered that for "hydrogen atom" on the three-dimensional sphere only discrete spectrum exists. Later, Infeld and Schild [11] found that in an open hyperbolic universe there is only a finite (but very large) number of bound states. Let us also mention some articles devoted to the investigation of various aspects of Kepler-Coulomb problem in space of constant curvature, for instance [12, 13, 14, 15, 16, 17].

The motion in Coulomb field on the one-sheet hyperboloid, as shown by Grosche [19] (see also [20]), has some peculiarities. The potential is not singular, and the discrete energy spectrum is infinitely degenerate. Quite recently in the articles [21, 22] the classical and quantum Kepler–Coulomb problem on the anti de Sitter AdS_3 configuration space have been discussed. In

particular it has been shown in [22] that for the classical motion, like the cases of flat Euclidean space, sphere, and pseudosphere, all bounded trajectories are closed and periodic.

The present paper aims at investigating the classical Kepler problem in hyperbolic space $H_1^3 = SO(3,1)/SO(2,1)$ (single-sheeted hyperboloid) which, to our knowledge, has not been elucidated in literature so far.

The classical Kepler problem on the one-sheeted hyperboloid $H_1^3 : -x_0^2 + x_1^2 + x_2^2 + x_3^2 = R^2$, where x_i $(i = 0, 1, 2, 3)$ are the Cartesian coordinates in the four-dimensional Minkowski space $M_{3,1}$, is defined by the potential [19]

$$V = -\frac{\alpha}{R}\frac{x_0}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = -\frac{\alpha}{R}\tanh\tau \tag{1}$$

where τ is the "quasi-radial angle" corresponding to the geodesic pseudo-spherical coordinate on H_1^3, namely

$$x_0 = R\sinh\tau,\ x_1 = R\cosh\tau\sin\theta\cos\phi,\ x_2 = R\cosh\tau\sin\theta\sin\phi,\ x_3 = R\cosh\tau\cos\theta, \tag{2}$$

$$\tau \in (-\infty, \infty), \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi).$$

The standard metric of Minkowski space $M_{3,1}$: $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ induces on H_1^3 the indefinite metric

$$\frac{ds^2}{R^2} = d\tau^2 - \cosh^2\tau\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3}$$

Then the kinetic energy is given by

$$T = \frac{R^2}{2}\left[\dot\tau^2 - \cosh^2\tau\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right)\right], \tag{4}$$

and the canonical momenta can be obtained in a usual way

$$p_\tau = \frac{\partial T}{\partial\dot\tau} = R^2\dot\tau, \quad p_\theta = \frac{\partial T}{\partial\dot\theta} = -R^2\cosh^2\tau\dot\theta, \quad p_\phi = \frac{\partial T}{\partial\dot\phi} = -R^2\cosh^2\tau\sin^2\theta\dot\phi. \tag{5}$$

Thus the Hamiltonian describing the Kepler problem in the pseudo-spherical phase space $(\tau, \theta, \phi; p_\tau, p_\theta, p_\phi)$ with respect to the canonical Poisson brackets

$$\{f, g\} = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right) \tag{6}$$

has the form

$$H = \frac{1}{2R^2}\left\{p_\tau^2 - \frac{1}{\cosh^2\tau}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)\right\} - \frac{\alpha}{R}\tanh\tau. \tag{7}$$

The Hamilton–Jacobi equation associated with the Hamiltonian (7) is obtained after the substitution $p_{\mu_i} \to \partial S/\partial\mu_i$, where $\mu_i = (\tau, \theta, \phi)$:

$$\frac{1}{2R^2}\left\{\left(\frac{\partial S}{\partial\tau}\right)^2 - \frac{1}{\cosh^2\tau}\left(\frac{\partial S}{\partial\theta}\right)^2 - \frac{1}{\cosh^2\tau\sin^2\theta}\left(\frac{\partial S}{\partial\phi}\right)^2\right\} - \frac{\alpha}{R}\tanh\tau = E. \tag{8}$$

This equation is completely separable, and the coordinate ϕ is cyclic. We seek solutions for the classical action $S(\tau,\ \theta,\ \phi,\ t)$ in the form

$$S(\tau, \theta, \phi, t) = -Et + p_\phi\phi + S_1(\tau) + S_2(\theta), \tag{9}$$

and obtain from (8) two ordinary differential equations

$$\left(\frac{dS_2}{d\theta}\right)^2 + \frac{p_\phi^2}{\sin^2\theta} = L^2, \tag{10}$$

$$\left(\frac{dS_1}{d\tau}\right)^2 + 2R^2V(\tau) - \frac{L^2}{\cosh^2\tau} = 2R^2E, \tag{11}$$

where L^2 is the separation constant which coincides with the square of the angular momentum. From equation (10) it follows that for a fixed $L^2 \geq 0$ the separation constant p_ϕ^2 is in the range $0 \leq p_\phi^2 \leq L^2$. The equation (11) describes the motion in the field of the effective potential (see also Fig.1)

$$V_{eff}(\tau) = -\frac{\alpha}{R}\tanh\tau - \frac{L^2}{2R^2\cosh^2\tau}. \tag{12}$$

In the case of $L^2 > \alpha R$ this potential has a minimum at $\tau_0 = \tanh^{-1}[\alpha R/L^2]$ and at this point

$$V_{eff}(\tau_0) = -\frac{1}{2R^2}\left(L^2 + \frac{\alpha^2R^2}{L^2}\right). \tag{13}$$

Using now equations (10) and (11) we get

$$S_1(\tau) = \int\sqrt{2R^2E - 2R^2V(\tau) + \frac{L^2}{\cosh^2\tau}}\,d\tau, \tag{14}$$

$$S_2(\tau) = \int\sqrt{L^2 - \frac{p_\phi^2}{\sin^2\theta}}\,d\theta. \tag{15}$$

Since only the trajectories are in the scope of our interest, we will follow the usual procedures [23] and consider the equations

$$\frac{\partial S}{\partial E} = \frac{\partial S_1}{\partial E} - t = -t_0, \quad \frac{\partial S}{\partial L^2} = \frac{\partial S_1}{\partial L^2} + \frac{\partial S_2}{\partial L^2} = \beta, \quad \frac{\partial S}{\partial p_\phi} = \phi + \frac{\partial S_2}{\partial p_\phi} = \phi_0, \tag{16}$$

where t_0, ϕ_0, and β are constants.

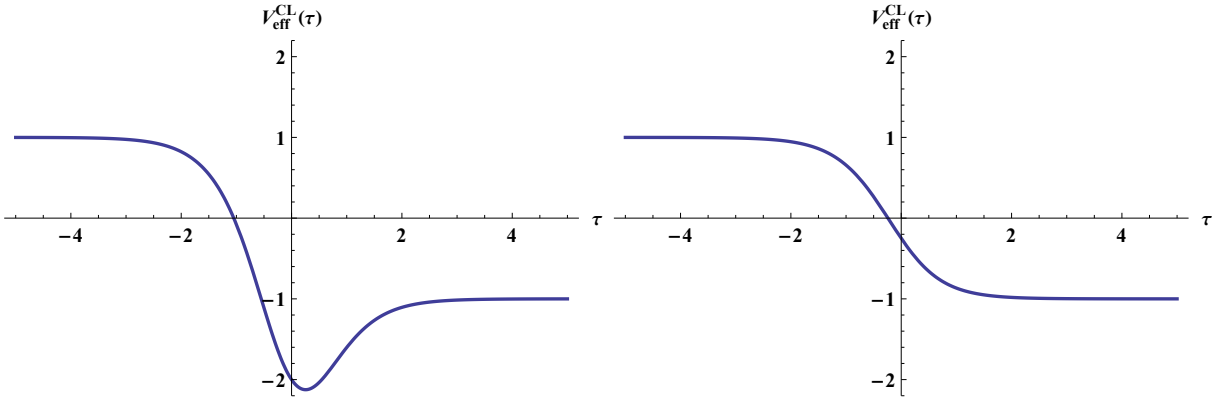


Figure 1: $V_{eff}^{Cl}(\tau) = -\frac{\alpha}{R}\tanh\tau - \frac{L^2}{2R^2\cosh^2\tau}$ for $L^2 = 4$ and $L^2 = 1/2$ ($\alpha = R = 1$)

From (14) and the first of equations (16) we obtain

$$t - t_0 = \frac{R^2}{L}\int\frac{dX}{(1-X^2)\sqrt{-X^2 + 2\alpha R/L^2X + 2R^2E/L^2 + 1}}, \tag{17}$$

where $X = \tanh\tau \in [-1, 1]$, and the roots of the subradical polynom in denominator are

$$X_{1,2} = \frac{\alpha R}{L^2} \pm \sqrt{\left(\frac{\alpha R}{L^2}\right)^2 + \frac{2R^2E}{L^2} + 1}\,. \tag{18}$$

The finite motion, when the range of τ is given by

$$\frac{\alpha R}{L^2} - \sqrt{\left(\frac{\alpha R}{L^2}\right)^2 + \frac{2R^2E}{L^2} + 1} < \tanh\tau < \frac{\alpha R}{L^2} + \sqrt{\left(\frac{\alpha R}{L^2}\right)^2 + \frac{2R^2E}{L^2} + 1}\,, \tag{19}$$

exists for the negative values of E in the range

$$-\frac{1}{2R^2}\left(L^2 + \frac{\alpha^2R^2}{L^2}\right) \le E < -\frac{\alpha}{R}. \tag{20}$$

In the case of $E = E_{\min} = -\left(L^2 + \alpha^2R^2/L^2\right)/(2R^2)$, the integral in (17) is not well defined and we turn directly to the equation (11). Then from equation

$$\left(\frac{dS_1}{d\tau}\right)^2 = -L^2\left(\tanh\tau - \frac{\alpha R}{L^2}\right)^2 \tag{21}$$

it follows that $dS_1/d\tau = 0$ and $\tanh\tau = \alpha R/L^2$. Therefore

$$\tau = \tanh^{-1}\left(\frac{R|E|}{\alpha} + \sqrt{\frac{R^2E^2}{\alpha^2} - 1}\right), \tag{22}$$

i.e. the paths are circles. In the case of $-\alpha/R \le E \le \alpha/R$, the range in which τ may vary is determined by $-1 + \alpha R/L^2 \le \tanh\tau \le 1$, that is the motion is limited from the left side but not limited from the right side, and particle can move to the infinity. In the case of $E \ge \alpha/R$ we have $X_1 = 1 + 2\alpha R/L^2$, $X_2 = -1$ and $-\infty < \tau < \infty$.

From equations (14) and (15) we obtain

$$\frac{\partial S_1}{\partial L^2} = \frac{1}{2}\int \frac{\cosh^{-2}\tau d\tau}{\sqrt{2R^2E + 2\alpha R\tanh\tau - L^2/\cosh^2\tau}} = \frac{1}{2L}\,\arcsin\frac{\tanh\tau - \alpha R/L^2}{\sqrt{(\alpha R/L^2)^2 + 2R^2E/L^2 + 1}}, \tag{23}$$

$$\frac{\partial S_2}{\partial L^2} = \frac{1}{2}\int \frac{d\theta}{\sqrt{L^2 - p_\phi^2/\sin^2\theta}} = \frac{1}{2L}\,\arcsin\frac{\cos\theta}{\sqrt{1 - p_\phi^2/L^2}}. \tag{24}$$

In addition to equation (20) we require $-\sqrt{1 - p_\phi^2/L^2} \le \cos\theta \le \sqrt{1 - p_\phi^2/L^2}$ and

$$|\tanh\tau - \alpha R/L^2| \le \sqrt{(\alpha R/L^2)^2 + 2R^2E/L^2 + 1}. \tag{25}$$

The last condition coincides with (19). Finally for $\partial S/\partial L^2$ we have

$$\frac{\partial S}{\partial L^2} = \frac{1}{2L}\left[\arcsin\frac{\tanh\tau - \alpha R/L^2}{\sqrt{(\alpha R/L^2)^2 + 2R^2E/L^2 + 1}} + \arcsin\frac{\cos\theta}{\sqrt{1 - p_\phi^2/L^2}}\right] = \beta. \tag{26}$$

At the next step, from equations (15) and (16) we obtain

$$\frac{\partial S}{\partial p_\phi} = \phi + \int \frac{p_\phi d\theta}{\sin^2\theta\sqrt{L^2 - p_\phi^2/\sin^2\theta}} = \phi + \arcsin\frac{\cot\theta}{\sqrt{L^2/p_\phi^2 - 1}} = \phi_0, \tag{27}$$

for $|\cot\theta| \le \sqrt{L^2/p_\phi^2 - 1}$ and hence

$$\cot\theta = \sqrt{L^2/p_\phi^2 - 1}\,\sin(\phi_0 - \phi). \tag{28}$$

It is easy to see that for the fixed values of L^2 and p_ϕ^2 the motion of particle on the one-sheeted hyperboloid is restricted by the additional condition $\frac{x_3}{x_1\sin\phi_0 - x_2\cos\phi_0} = \sqrt{L^2/p_\phi^2 - 1}$. Therefore, without the loss of generality we can choose $L^2 = p_\phi^2$ or, which is the same, $\theta = \pi/2$, and $x_3 = 0$. Thereby the path of the motion lies on the two-dimensional one-sheeted hyperboloid: $x_0^2 - x_1^2 - x_2^2 = -R^2$. Eliminating the dependence on θ in equation (26) with the help of (28), and putting $L^2 = p_\phi^2$, we can write the equation of the path in the form

$$\frac{1}{1-\tanh\tau} = \frac{p}{1+\varepsilon(R)\,\cos\phi}, \tag{29}$$

where we use the notations

$$p = \left(1 - \frac{\alpha R}{L^2}\right)^{-1} > 0, \qquad \varepsilon(R) = \sqrt{1 + \frac{2\alpha R}{L^2}\,\frac{(1+RE/\alpha)}{(1-\alpha R/L^2)^2}}, \tag{30}$$

and choose $\phi_0 = 2\beta L + \frac{\pi}{2}$ so that the point $\phi = 0$ will be the nearest to the center. When $0 \le \varepsilon(R) < 1$, the orbits are ellipses (circles for $\varepsilon(R) = 0$), when $\varepsilon(R) = 1$, the path is a parabola and when $\varepsilon(R) > 1$ the path is a hyperbola.

Now we summarize some properties of particles orbits.

A. The elliptic orbit is only possible if $-\frac{1}{2R^2}\left(L^2 + \frac{\alpha^2R^2}{L^2}\right) < E < -\frac{\alpha}{R}$, and then

$$0 \le \varepsilon(R) = \sqrt{1 + \frac{2\alpha R}{L^2}\,\frac{(1-R|E|/\alpha)}{(1-\alpha R/L^2)^2}} < 1. \tag{31}$$

Let $\tau_{\min}$ and $\tau_{\max}$ denote, respectively, the minimal and maximal angular distances from the center of field. It is obvious that they correspond to the angles $\phi = 0$ and $\phi = \pi$. Accordingly to equations (29) and (30) we have

$$\tau_{\min} = \tanh^{-1}\left\{\frac{\alpha R}{L^2} - \sqrt{\left(\frac{\alpha R}{L^2}\right)^2 - \frac{2R^2|E|}{L^2} + 1}\right\}, \tag{32}$$

$$\tau_{\max} = \tanh^{-1}\left\{\frac{\alpha R}{L^2} + \sqrt{\left(\frac{\alpha R}{L^2}\right)^2 - \frac{2R^2|E|}{L^2} + 1}\right\}. \tag{33}$$

Let us denote by $2a$ the length of the major axis of the ellipse and by $2c$ the focal distance. It is clear that

$$\tanh 2a = \tanh(\tau_{\max} + \tau_{\min}) = \frac{\tanh\tau_{\min} + \tanh\tau_{\max}}{1 + \tanh\tau_{\min}\cdot\tanh\tau_{\max}} = -\frac{\alpha}{RE}, \tag{34}$$

$$\tanh 2c = \tanh(\tau_{\max} - \tau_{\min}) = \frac{\tanh\tau_{\min} - \tanh\tau_{\max}}{1 - \tanh\tau_{\min}\cdot\tanh\tau_{\max}} = \frac{\varepsilon(R)}{p - \coth 2a}.$$

Thus we get the result true also in Euclidean space, that the length of the major axis of the ellipse depends only on the energy. The bounded orbits are shown in the Fig. 2.

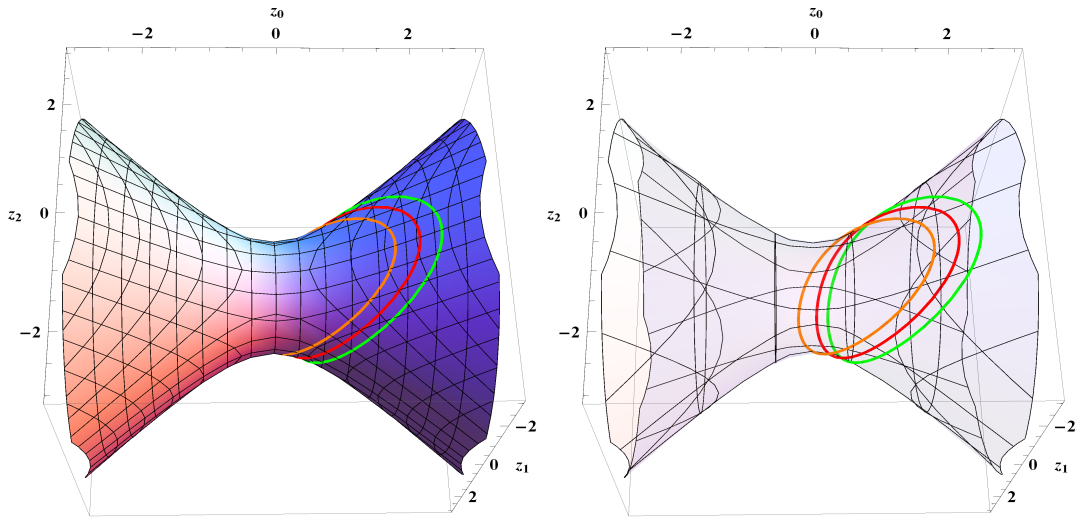


Figure 2: The paths of motion for $\epsilon = 0.8$ and $p = 1.5\,,\ 2\,,\ 2.5$

We can easily calculate the period of the elliptic motion. Using equations (17), (30) and (32) and taking into account the last formula, we obtain

$$\begin{aligned} T &= \frac{R^2}{L}\int_{\tanh\tau_{min}}^{\tanh\tau_{max}} \frac{dX}{(1-X^2)\sqrt{-X^2+2\alpha R/L^2 X+2R^2E/L^2+1}} \\ &= \frac{\pi R}{\sqrt{8|E|}}\left[\frac{1}{\sqrt{1+\alpha/R|E|}}+\frac{1}{\sqrt{1-\alpha/R|E|}}\right]. \end{aligned} \tag{35}$$

Thus, the period T depends only on energy. Using now equation (34) we can rewrite equation (35) in the form

$$T^2 = \frac{\pi^2 R^3}{\alpha}\sinh a\cosh^3 a, \tag{36}$$

which shows that the square of period depends only on the major axis of the orbit (*the third Kepler's law*). In the case of the minimal energy (see equation (20)) we have $\varepsilon = 0$ and again find that the orbits are circles.

B. For $E = -\alpha/R$ we have $\varepsilon(R) = 1$ and the particle moves from infinity ($\tau \sim \infty$) at $\phi = \pi$, is reaches the turning point $\tau = \tanh^{-1}(2\alpha R/L^2 - 1)$ ($\phi = 0$) and returns to infinity. The path of the motion is a parabola (see Fig.3).

C. For $-\alpha/R < E < \alpha/R$, the path of the motion is a hyperbola. The particle again moves from infinity at $\phi = \pi/2 - \sin^{-1}[\varepsilon(R)^{-1}]$, turns at the point $\tau = \tanh^{-1}(\alpha R/L^2 - \sqrt{(\alpha R/L^2)^2 + 2R^2E/L^2 + 1})$ ($\phi = 0$) and goes back to infinity (see Fig.4 and Fig. 5).

D. For $E \geq \alpha/R$, $\tau \in (-\infty, \infty)$ (see Fig.6 and Fig. 7).

Acknowledgments

The work of Yu.A.K., L.G.M, V.S.O., and G.S.P. was supported in part by the Armenian-Belarusian grants Nos. 13RB-035 and Ph14ARM-029 from SCS and FFR.

References

[1] G. t' Hooft. Non-perturbative 2 particle scattering amplitudes in 2+1 dimensional quantum gravity. // Commun. Math. Phys. 1988. Vol. 117, no. 4. P. 685–700.

[2] G.W. Gibbons. Anti-de-Sitter spacetime and its uses. // arXiv:hep-th/1110.1206

[3] P.F. Bracken. Hamiltonians for the Quantum Hall Effect on Spaces with Non-Constant Metrics. // Int. J. Theor. Phys. 2007. Vol. 46, no. 1. P. 116–129.

[4] J.P. Gazeau and W. Piechocki. Coherent state quantization of a particle in de Sitter space. // J. Phys. A: Math. Gen. 2004. Vol. 37, no. 27. P. 6977–6986.

[5] N.A. Chernikov. The Kepler problem in the Lobachevsky space and its solution. // Acta Physica Polonica B. 1992. Vol. 23, no. 2. P. 115-–122.

[6] V.V. Kozlov and A.O. Harin. Kepler's problem in constant curvature spaces. // Celestial Mechanics and Dynamical Astronomy. 1992. Vol. 54, no. 4. P.393–399.

[7] V.V. Kozlov. About dynamics in spaces of constant curvature. // Vestnik Moskovskogo Universiteta Seria 1 Matematika, Mekhanika. 1994. No. 2. P. 28–35.

[8] P. Dombrowski and J. Zitterbarth. On the planetary motion in the three-dimensional standard spaces M^3_κ of constant curvature $\kappa \in \mathbf{R}$. // Demonstratio Mathematica. 1991. Vol. 24, no. 3-4. P. 375–458.

[9] J.J. Slawianowski. Bertrand systems on spaces of constant sectional curvature. The action-angle analysis. // Reports on Mathematical Physics. 2000. Vol. 46, no. 3. P. 429–460.

[10] E. Schrödinger. A method of determining quantum-mechanical eigenvalues and eigenfunctions // Proc. Roy. Irish Acad. A. 1940. Vol. 46. P. 9–16. 1941. Vol. 46. P. 183–206. Vol. 47. P. 53–54.

[11] L. Infeld and A. Schild. A note on the Kepler problem in a space of constant negative curvature. // Phys. Rev. 1945. Vol. 67, no. 2. P. 121–122.

[12] P.W. Higgs. Dynamical symmetries in a spherical geometry I. // J. Phys. A: Math. and Gen. 1979. Vol. 12, no 3. P. 309–323.

[13] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik. On the quantum-mechanical Kepler problem in a three-dimensional Lobachevsky space. // Dokl. AN BSSR. 1980. Vol. 24, no. 1. P. 19–22.

[14] A.A. Bogush, V.S. Otchik, V.M. Red'kov. Separation of variables in Schrödinger equation and normalized state functions for Kepler problem in three-dimensional spaces of constant curvature. // Izv. AN BSSR, ser. fiz.-mat. nauk. 1983. No. 3. P. 56–62.

[15] A.O. Barut, A. Inomata, G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature: II. Hyperbolic space. // J. Phys. A: Math. Gen. 1990. Vol. 23, no. P. 1179–1190.

[16] C. Grosche. Path integral for the Kepler problem on the pseudosphere. // Annals of Physics. 1990. Vol. 204, no. 1. P. 208–222.

[17] C. Grosche, G.S. Pogosyan and A.N. Sissakian. Path integral discussion for Smorodinsky-Winternitz potentials: I. two- and three-dimensional Euclidean space. Fortschr. Phys. 1995. Vol. 43, no. 6. P. 453–521; Path integral discussion for Smorodinsky-Winternitz potentials: II. The two- and three-dimensional sphere. Fortschr. Phys. 1995. Vol. 43, no. 6. P. 523–563; Path-integral approach to superintegrable potentials on the two-dimensional hyperboloid. Phys. Element. Part. At. Nucl. 1997. Vol. 28, no. 2. P. 244–278; Path integral appraoach for superintegrable potentials on the three-dimensional hyperboloid. // Phys. Element. Part. At. Nucl. 1997. Vol. 28, no. 5. P. 1229–1294.

[18] E.G. Kalnins, J.M. Kress, W. Miller Jr., G.S. Pogosyan. Completeness of Superintegrability in Two-Dimensional Constant-Curvature Spaces. J. Phys. A: Math. and Gen. 2001. Vol. 34, no. 22. P. 4705–4720.

[19] C. Grosche. On the Path Integral in Imaginary Lobachevsky Space. // J. Phys. A: Math. and Gen. 1994. Vol. 27, no. 10. P. 3475–3490.

[20] Yu.A. Kurochkin, V.S. Otchik, E.M. Ovsiyuk, Dz.V. Shoukavy. On some integrable systems in the extended Lobachevsky space. // Phys. At. Nucl. 2011. Vol. 74, no. 6. P. 944–948.

[21] D.R. Petrosyan and G.S. Pogosyan. The Kepler-Coulomb problem on $SO(2,2)$ hyperboloid. // Phys. At. Nucl. 2012. Vol. 75, no. 10. P. 1272-1278.

[22] D.R. Petrosyan and G.S. Pogosyan. Classical Kepler-Coulomb problem on $SO(2,2)$ hyperboloid. // Phys. At. Nucl. 2012. Vol. 76, no. 10. P. 1273-1280.

[23] L.D. Landau and E.M. Lifshitz. Mechanics. Course of Theoretical Physics, Vol 1. – Pergamon Press, 1969. 165 p.

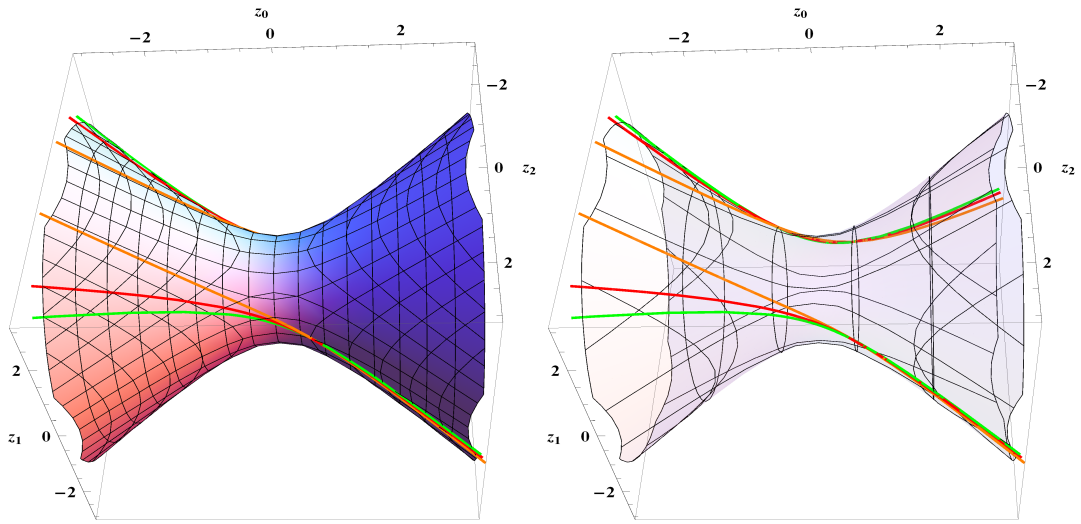


Figure 3: The paths of motion for $L^2 = 2$ and $E = -1/2\,,\, 0\,,\, 1/2$ ($\alpha = R = 1$)

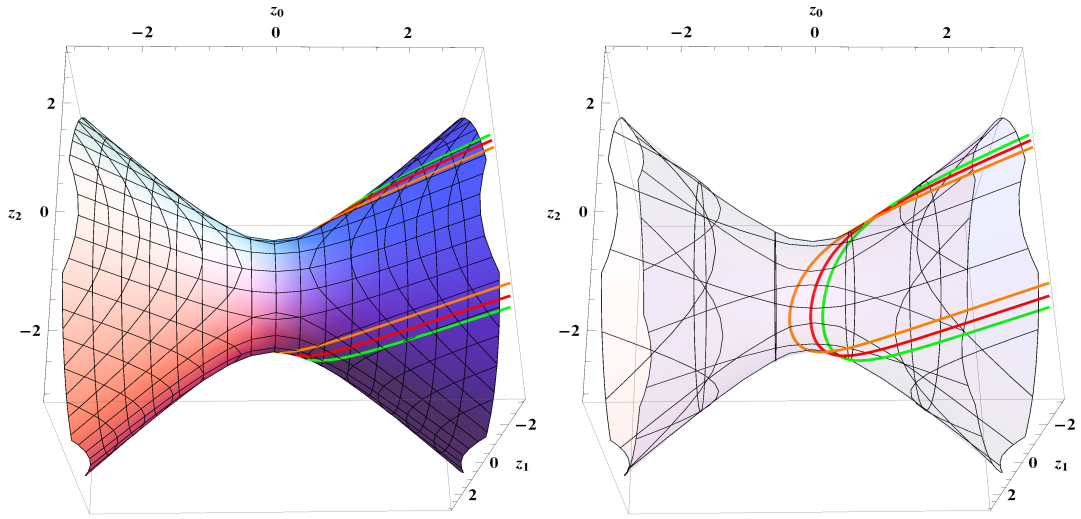


Figure 4: The paths of motion for $\epsilon = 1$ and $p = 1.5\,,\ 2\,,\ 2.5$

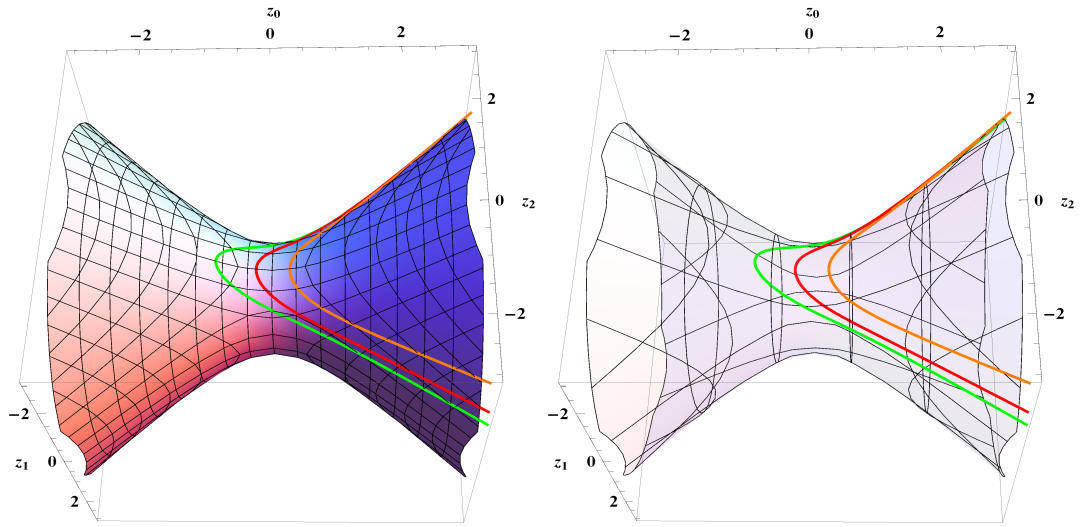


Figure 5: The paths of motion for $L^2 = 1/2$ and $E = -1/2\,,\ 0\,,\ 1/2$ ($\alpha = R = 1$)

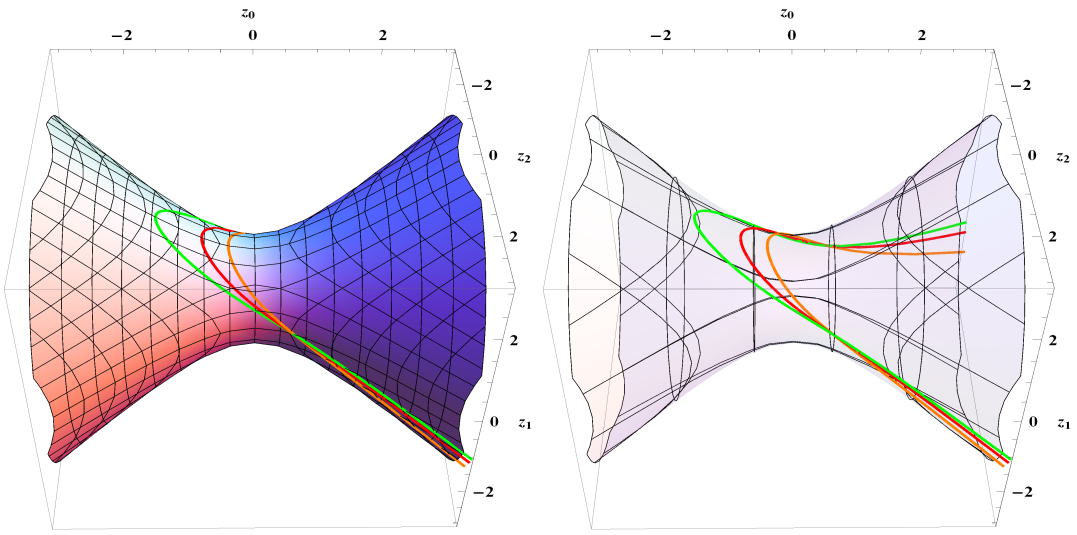


Figure 6: The paths of motion for $L^2 = 2$ and $E = -1/2\,,\, 0\,,\, 1/2$ ($\alpha = R = 1$)

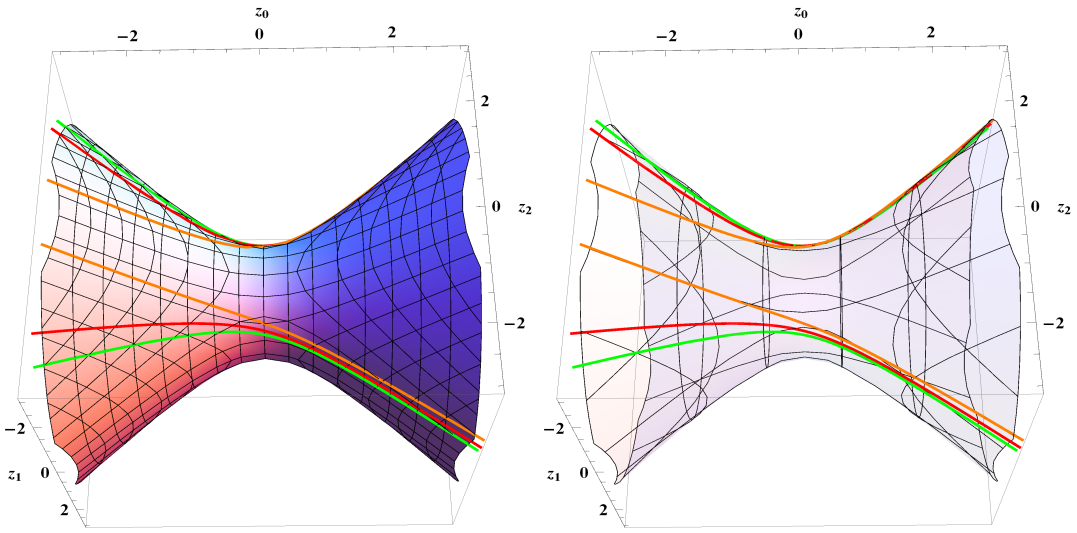


Figure 7: The paths of motion for $L^2 = 1/2$ and $E = -1/2\,,\, 0\,,\, 1/2$ ($\alpha = R = 1$)

Применение неевклидовой геометрии в задачах релятивистской физики

Е.А. Ушаков
ushakovea@bsu.by; Белорусский государственный университет, Минск

Аннотация

Гравитационные эффекты в общей теории относительности интерпретируются как движения, ассоциированные с циклом в смысле Э. Картана [1] относительно риманова тензора кривизны. Этот подход можно обобщить, рассматривая физические эффекты в других областях классической физики как движения, ассоциированные с циклом, относительно тензора кривизны некоторого эффективного неевклидова пространства, моделирующего задачу [2].

Методом точного в первом порядке по линеаризованной гравитации решения обобщенного скалярного уравнения Гросс-Питаевского, пригодного для описания широкого класса интерференционных систем, получены гравитационно-индуцированные фазовые сдвиги в различных метриках.

Проведено сравнение полученных результатов с имеющимися в литературе.

1 Картанова проблема и движения, ассоциированные с циклом

Одна из основных проблем в общей теории относительности – установление возможности (принципиальной) измерения величин, входящих в ее уравнения, проблема Картана [1]. Всякое измерение должно удовлетворять трем аксиомам равенства, в частности, транзитивности. Физическая же реализация последнего условия требует физической процедуры, геометрически представленной конфигурацией Дезарга.

Физическая ситуация, характеризуемая появлением тензора Римана-Кристоффеля, автоматически приводит к недезарговой геометрии,и, следовательно, невозможности прямого измерения. Это позволяет так формулировать картанову проблему: осуществить измерение, (в принципе требующее транзитивности равенства) в физической ситуации, когда, вообще говоря, такая транзитивность отсутствует.

2 Связь движений, ассоциированных с циклом, с уравнениями Эйнштейна, уравнениями девиации геодезических и неинтегрируемым фазовым фактором

В приложениях неевклидовых геометрий в физике фундаментальную роль играют три уравнения, включающие кривизну пространства-времени: уравнения Эйнштейна

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{1}$$

уравнения девиации геодезических

$$\frac{D^2\eta^\nu}{ds^2} = R^\mu{}_{\nu\lambda\omega} u^\nu \eta^\lambda u^\omega, \tag{2}$$

и уравнение, описывающее изменение вектора при квазипараллельном переносе его по бесконечно малому замкнутому контуру – движение, ассоциированное с циклом в смысле Э. Картана

$$DA^\mu = R^\mu{}_{\nu\lambda\omega} A^\nu \, dS^{\lambda\omega}. \tag{3}$$

В работе [2] показано, что уравнение Эйнштейна в виде (1) утверждает, что среднее значение перемещения, ассоциированного с циклом, любого вектора при параллельном переносе по трем некомпланарным инфинитезимальным параллелограммам, обладающим одной общей стороной, не произвольно, но определяется тензором энергии-импульса полей, за исключением гравитационного, в которых происходит квазипараллельное перемещение. В этом физический смысл уравнений Эйнштейна в рамках концепции движений, ассоциированных с циклом.

Если инфинитезимальный цикл образовать отрезками близких друг к другу геодезических линий и линий нумерующих их, то уравнение девиации геодезических получается как движение, ассоциированное с циклом для вектора 4-скорости u^μ [4].

Особую роль концепция движений, ассоциированных с циклом, приобретает в связи с тем, что наряду с ОТО, созданы многочисленные конкурирующие теории гравитации и теории, обобщающие и развивающие ОТО [5]. Так в пространстве Римана-Картана движение, ассоциированное с циклом, включает как вращение вектора, обусловленное кривизной, так и трансляцию точки, обусловленную кручением. Фундаментальные величины пространства Римана-Картана – кривизна $R^\alpha{}_{\beta\mu\nu}$ и кручение $S^\alpha{}_{\mu\nu}$ – естественным образом входят в дифференциальный формализм, в котором обыкновенная производная ∂_μ заменяется ковариантной ∇_μ. Коммутатор ковариантных производных

$$[\nabla_\nu, \nabla_\nu]\, t = \left(iR^\alpha{}_{\beta\mu\nu} U_\alpha{}^\beta + S^\alpha{}_{\mu\nu} \nabla_\alpha \right) t \tag{4}$$

выражается через обе эти величины. Здесь $U_\alpha{}^\beta$ – генераторы $GL(4,R)$ алгебры Ли для тензорного представления, к которому принадлежит t. Эта ситуация контрастирует с интегральным формализмом, в который естественным образом входит только кривизна. При развертке вдоль кривой τ на касательное пространство Минковского с помощью параллельного переноса получаем следующее поведение базиса и точки:

$$e(t) = e(t_1) T \exp\left(-\int_t^{t_1} \Gamma_\nu(x) \frac{dx^\nu}{dt} dt \right), \tag{5}$$

$$\vec{r}(t) = -\int_t^{t_1} \vec{e}_i(t) \frac{dx^i}{dt} dt. \tag{6}$$

Для произвольного вектора $\overline{V}(t)$ имеем

$$\overline{V}(t) = T \exp\left(-\int_t^{t_1} \Gamma_\nu(x) \frac{dx^\nu}{dt} dt \right) V(t), \tag{7}$$

где Γ_ν – 4×4 матрицы

$$\Gamma_\nu = iU_\alpha{}^\beta \Gamma^\alpha{}_{\beta\gamma}. \tag{8}$$

$T\exp$ – означает упорядоченную экспоненту, V и $\overline{V}$ рассматриваются как 4×1 матрицы. Из формул (5) и (7) видно, что поведение вектора может быть объяснено посредством неинтегрируемого фазового фактора относительно $GL(4,R)$. Однако поведение точни не может быть описано неинтегрируемым фазовым фактором. Это означает, как отмечено Хаяши и Ширафуджи [5], что для гравитации концепция движения, ассоциированного с циклом, более важна, чем неинтегрируемый фазовый фактор.

Для случая, когда τ – бесконечно малая петля, формулы (3) и (4) с помощью итераций могут быть сведены к известным соотношениям, выражающим изменение вектора и трансляцию точки. ассоциированные с этой петлей:

$$\Delta V^{\lambda} = R^{\lambda}{}_{\mu\nu\omega} V^{\mu} \oint x^{\nu} dx^{\omega}, \tag{9}$$

$$\vec{r}(\tau) = \frac{1}{2} S^{\lambda}{}_{\mu\nu} \vec{e}_{\lambda} \oint x^{\mu} dx^{\nu}. \tag{10}$$

3 Отклонение луча света в полях Шварцшильда, Нордстрема-Рейснера и Керра

Аналитически перемещение, ассоциированное с циклом, выражается соотношением

$$\underset{\mu}{e}_{\mu} \underset{\sigma\rho}{D} \underset{\nu}{e}^{\mu} = R^{\mu}{}_{\gamma\kappa\omega} \underset{\nu}{e}^{\gamma} \underset{\mu}{e}_{\mu} \underset{\sigma}{dx}^{\nu} \underset{\rho}{dx}^{\omega}, \tag{11}$$

где $\underset{\mu}{e}_{\nu}$ – базисные векторы (тетрады) в схоутеновской символике; греческие индексы пробегают значения 0, 1, 2, 3, латинские 1, 2, 3; подиндексы относятся к физическому базису и указывают направление перемещения [2].

Пусть поток невзаимодействующих ультрарелятивистских частиц движется со скоростью v_0 параллельно оси x^1 с прицельным параметром b. В процессе рассеяния в плоскости x^1x^2 появляются проекции векторов сопутствующего базиса на направление x^2, значения которых определяются согласно (11)

$$\underset{2}{e}_{2} \underset{20}{D} \underset{0}{e}^{2} = R^{2}{}_{020} \underset{0}{e}^{0} \underset{2}{e}_{2} \underset{2}{dx}^{2} \underset{0}{dx}^{0}, \tag{12}$$

$$\underset{2}{e}_{2} \underset{21}{D} \underset{1}{e}^{2} = R^{2}{}_{121} \underset{1}{e}^{1} \underset{2}{e}_{2} \underset{2}{dx}^{2} \underset{1}{dx}^{1}. \tag{13}$$

Другие компоненты $R^{\mu}{}_{\gamma\kappa\omega}$ вклад в рассеяние не дают.

В формуле (12) при рассеянии на малые углы можно перейти к элементу площади dx^1dx^2, если учесть, что

$$dx^0 = \frac{dx^1}{\beta}, \quad \beta = \frac{v_0}{c}. \tag{14}$$

Тогда для метрики Нордстрема-Рейснера

$$R^{2}{}_{020} \approx \frac{m}{r^3} - \frac{k}{r^4} - \frac{3my^2}{r^5} + \frac{4ky^2}{r^6}, \quad R^{2}{}_{121} \approx -\frac{m}{r^3} + \frac{k}{r^4}. \tag{15}$$

Подставляя (15) в (12) и (13) и интегрируя по квадрату, образованному осью x^2 и прямой $x^2 = b$ после удвоения результата в силу симметрии задачи, получим:

$$\theta = \frac{2m}{b\beta^2}\left(1+\beta^2\right) - \frac{k\pi}{4b^2\beta^2}\left(2+\beta^2\right). \tag{16}$$

Для света $\beta = 1$ и (16) дает

$$\theta = \frac{4m}{b} - \frac{3\pi k}{4b^2}, \tag{17}$$

где $k = \gamma q^2 c^{-4}$ – параметр, связанный с зарядом, m – гравитационный радиус. Полагая $k = 0$, приходим к эйнштейновскому эффекту отклонения луча света в поле Шварцшильда.

В метрике Керра вклад в рассеяние вносят следующие компоненты тензора кривизны:

$$R^2{}_{020} \approx \frac{m}{r^3} - \frac{3my^2}{r^5}, \quad R^2{}_{121} \approx -\frac{m}{r^3}$$
$$R^2{}_{120} \approx \partial_y \left(\frac{L}{r^3}\right), \quad R^2{}_{021} \approx \partial_y \left(\frac{L}{r^3}\right). \tag{18}$$

Проводя расчет аналогично предыдущему, получим [6]:

$$\theta = \left(\frac{2m}{b\beta^2} + \frac{2L}{b^2\beta^2}\right)\left(1 + \beta^2\right), \tag{19}$$

где $L = \gamma S c^{-3}$, $S = \frac{2}{5} M R^2 \omega$ – угловой момент шара радиуса R.

Для отклонения луча света из (19) следует:

$$\theta = \left(\frac{4m}{b} + \frac{2L}{b^2\beta^2}\right). \tag{20}$$

Результаты (17) и (20) совпадает с имеющимися в литературе, но полученными интегрированием по траектории.

4 Расчет гравитационно-индуцированного фазового сдвига

Проникновение гравитационных экспериментов в квантовую область стимулирует изучение квантовых гравитационных эффектов. Актуальным становится вопрос о теоретическом расчете величин этих эффектов [7].

Квантовый отклик частицы на воздействие гравитационного поля проявляется во влиянии гравитации на суперпозицию ее квантовых состояний. Сопоставляя квантовой частице волну де-Бройля и рассматривая суперпозицию

$$\Psi = \Psi_1 + \Psi_2, \tag{21}$$

можно интерпретировать сдвиг интерференционной картины, например, в нейтронной интерферометрии, различным действием гравитационного поля на квантовые альтернативы Ψ_1 и Ψ_2. Эффект кривизны оказывается пропорциональным величине пространственного разделения интерферирующих пучков.

В нерелятивистских экспериментах типа COW [8] обнаружено воздействие ньютонова потенциала Земли и вращения на волновую функцию нейтрона, причем индикатором этого воздействия является фаза волновой функции. В [9] предложено рассмотрение релятивистских эффектов гравитации с привлечением электронной интерферометрии.

В основу описания взаимодействия квантовой системы с гравитационным полем положим концепцию Э. Картана движений, ассоциированных с циклом [10].

Пусть квантовая система описывается волновой функцией, удовлетворяющей уравнению вида

$$\left[\left(\nabla_\mu - i\frac{e}{c}A_\mu\right)^2 - \frac{m^2c^2}{\hbar^2}\right]\Psi(x) = \beta\left|\Psi(x)\right|^2\Psi(x), \tag{22}$$

пригодного для описания широкого класса интерференционных систем.

Используем метод точного в первом порядке по линеаризованной гравитации решения обобщенного скалярного уравнения (22), когда решение находится в квадратурах из решения в плоском пространстве-времени $\Phi(x)$ в виде

$$\Psi(x) = e^{-i\chi(x)}\Phi(x), \tag{23}$$

где фазовый фактор χ определяется как

$$\chi = -\frac{1}{4}\int\limits_p^x dz^\lambda \left[\Gamma_{\alpha,\lambda\beta} - \Gamma_{\beta,\lambda\alpha}\right]\hat{J}^{\alpha\beta}(z) + \frac{1}{2}\int\limits_p^x dz^\lambda \gamma_{\lambda\beta}\hat{P}^\beta, \tag{24}$$

здесь $\hat{J}^{\alpha\beta}(z)$ – генератор группы Лоренца:

$$\hat{J}^{\alpha\beta}(z)F(x) = i\left[(x^\alpha - z^\alpha)\partial^\beta F(x) - (x^\beta - z^\beta)\partial^\alpha F(x)\right], \tag{25}$$

а $\hat{P}^\alpha$ – генератор группы сдвигов:

$$\hat{P}^\alpha F(x) = i\partial^\alpha F(x). \tag{26}$$

В линеаризованной гравитации

$$\begin{aligned} \gamma_{\mu\nu} &= g_{\mu\nu} - \eta_{\mu\nu}, \\ \Gamma_{\alpha,\lambda\beta} &= \frac{1}{2}\left(\gamma_{\alpha\lambda,\beta} + \gamma_{\alpha\beta,\lambda} - \gamma_{\lambda\beta,\alpha}\right) \\ R_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(\gamma_{\mu\alpha,\nu\beta} - \gamma_{\nu\alpha,\mu\beta} - \gamma_{\mu\beta,\nu\alpha} + \gamma_{\nu\beta,\mu\alpha}\right). \end{aligned} \tag{27}$$

Можно показать, что если $\Phi(x)$ удовлетворяет уравнению

$$\eta^{\mu\nu}\Phi_{,\mu\nu} - \frac{m^2c^2}{\hbar^2}\Phi = \beta\left|\Phi\right|^2\Phi, \tag{28}$$

то

$$\Psi(x) = e^{-i\chi}\Phi(x) \tag{29}$$

удовлетворяет (22) с $e = 0$. Т.е. эффект слабого гравитационного поля может быть описан фазовым фактором, если известно решение в отсутствие гравитации.

Эффект электромагнитного поля может быть учтен через фазовый фактор дополнительным членом

$$-\frac{e}{\hbar c}\int\limits_p^x dz^\lambda A_\lambda(z). \tag{30}$$

Решение (29) в первом порядке калибровочно-инвариантно и инвариантно при преобразрваниях

$$x'^\mu = x^\mu + \xi^\mu, \tag{31}$$

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}, \tag{32}$$

где ξ_μ – достаточно гладкие функции, обеспечивающие малость $\gamma_{\mu\nu}$.

Тогда [11], по теореме Стокса

$$\chi = -\frac{1}{4}\int\limits_{S_p} R_{\mu\nu\alpha\beta}J^{\alpha\beta}df^{\mu\nu} - \frac{e}{\hbar c}\int\limits_{S_p} F_{\mu\nu}df^{\mu\nu}. \tag{33}$$

Расчет фазового сдвига для интерференции скалярных частиц во вращающейся системе отсчета в плоском пространстве-времени и в стационарной системе отсчета в поле Шварцшильда дает следующие результаты [7]:

$$\Delta\chi = -\frac{ml^2\omega}{\hbar}\left(2 + \frac{1}{2}\frac{v^2}{c^2}\right) \tag{34}$$

и

$$\Delta\chi = 2\pi\lambda\frac{g}{h^2}m^2S\sin\delta\left(1 + \frac{1}{2}\beta^2\right), \tag{35}$$

где l – длина плеча квадратного интерферометра, покоящегося во вращающейся с угловой скоростью ω системе отсчета, δ – угол фиксирующий ориентацию интерферометра.

Полученные выражения отличаются от результатов Анандана во втором порядке по $\beta = \frac{v}{c}$, подтверждая некорректность рассмотрения фазы скалярной частицы по аналогии с фазой спинорных частиц, отмеченную впервые в работе [12].

Порядок эффекта позволяет надеяться, что развитие атомной интерферометрии даст возможность зарегистрировать его в ближайшем будущем.

Примененный подход дает новую трактовку процесса взаимодействия квантовой частицы с гравитационным полем, что может оказаться существенным при построении квантовой теории гравитации.

Список литературы

[1] E. Cartan, L'enselgment mathematique, 1924, 24, 5.

[2] Левашев А.Е. Рассеяние частиц силовым центром как перемещение, ассоциированное с циклом по Э. Картану / А.Е. Левашев, Е.А. Ушаков // Теория относительности и гравитация. – Москва, Наука, 1976. – С. 64-73

[3] Ушаков Е.А. Расчет гравитационно-индуцированного фазового сдвига // Е.А. Ушаков // Сб. материалов V-й международной конференции "Современные проблемы преподавания теоретической и математической физики, физики конденсированных сред и астрономии". – Брест, 15-16 октября 2015 г. – С. 115-116.

[4] Мицкевич Н.В. Физическое поле в общей теории относительности. – Москва, Наука, 1969. – С.85.

[5] Hayashi K. Gravity, Nonintegrable Phase Factor and Motion, Associated with a Closed Curve / K. Hayashi , T. Shirafuji // Progr. Theor. Phys., 1977. – V. 58, p. 353.

[6] Ушаков Е.А. Отклонение луча света в полях Рейснера-Нордстрема и Керра как перемещение, ассоциированное с циклом. // Е.А. Ушаков, Н.Н. Костюкович // Изв. АН БССР, серия физ.-мат. наук, 1976. – №̲ 3, с. 113-116.

[7] Сивцов И.А. К вопросу об интерференционном эксперименте в гравитационном поле / И.А. Сивцов, Е.А. Ушаков // Сб. материалов международной научно-практической конференции, 22-29 октября 2003 г. – "Современные научные проблемы и вопросы преподавания теоретической и математической физики, физики конденсированных сред и астрономии". – Брест, 2003. – С. 85-87.

[8] Collela R. Observation of Gravitationally Induced Quantum Interference / R. Colella, A. W. Overhauser, and S. A. Werner // Phys. Rev. Lett. – 1975. – Vol. 34. – P. 1472.

[9] Anandan J. New Relativistic Gravitational Effects Using Charged Particel / J. Anandan // General Relativity and Gravitation. – 1984. – Vol. 16. – PP. 33-41.

[10] Ушаков Е.А. О гравитационных эффектов в квантовой области / Е.А. Ушаков // Сб. статей "Гравитация и электромагнетизм". – Минск. – 1998. – С. 192-195.

[11] Papini G. Particle Wave Functions in Weak Gravitational Fields / G. Papini // Nuovo Cimento. – 1967. – Vol. 52B. – PP. 136-141.

[12] Alsing P.M. The Phase of a Quantum Mechanical Particle in Curved Spacetime / P.M. Alsing, J.C. Evans, K.K. Nandi // Gen. Rev. Grav. – 2001. – V. 33. – PP. 1459-1487.

On the possible trajectories of spinning particles in an external electromagnetic fields

A. N. Tarakanov
tarak-ph@mail.ru; Institute of Informational Technologies,
Belarusian State University of Informatics and Radioelectronics, Belarus

Abstract

By means of the method of moving Frenet-Serret frame the set of equations of motion is derived for spinning particle in an arbitrary external field, which is determined by potential depend-ing from both position and the state of movement, as well as by two pseudo-vectors one of which is easily associated with external magnetic field, and another still remains undetermined. It is shown that description of the motion of both massive and massless particles with spin is possible. All solutions of the equations of motion of spinning particle in the absence of external fields were found, and besides, we give more precise definition of a free object. It turns out that the massive particles always possess a longitudinal polarization. There are possible transversal motions of the following types: 1) oscillatory motion with proper frequency, 2) circular motion, and 3) complicated motion along rosette trajectories round the center of inertia with the velocity, varying in the limits $v_{\min} < v < v_{\max}$. Free massless particles can either fluctuate or move along ellipses around fixed centers of balance, with the spin of particles can have any direction. The behavior of spin particles in a constant homogeneous magnetic field is also considered and all types of trajectories are found.

1 Introduction

It became clear after theoretical discovery of Zitterbewegung by Schrödinger and numerous papers devoted to it, that spin makes essential impact on a trajectory even for the free particles. However a possibility of taking account of spin in terms of classical nonrelativistic theory has not been properly analyzed. In the papers [1]-[3] an idea is proposed for description of the objects with internal degrees of freedom by generalization of the Second Newton's Law, and classic non-relativistic theory of such objects interacting with external fields by means of potential depending on both position and the state of motion of the object has been developed.

In [2]-[3] solutions are found for non-relativistic equation of motion for free mass point in the center-of-inertia reference frame (r. f.), which generally does not coincide with the center-of-mass r. f. moving through the complicated trajectory around the direction of movement of the center of inertia that can be interpreted as Zitterbewegung. In this connection a particle should be considered as non-inertial r. f. with some structure, whose spin is classical proper angular momentum. The main conclusion lies in the fact that the electric charge should not be seen as a physical quantity characterizing the electromagnetic interaction, but as a consequence of the presence of the particles spin. Then electromagnetic interaction can be interpreted as interaction of the spin with external field. So the question arises, how the motion of spinning particles in an electromagnetic field and their interaction with each other can be associated with the behavior of charged particles considered in the framework of the standard (classical or quantum) electrodynamics. In this paper, under the proposed equations of motion the effect of spin on the behavior of both free particles and particles in an external magnetic field is studied.

2 Equations of Motion in Arbitrary External Field

The equation of motion of a particle with spin in an external field can be written as the Second Newton's Law $d\mathbf{P}/dt = \mathbf{F} = \mathbf{F}_{\rm g} + \mathbf{F}^{\rm ext}$ ([1]-[3]), or

$$\frac{d}{dt}\left(m_0\mathbf{V} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}]\right) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}] = \\ -\frac{\partial U}{\partial \mathbf{R}} + \frac{d}{dt}\frac{\partial U}{\partial \mathbf{V}} + [\mathbf{C}^{\rm ext}\times\mathbf{V}] - \frac{d}{dt}[\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}]\ , \tag{1}$$

where $\mathbf{F}_{\rm g} = -\varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}]$ is a gyroscopic force, arising due to non-inertial object, $\mathbf{F}^{\rm ext} = -\partial U/\partial\mathbf{R} + [\mathbf{C}^{\rm ext}\times\mathbf{V}]$ is external force,

$$U = U_0 + \varsigma\Omega_0^2([\mathbf{R}\times\mathbf{V}]\cdot\mathbf{s}) - ([\mathbf{R}\times\mathbf{V}]\cdot\mathbf{C}^{\rm ext})$$

is potential energy of interaction, $\mathbf{\Omega}_0$ is Zitterbewegung frequency of free particle, $\mathbf{C}^{\rm ext}$ and $\mathbf{S}^{\rm ext}$ are some pseudo-vectors due to external field.

Dynamical momentum of the object $\mathbf{P} = \mathbf{P}_{\rm kin} + \mathbf{A}$ contains proper kinetic momentum $\mathbf{P}_{\rm kin} = \\ = m_0\mathbf{V} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}]$ and addition $\mathbf{A} = -\partial U/\partial\mathbf{V} + [\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}]$, arising due to interaction with external fields. This addition leads to a re-normalization of the force $\mathbf{F}$, so that the change of the proper kinetic momentum $\mathbf{P}_{\rm kin}$ is determined by the right-hand side of the equation of motion (1), which can be compared with the Lorentz force, if we put

$$\mathbf{E} = -\frac{\partial U}{\partial\mathbf{R}} + \frac{d}{dt}\left(\frac{\partial U}{\partial\mathbf{V}} - [\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}]\right) = -\frac{\partial U}{\partial\mathbf{R}} - \frac{d\mathbf{A}}{dt}\ ,\ \mathbf{B} = -\mathbf{C}^{\rm ext}\ , \tag{2}$$

which shows that addition $\mathbf{A}$ has a role of vector potential, introduced by F. E. Neumann. If we assume the definition (2), the equation of motion (1) takes the form

$$\frac{d}{dt}\left(m_0\mathbf{V} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}]\right) + \varsigma[\mathbf{s}\times\mathbf{V}] = \mathbf{E} + [\mathbf{V}\times\mathbf{B}]\ , \tag{3}$$

and the equation of motion of spin is given by

$$\dot{\mathbf{s}} = [\mathbf{\Omega}\times\mathbf{s}]\ , \tag{4}$$

where $\mathbf{\Omega}$ is angular velocity of spin precession.

Equation (1) leads to the balance of total energy

$$\mathcal{E} = \frac{m_0\mathbf{V}^2}{2} + \varsigma(\mathbf{V}\cdot[\mathbf{s}\times\dot{\mathbf{V}}]) + (\mathbf{V}\cdot[\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}]) - (\mathbf{V}\cdot\frac{\partial U}{\partial\mathbf{V}}) + U\ . \tag{5}$$

Let r. f. K′, whose origin in K is specified by radius vector $\mathbf{R}_{(\rm K')}$, be the r. f. moving relative to absolute one with the velocity $\mathbf{V}_{(\rm K')}$. We write down the equation (1), written in absolute r. f. K, in moving r. f. K′. Then $\mathbf{R} = \mathbf{R}_{(\rm K')} + \mathbf{r}$, $\mathbf{V} = \mathbf{V}_{(\rm K')} + \mathbf{v}$, ..., where $\mathbf{r}$ and $\mathbf{v}$ are radius vector and velocity of the object relative to the origin of K′. It can be shown that

$$\frac{\partial U}{\partial\mathbf{R}} = \frac{\partial U}{\partial\mathbf{r}}\ ,\frac{\partial U}{\partial\mathbf{V}} = \frac{\partial U}{\partial\mathbf{v}}\ ,\ \dots\ . \tag{6}$$

Substitution of (6) into (1) leads it to the form

$$\frac{d}{dt}(m_0\mathbf{v} + \varsigma[\mathbf{s}\times\dot{\mathbf{v}}]) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{v}] + \frac{d}{dt}(m_0\mathbf{V}_{(\rm K')} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}_{(\rm K')}]) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}_{(\rm K')}] = \\ = -\frac{\partial U}{\partial\mathbf{r}} + \frac{d}{dt}\frac{\partial U}{\partial\mathbf{v}} + [\mathbf{v}\times\mathbf{B}] - \frac{d}{dt}[\mathbf{S}^{\rm ext}\times\dot{\mathbf{v}}] + [\mathbf{V}_{(\rm K')}\times\mathbf{B}] - \frac{d}{dt}[\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}_{(\rm K')}]\ . \tag{7}$$

There is no necessity to use vector potential to describe the motion in a constant electromagnetic field. In this case electric field is defined as

$$\mathbf{E} = -\frac{\partial U}{\partial \mathbf{R}} = -\frac{\partial U}{\partial \mathbf{r}} \ , \tag{8}$$

whence it follows

$$U = -\int (\mathbf{E}\cdot\mathbf{R}) + u(\mathbf{V}, \dot{\mathbf{V}}, ..., \dot{\mathbf{V}}^{(N)}) = -\int (\mathbf{E}\cdot\mathbf{r}) + u(\mathbf{v}, \dot{\mathbf{v}}, ..., \dot{\mathbf{v}}^{(N)}) \ . \tag{9}$$

On the other hand, in the case of varying field its dependence on time may be considered in potential function together with conservation of the definition (8) instead of generally accepted definition $\mathbf{E} = -\nabla U - \partial\mathbf{A}/\partial t$ in Maxwell electrodynamics. Then the equation (1) will become

$$\frac{d}{dt}\left(m_0\mathbf{V} - \frac{\partial u}{\partial \mathbf{V}} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}] + [\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}]\right) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}] = \mathbf{E} + [\mathbf{V}\times\mathbf{B}] \ . \tag{10}$$

In the moving r. f. K$'$ relations $\mathbf{V}_{(\mathrm{K}')} = \mathbf{0}$, $\dot{\mathbf{V}}_{(\mathrm{K}')} = \mathbf{0}$ should be fulfilled. Therefore in view of (6), (8) equation (7) splits into two equations

$$\frac{d}{dt}\left(m_0\mathbf{v} - \frac{\partial u}{\partial \mathbf{v}} + \varsigma[\mathbf{s}\times\dot{\mathbf{v}}] + [\mathbf{S}^{\rm ext}\times\dot{\mathbf{v}}]\right) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{v}] = \mathbf{E} + [\mathbf{v}\times\mathbf{B}] \ , \tag{11}$$

$$\frac{d}{dt}\left(m_0\mathbf{V}_{(\mathrm{K}')} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}_{(\mathrm{K}')}] + [\mathbf{S}^{\rm ext}\times\dot{\mathbf{V}}_{(\mathrm{K}')}]\right) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}_{(\mathrm{K}')}] = [\mathbf{V}_{(\mathrm{K}')}\times\mathbf{B}] \ . \tag{12}$$

We shall introduce in K$'$ orthonormal basis Frenet-Serret

$$\mathbf{e}_\tau = \frac{\mathbf{v}}{v} = [\mathbf{e}_{\rm n}\times\mathbf{e}_{\rm b}] \ , \ \mathbf{e}_{\rm n} = \frac{[\mathbf{v}\times[\dot{\mathbf{v}}\times\mathbf{v}]]}{|[\mathbf{v}\times[\dot{\mathbf{v}}\times\mathbf{v}]]|} = [\mathbf{e}_{\rm b}\times\mathbf{e}_\tau] \ , \ \mathbf{e}_{\rm b} = \frac{[\mathbf{v}\times\dot{\mathbf{v}}]}{|[\mathbf{v}\times\dot{\mathbf{v}}]|} = [\mathbf{e}_\tau\times\mathbf{e}_{\rm n}] \ , \tag{13}$$

with equations of motion

$$\dot{\mathbf{e}}_\tau = [\mathbf{\Omega}_{\rm D}\times\mathbf{e}_\tau] = vK\mathbf{e}_{\rm n} \ , \ \dot{\mathbf{e}}_{\rm n} = [\mathbf{\Omega}_{\rm D}\times\mathbf{e}_{\rm n}] = -vK\mathbf{e}_\tau + vT\mathbf{e}_{\rm b} \ , \ \dot{\mathbf{e}}_{\rm b} = [\mathbf{\Omega}_{\rm D}\times\mathbf{e}_{\rm b}] = -vT\mathbf{e}_{\rm n} \ , \tag{14}$$

where $\mathbf{\Omega}_{\rm D} = v(T\mathbf{e}_\tau + K\mathbf{e}_{\rm b})$ is Darboux vector defining an angular velocity of moving frame, $K = |[\mathbf{v}\times\dot{\mathbf{v}}]|/v^3$ is curvature, $T = (\mathbf{v}\cdot[\dot{\mathbf{v}}\times\ddot{\mathbf{v}}])/[\mathbf{v}\times\dot{\mathbf{v}}]^2$ is torsion of the trajectory. If we decompose spin pseudo-vector in the basis (13) we find from (4) that $\mathbf{\Omega} = \mathbf{\Omega}_{\rm D}$, and spin components s_τ, $s_{\rm n}$ and $s_{\rm b}$ are constant.

We choose r. f. K$'$ so that its velocity has to be orthogonal to the plane of vectors $\mathbf{v}$ and $\dot{\mathbf{v}}$, $\mathbf{V}_{(\mathrm{K}')} = V_{(\mathrm{K}')}\mathbf{e}_{\rm b}$. When it is inertial frame, then $\dot{\mathbf{V}}_{(\mathrm{K}')} = \mathbf{0}$, the torsion T vanishes, and the binormal direction is kept constant, which greatly simplifies the equation of motion.

For the sake of simplicity we assume that the source of the field U is at rest and function u does not depend on accelerations $\dot{\mathbf{v}}, \ddot{\mathbf{v}}, ..., \mathbf{v}^{(N)}$ of the particle. Then

$$\frac{d}{dt}\frac{\partial u}{\partial \mathbf{v}} = \frac{d}{dt}\left(\frac{du}{vdv}\mathbf{v}\right) = \frac{du}{vdv}\dot{\mathbf{v}} + (\mathbf{v}\cdot\dot{\mathbf{v}})\frac{d}{vdv}\left(\frac{du}{vdv}\right)\mathbf{v} \ . \tag{15}$$

Further, we seek a solution of equation (11) in the form

$$\mathbf{v}(t) = v(t)[\cos\Phi(t)\mathbf{e}_X + \sin\Phi(t)\mathbf{e}_Y] \ . \tag{16}$$

where $\mathbf{e}_X$, $\mathbf{e}_Y$ are unit vectors of coordinate system in the plane which is orthogonal to the Z axis in absolute r. f., $\mathbf{e}_Z = \mathbf{e}_{\rm b}$. Then the spin of the particle is equal to

$$\mathbf{s} = (s_\tau - s_{\rm n}\sin\Phi)\mathbf{e}_X + (s_\tau + s_{\rm n}\cos\Phi)\mathbf{e}_Y + s_{\rm b}\mathbf{e}_Z \ . \tag{17}$$

In view of (15)-(17) equations of motion (11), (12) and spin precession (4) are

$$\left\{\frac{d}{dt}\left[\left(m_0v-\frac{du}{dv}-(\varsigma s_{\rm b}+S_Z^{\rm ext})v\dot\Phi\right)\cos\Phi-(\varsigma s_{\rm b}+S_Z^{\rm ext})\dot v\sin\Phi\right]-\varsigma s_{\rm b}v\Omega_0^2\sin\Phi\right\}\mathbf{e}_X+$$

$$+\left\{\frac{d}{dt}\left[\left(m_0v-\frac{du}{dv}-(\varsigma s_{\rm b}+S_Z^{\rm ext})v\dot\Phi\right)\sin\Phi+(\varsigma s_{\rm b}+S_Z^{\rm ext})\dot v\cos\Phi\right]+\varsigma s_{\rm b}v\Omega_0^2\cos\Phi\right\}\mathbf{e}_Y+$$

$$+\frac{d}{dt}\left[(S_X^{\rm ext}\dot v+S_Y^{\rm ext}v\dot\Phi)\sin\Phi+(S_X^{\rm ext}v\dot\Phi-S_Y^{\rm ext}\dot v)\cos\Phi\right]\mathbf{e}_Z+$$
$$+\left[\varsigma s_\tau(v\ddot\Phi+\dot v\dot\Phi)-\varsigma s_{\rm n}(\ddot v+\Omega_0^2v)\right]\mathbf{e}_Z=(E_X+vB_Z\sin\Phi)\mathbf{e}_X+$$
$$+(E_Y-vB_Z\cos\Phi)\mathbf{e}_Y+(E_Z+vB_Y\cos\Phi-vB_X\sin\Phi)\mathbf{e}_Z\ , \tag{18}$$

$$\frac{d}{dt}\left[\dot V_{(\rm K')}(S_Y^{\rm ext}+\varsigma s_\tau\sin\Phi+\varsigma s_{\rm n}\cos\Phi)\right]\mathbf{e}_X+\left[\varsigma s_\tau\sin\Phi+\varsigma s_{\rm n}\cos\Phi\right]\Omega_0^2V_{(\rm K')}\mathbf{e}_X-$$
$$-\frac{d}{dt}\left[\dot V_{(\rm K')}(S_X^{\rm ext}+\varsigma s_\tau\cos\Phi-\varsigma s_{\rm n}\sin\Phi)\right]\mathbf{e}_Y+\left[\varsigma s_{\rm n}\sin\Phi-\varsigma s_\tau\cos\Phi\right]\Omega_0^2V_{(\rm K')}\mathbf{e}_Y+$$

$$+m_0\dot V_{(\rm K')}\mathbf{e}_Z=-V_{(\rm K')}B_Y\mathbf{e}_X+V_{(\rm K')}B_X\mathbf{e}_Y\ , \tag{19}$$

$$\dot{\mathbf{s}}=[\mathbf{\Omega}_{\rm D}\times\mathbf{s}]=\dot\Phi\left[-(s_\tau\sin\Phi+s_{\rm n}\cos\Phi)\mathbf{e}_X+(s_\tau\cos\Phi-s_{\rm n}\sin\Phi)\mathbf{e}_Y\right]\ , \tag{20}$$

where $\mathbf{\Omega}_{\rm D}=\Omega_{\rm b}\mathbf{e}_{\rm b}=\dot\Phi\mathbf{e}_Z$.

It follows from (19), that $\dot V_{(\rm K')}=0$ at $m_0\neq0$, so that (19) is equivalent to equations

$$\varsigma s_{\rm n}\Omega_0^2=B_X\sin\Phi-B_Y\cos\Phi\ ,\ \varsigma s_\tau\Omega_0^2=-B_X\cos\Phi-B_Y\sin\Phi\ ; \tag{21}$$

whence it follows $\varsigma^2(s_\tau^2+s_{\rm n}^2)\Omega_0^4=B_X^2+B_Y^2$, as well as

$$\sin\Phi=\frac{\varsigma\Omega_0^2(s_{\rm n}B_X-s_\tau B_Y)}{B_X^2+B_Y^2}\ ,\quad \cos\Phi=-\frac{\varsigma\Omega_0^2(s_\tau B_X+s_{\rm n}B_Y)}{B_X^2+B_Y^2}\ . \tag{22}$$

In the case $m_0=0$ the moving r. f. K′ may be non-inertial one. Then (19) reduces to the set of two equations

$$\frac{d}{dt}\left[\dot V_{(\rm K')}S_Y^{\rm ext}\right]+\left[\varsigma s_\tau(\ddot V_{(\rm K')}+\Omega_0^2V_{(\rm K')})-\varsigma s_{\rm n}\dot V_{(\rm K')}\dot\Phi\right]\sin\Phi+$$
$$+\left[\varsigma s_{\rm n}(\ddot V_{(\rm K')}+\Omega_0^2V_{(\rm K')})+\varsigma s_\tau\dot V_{(\rm K')}\dot\Phi\right]\cos\Phi+B_YV_{(\rm K')}=0\ , \tag{23}$$

$$\frac{d}{dt}\left[\dot V_{(\rm K')}S_X^{\rm ext}\right]+\left[\varsigma s_\tau(\ddot V_{(\rm K')}+\Omega_0^2V_{(\rm K')})-\varsigma s_{\rm n}\dot V_{(\rm K')}\dot\Phi\right]\cos\Phi-$$
$$-\left[\varsigma s_{\rm n}(\ddot V_{(\rm K')}+\Omega_0^2V_{(\rm K')})+\varsigma s_\tau\dot V_{(\rm K')}\dot\Phi\right]\sin\Phi+B_XV_{(\rm K')}=0\ . \tag{24}$$

In the r. f. K′ the energy (5) is conserved and is the self-energy of particle,

$$\mathcal{E}=\mathcal{E}_0=\frac{m_0\mathbf{v}^2}{2}-(\mathbf{v}\cdot\frac{\partial U}{\partial\mathbf{v}})+U+(\mathbf{v}\cdot[(\varsigma\mathbf{s}+\mathbf{S}^{\rm ext})\times\dot{\mathbf{v}}])=$$
$$=\frac{m_0v^2}{2}-v\frac{du}{dv}+u-v^2\dot\Phi(\varsigma s_{\rm b}+S_Z^{\rm ext})\ , \tag{25}$$

which gives the relation the relation

$$(\varsigma s_{\rm b} + S_Z^{\rm ext})(v\ddot{\Phi} + 2\dot{v}\dot{\Phi}) + v\dot{S}_Z^{\rm ext}\dot{\Phi} = \frac{d}{dt}\left(m_0 v - \frac{du}{dv}\right) . \tag{26}$$

In view of (21), (26) and $\dot{V}_{(\rm K')} = 0$ for massive particles (18) reduces to the set of equations

$$E_X \cos\Phi + E_Y \sin\Phi = 0 , \tag{27}$$

$$\left(m_0 v - \frac{du}{dv}\right) + (\varsigma s_{\rm b} + S_Z^{\rm ext})(\ddot{v} - v\dot{\Phi}^2) + \dot{S}_Z^{\rm ext}\dot{v} + (\varsigma s_{\rm b}\Omega_0^2 + B_Z)v =$$

$$= -E_X \sin\Phi + E_Y \cos\Phi , \tag{28}$$

$$\frac{d}{dt}\left[\varsigma s_\tau v\dot{\Phi} - \varsigma s_{\rm n}\dot{v} + (S_X^{\rm ext}\dot{v} + S_Y^{\rm ext}v\dot{\Phi})\sin\Phi + (S_X^{\rm ext}v\dot{\Phi} - S_Y^{\rm ext}\dot{v})\cos\Phi\right] = E_Z . \tag{29}$$

For massless particles one should set $m_0 = 0$ in (28) and instead of (21) one should use (23)-(24). We emphasize that obtained system of equations holds for an arbitrary external field.

3 The motion of free particle

In [2]-[3] there considered equations of motion (1) and (4) for the case $U_0 = 0$, $\mathbf{S}^{\rm ext} = \mathbf{0}$, $\mathbf{C}^{\rm ext} = \mathbf{0}$, and all solutions of them are found. By definition an object should be free if the latter two conditions are fulfilled and the force $\mathbf{F}$ vanishes at all time. Then $U_0 = 0$ is special case of the condition $\partial U_0/\partial\mathbf{R} = \mathbf{0}$, whence it follows that U_0 may be function of the velocity and accelerations. On the other hand, if free object will be defined by the condition $\partial U/\partial\mathbf{R} = \mathbf{0}$, whence $U = u(t, \mathbf{V}, \dot{\mathbf{V}}, \ddot{\mathbf{V}}, ..., \dot{\mathbf{V}}^{(N)})$, then in the l. h. s. of (1) a term remains that make sense of gyroscopic force. If U does not depend explicitly on time, and velocity, and accelerations, then it is constant, and it may be set zero. It is easy to show in this case that joint solution of (1) and (4) leads to the fact that spin of the object is always collinear to the velocity $\mathbf{V}$ which remains constant, $\dot{\mathbf{V}} = \mathbf{0}$. Hence, the force $\mathbf{F}$ vanishes, $U = U_0$, so that the object in question moves inertially in total concordance with standard Newton's mechanics.

If $U = u \neq 0$, then one should be based on (27)-(29) and (21) for massive particles or (23), (24) for massless particles at $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ and $\mathbf{S}^{\rm ext} = \mathbf{0}$. It follows from (21) that $s_\tau = s_{\rm n} = 0$ and (23), (24) reduce to the set

$$s_\tau(\ddot{V}_{(\rm K')} + \Omega_0^2 V_{(\rm K')}) = s_{\rm n}\dot{V}_{(\rm K')}\dot{\Phi} , \; s_{\rm n}(\ddot{V}_{(\rm K')} + \Omega_0^2 V_{(\rm K')}) = -s_\tau\dot{V}_{(\rm K')}\dot{\Phi} , \tag{30}$$

which are valid also when $s_\tau \neq 0$, $s_{\rm n} \neq 0$. Equations (27)-(29) reduce to

$$\left(m_0 v - \frac{du}{dv}\right)\dot{\Phi} = -\varsigma s_{\rm b}[\ddot{v} + (\Omega_0^2 - \dot{\Phi}^2)v] , \tag{31}$$

$$s_{\rm n}\ddot{v} = s_\tau(v\ddot{\Phi} + \dot{v}\dot{\Phi}) , \;\; s_{\rm n}\dot{v} - s_\tau v\dot{\Phi} = \text{const} , \tag{32}$$

where (32) takes place for massless particle and becomes identity for massive one.

Substitution of (26) at $S_Z^{\rm ext} = 0$ into (31) leads to

$$\frac{d}{dt}\left(\frac{\ddot{v} + \Omega_0^2 v}{\dot{\Phi}}\right) + \dot{v}\dot{\Phi} = 0 , \tag{33}$$

whence it follows the first integral

$$\frac{(\ddot{v}+\Omega_0^2 v)^2}{\dot{\Phi}^2}+\dot{v}^2+\Omega_0^2 v^2=D^2=\text{const}\ , \tag{34}$$

and

$$\dot{\Phi}=\frac{\ddot{v}+\Omega_0^2 v}{\sqrt{D^2-\dot{v}^2-\Omega_0^2 v^2}}\ , \tag{35}$$

Now substituting (35) into (31), we obtain the equation

$$\frac{m_0 v-du/dv}{\sqrt{D^2-\dot{v}^2-\Omega_0^2 v^2}}(\ddot{v}+\Omega_0^2 v)=-\varsigma s_{\rm b}(\ddot{v}+\Omega_0^2 v)\left[1-\frac{(\ddot{v}+\Omega_0^2 v)v}{D^2-\dot{v}^2-\Omega_0^2 v^2}\right]\ , \tag{36}$$

which is valid in two cases: I. $\ddot{v}+\Omega_0^2 v=0$, $\dot{\Phi}=0$, and II. $\ddot{v}+\Omega_0^2 v\neq 0$, $\dot{\Phi}\neq 0$. Let us consider them in details.

I.1.

$$\ddot{v}+\Omega_0^2 v=0,\ \dot{\Phi}=0,\ s_\tau=0,\ s_{\rm n}=0, m_0\geq 0.$$

In this case we have $v=v_0\cos(\Omega_0 t+\varphi_0)$, whence the equation of the trajectory in the absolute r. f. is given by

$$\mathbf{R}(t)=\mathbf{R}(0)+\frac{v_0}{\Omega_0}[\sin(\Omega_0 t+\varphi_0)-\sin\varphi_0](\cos\Phi\mathbf{e}_X+\sin\Phi\mathbf{e}_Y)+V_{(\mathrm{K}')}t\mathbf{e}_Z\ , \tag{37}$$

i. e. the particle has a longitudinal polarization and oscillates in the plane (X,Y) around the center of inertia moving along the Z axis with velocity $\mathbf{V}_{(\mathrm{C})}=\mathbf{V}_{(\mathrm{K}')}$. At $\Omega_0=0$ oscillations are absent, and particle moves along the Z axis in accordance with the law of inertia of Galileo-Newton.

I.2. $\ddot{v}+\Omega_0^2 v=0$, $\dot{\Phi}=0$, $m_0=0$. Equations (30) and (32) give

$$\ddot{V}_{(\mathrm{K}')}+\Omega_0^2 V_{(\mathrm{K}')}=0,\ V_{(\mathrm{K}')}=V_{(\mathrm{K}')0}\cos(\Omega_0 t+\varphi_0),\ s_{\rm n}=0,\ s_\tau\neq 0.$$

Thus, we obtain a trajectory

$$\mathbf{R}(t)=\mathbf{R}(0)+\frac{v_0}{\Omega_0}\sin(\Omega_0 t+\varphi_0)(\cos\Phi\mathbf{e}_X+\sin\Phi\mathbf{e}_Y)+\frac{V_{(\mathrm{K}')0}}{\Omega_0}\sin(\Omega_0 t+\varphi_1)\mathbf{e}_Z\ , \tag{38}$$

which in general is an ellipse, transforming into either circle at $\Delta\varphi=\varphi_1-\varphi_0=(2m+1)\pi/2$ or line segment at $\Delta\varphi=m\pi$, where m is integer.

II. The second case of equation (36) for function $u(v)$ is specified by conditions $\ddot{v}+\Omega_0^2 v\neq 0$, $\dot{\Phi}\neq 0$. We have

$$\frac{du}{dv}=\left[m_0-\frac{\varsigma s_{\rm b}(\ddot{v}+\Omega_0^2 v)}{\sqrt{D^2-\dot{v}^2-\Omega_0^2 v^2}}\right]v+\varsigma s_{\rm b}\sqrt{D^2-\dot{v}^2-\Omega_0^2 v^2}\ . \tag{39}$$

whence it follows

$$u(v)=\frac{m_0 v^2}{2}+\varsigma s_{\rm b} v\sqrt{D^2-\dot{v}^2-\Omega_0^2 v^2}+\mathcal{E}_0\ . \tag{40}$$

Equation of motion has the infinite set of solutions, one of which corresponds to $\Phi = \Omega_{\rm D} t$, where $\Omega_{\rm D} = \pm\sqrt{\mathbf{\Omega}_{\rm D}^2} = \text{const}$. In this case equation (35) admits first integral of motion $\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2} + \Omega_{\rm D} v = F$, which reduces to equation for velocity

$$\dot{v} = \pm\sqrt{D^2 - F^2 + 2F\Omega_{\rm D} v - (\Omega_{\rm D}^2 + \Omega_0^2)v^2}\ . \tag{41}$$

Substitution of (41) into (35) leads to equation

$$\ddot{v} + (\Omega_{\rm D}^2 + \Omega_0^2)v = F\Omega_{\rm D}\ , \tag{42}$$

which has general solution

$$v(t) = \frac{F\Omega_{\rm D}}{\chi^2} + v_0 \cos(\chi t + \varphi_0)\ ,\ \chi = \sqrt{\Omega_{\rm D}^2 + \Omega_0^2}\ , \tag{43}$$

It follows from (41) and (43), that velocity may vary in the limits $v_{\min} < v < v_{\max}$, where

$$v_{\min} = \frac{F\Omega_{\rm D}}{\chi^2} - v_0 \geq 0\ ,\ \ v_{\max} = \frac{F\Omega_{\rm D}}{\chi^2} + v_0\ ,\ \ v_0 = \frac{\sqrt{D^2\chi^2 - F^2\Omega_0^2}}{\chi^2}\ . \tag{44}$$

For $u(v)$ we obtain the expression

$$u(v) = \frac{m_0 v^2}{2} + \varsigma s_{\rm b} v|F - \Omega_{\rm D} v| + \mathcal{E}_0\ . \tag{45}$$

The law of motion for both massive and massless particles with longitudinal spin polarization looks like

$$\begin{aligned}\mathbf{R}(t) = \mathbf{R}(0) + \frac{F}{\chi^2}(\sin\Omega_{\rm D} t\mathbf{e}_X - \cos\Omega_{\rm D} t\mathbf{e}_Y) + V_{(\rm K')} t\mathbf{e}_Z + \\ + \frac{v_0}{2\Omega_0^2}\{(\chi - \Omega_{\rm D})\sin[(\chi + \Omega_{\rm D})t] + (\chi + \Omega_{\rm D})\sin[(\chi - \Omega_{\rm D})t]\}\mathbf{e}_X - \\ - \frac{v_0}{2\Omega_0^2}\{(\chi - \Omega_{\rm D})\cos[(\chi + \Omega_{\rm D})t] - (\chi + \Omega_{\rm D})\cos[(\chi - \Omega_{\rm D})t]\}\mathbf{e}_Y\ .\end{aligned} \tag{46}$$

Note that there is another possibility that the equation of motion (32)-(33) are satisfied.

III. $\dot{v} = 0$, $v = v_0 = \text{const}$. Then it follows from (25), (26)

$$\frac{d}{dt}\left(\frac{du}{dv} + \varsigma s_{\rm b} v\dot{\Phi}\right) = 0\ ,\ \ \frac{du}{dv} + \varsigma s_{\rm b} v\dot{\Phi} = C\ ,\ u(v) = \mathcal{E}_0 - \frac{m_0 v^2}{2} + Cv\ . \tag{47}$$

Equation (31) reduces to

$$m_0 v + \varsigma s_{\rm b}\frac{\Omega_0^2 v}{\dot{\Phi}} = \frac{du}{dv} + \varsigma s_{\rm b} v\dot{\Phi} = C\ , \tag{48}$$

whence $\dot{\Phi} = \Omega_{\rm D} = \varsigma s_{\rm b}\Omega_0^2 v_0/(C - m_0 v_0) = \text{const}$. As a result we obtain following solutions.

III.1. $v = v_0$, $\Phi = \Omega_{\rm D} t$, $s_\tau = s_{\rm n} = 0$, $m_0 \geq 0$. Trajectory looks like helix along the Z axis

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_0}[\sin\Omega_{\rm D} t\mathbf{e}_X + (1 - \cos\Omega_{\rm D} t)\mathbf{e}_Y] + V_{(\rm K')} t\mathbf{e}_Z\ . \tag{49}$$

De facto this solution coincides with solutions I.1, I.2 from [2], [3].

III.2. $v = v_0$, $\Phi = \Omega_{\rm D} t$, $m_0 = 0$, $s_\tau = s_{\rm n} = 0$. From (3.1) we find $\ddot{V}_{({\rm K}')} + \Omega_0^2 V_{({\rm K}')} = 0$, $V_{({\rm K}')} = = V_{({\rm K}')0} \cos(\Omega_0 t + \varphi_1)$. Trajectory is represented by radius vector

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_{\rm D}} [\sin \Omega_{\rm D} t \mathbf{e}_X + (1 - \cos \Omega_{\rm D} t) \mathbf{e}_Y] + \frac{V_{({\rm K}')0}}{\Omega_0} [\sin(\Omega_0 t + \varphi_1) - \sin \varphi_1] \mathbf{e}_Z \ , \quad (50)$$

i. e. the particle performs complex movement around the stationary center of balance. At $\Omega_{\rm D} = \Omega_0$ the trajectory, which looks like three-dimensional Lissajous figure, becomes an ellipse ($V_{({\rm K}')0} \neq v_0$) or circle (at $V_{({\rm K}')0} = v_0$ or $V_{({\rm K}')0} = 0$).

We summarize the results of this section. We suppose that in a homogeneous isotropic space behavior of a free particle with spin is described by equations (3) (with $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$) or (10) (with $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{S}^{\rm ext} = \mathbf{0}$) and (4), which lead to the following types of motion.

1. Oscillatory motion of the particle with longitudinal polarization in the plane, which is orthogonal to direction of movement of the center of inertia, described by radius vector (37). At $\Omega_0 = 0$ oscillations are absent and particle moves uniformly in a straight line. The self-energy $\mathcal{E}_0$ of the particle may have any constant value, $u(v) = m_0 v^2/2 + + Cv + \mathcal{E}_0$.

2. Oscillatory or cyclic motion of massless particles around fixed center of balance, described by the law (38). Spin $\mathbf{s} = s_\tau(\cos\Phi \mathbf{e}_X + \sin\Phi \mathbf{e}_Y) + s_{\rm b}\mathbf{e}_Z$ has a constant direction. The self-energy $\mathcal{E}_0$ of the particle may have any constant value, $u(v) = Cv + \mathcal{E}_0$.

3. A motion of particle with longitudinal spin polarization along complicated trajectory (46) clockwise ($\Omega_{\rm D} > 0$) or counter-clockwise ($\Omega_{\rm D} < 0$). The velocity of particle relative to the center of inertia varies in the limits $v_{\min} < v < v_{\max}$.

4. A motion of massive or massless particle with longitudinal spin polarization along helix (49) clockwise ($\Omega_{\rm D} > 0$, $s_{\rm b} < 0$) or counter-clockwise ($\Omega_{\rm D} < 0$, $s_{\rm b} > 0$) with constant velocity v_0 relative to the center of inertia, which moves in absolute r. f. with constant velocity $\mathbf{V}_{\rm C} = \mathbf{V}_{({\rm K}')} = V_{({\rm K}')}\mathbf{e}_Z$.

5. A motion of massless particle along complicated trajectory (50) around fixed center of balance. Spin $\mathbf{s} = s_{\rm b}\mathbf{e}_Z$ is precessing around Z axis with angular velocity $\Omega_{\rm D}$.

4 Motion in a constant homogeneous magnetic field

Now we consider the behavior of a spinning particle in a constant homogeneous magnetic field. The corresponding equations of motion are given by (27)-(29) and (21), $\dot{V}_{({\rm K}')} = 0$ for massive particles or (23)-(24), $\dot{V}_{({\rm K}')} \neq 0$ for massless particles at $\mathbf{E} = \mathbf{0}$, $\mathbf{S}^{\rm ext} = \mathbf{0}$. From the set of equations above we have several cases determined by the relation between the mass, spin and magnetic field.

I. $m_0 \neq 0$. Equation (28) is

$$\left(m_0 v - \frac{du}{dv}\right) + \varsigma s_{\rm b} \left[\ddot{v} + (\Omega_0^2 + B_Z/\varsigma s_{\rm b} - \dot{\Phi}^2) v\right] = 0 \ , \quad (51)$$

From (21) for constant field $\mathbf{B}$ we have two possibilities: I. $\dot{\Phi} = 0$, and II. $\dot{\Phi} \neq 0$, $B_X = B_Y = 0$, which lead to the following solutions.

I.1.

$$m_0 \neq 0, \ \dot{\Phi} = 0, \ s_{\rm n} = 0, \ s_\tau \neq 0, \ \ddot{v} \neq 0, \ \Omega^2 = \Omega_0^2 + B_Z/\varsigma s_{\rm b} > 0.$$

Spin of a particle according to (17) and (22) is given by

$$\mathbf{s} = -\frac{1}{\varsigma \Omega_0^2}(B_X \mathbf{e}_X + B_Y \mathbf{e}_Y) + s_{\rm b}\mathbf{e}_Z \ , \quad (52)$$

Then (51) has a solution $v(t) = v_0 \cos(\Omega t + \varphi_0)$, and trajectory is represented by

$$\mathbf{R}(t) = \mathbf{R}(0) - \frac{v_0}{\varsigma s_\tau \Omega_0^2 \Omega} \left[\sin(\Omega t + \varphi_0) - \sin\varphi_0\right] (B_X \mathbf{e}_X + B_Y \mathbf{e}_Y) + V_{(\mathrm{K}')} t \mathbf{e}_Z \ , \tag{53}$$

i. e. particle with spin (52) oscillates in the plane (XY), and its center of inertia moves uniformly along Z axis.

I.2. $m_0 \neq 0$, $\dot{\Phi} = 0$, $s_\mathrm{n} = 0$, $s_\tau \neq 0$, $\ddot{v} \neq 0$, $\tilde{\Omega}^2 = -\Omega_0^2 - B_Z/\varsigma s_\mathrm{b} > 0$. Spin of the particle is defined by (52). Equation (51) has a solution $v(t) = v_0 \cosh(\tilde{\Omega} t + \varphi_0)$, whence

$$\mathbf{R}(t) = \mathbf{R}(0) - \frac{v_0}{\varsigma s_\tau \Omega_0^2 \tilde{\Omega}} \left[\sinh(\tilde{\Omega} t + \varphi_0) - \sinh\varphi_0\right] (B_X \mathbf{e}_X + B_Y \mathbf{e}_Y) + V_{(\mathrm{K}')} t \mathbf{e}_Z \ , \tag{54}$$

The path (54) is almost a straight line and deviates from it near $t = 0$.

I.3. $m_0 \neq 0$, $\dot{\Phi} = 0$, $s_\mathrm{n} \neq 0$, s_τ may have any value. (51) and (32) lead to $\ddot{v} = 0$, $v(t) = v_0 + wt$, $w = \mathrm{const}$, $\Omega_0^2 = -B_Z/\varsigma s_\mathrm{b}$. Spin of the particle is defined by (52). The trajectory is parabolic

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{B_Z(v_0 t + wt^2/2)}{s_\mathrm{b}(B_X^2 + B_Y^2)} \left[(s_\tau B_X + s_\mathrm{n} B_Y)\mathbf{e}_X - (s_\mathrm{n} B_X - s_\tau B_Y)\mathbf{e}_Y\right] + V_{(\mathrm{K}')} t \mathbf{e}_Z \ . \tag{55}$$

or straight line at $w = 0$.

In cases I.1-I.3 we find from (25)

$$u(v) = \frac{m_0 v^2}{2} + C_1 v + \mathcal{E}_0 \ . \tag{56}$$

II. $m_0 \neq 0$, $\dot{\Phi} \neq 0$. In this case the solution of (51) is consistent with (21), only when $s_\tau = s_\mathrm{n} = 0$, $B_X = B_Y = 0$, i. e. magnetic field $\mathbf{B}$ and spin $\mathbf{s}$ are collinear with the Z axis. Then (25) gives

$$\varsigma s_\mathrm{b} \dot{\Phi} = \frac{m_0}{2} - \frac{d}{dv}\frac{u}{v} - \frac{\mathcal{E}_0}{v^2} \ . \tag{57}$$

Equations (51) and (57) are two equations for three unknown functions $\Phi(t)$, $v(t)$ and potential function $u(v)$. Here we consider a particular solution corresponding to constant cyclotron frequency $\dot{\Phi} = \Omega_B = \mathrm{const}$. Then from (57) we find the potential function

$$u(v) = \frac{(m_0 - 2\varsigma s_\mathrm{b} \Omega_B) v^2}{2} + C_1 v + \mathcal{E}_0 \ , \tag{58}$$

and (51) reduces to

$$\ddot{v} + (\Omega_0^2 + \Omega_B^2 + B_Z/\varsigma s_\mathrm{b}) = C_1 \Omega_B \ , \tag{59}$$

which gives three types of solutions.

II.1.

$$m_0 \neq 0, \ \dot{\Phi} = \Omega_B, \ s_\tau = s_\mathrm{n} = 0, \ B_X = B_Y = 0, \ \Omega^2 = \Omega_0^2 + \Omega_B^2 + B_Z/\varsigma s_\mathrm{b} > 0.$$

Equation (59) has the following solution $v(t) = v_1 + v_0 \sin(\Omega t + \varphi_0)$. The relevant trajectory is described by radius vector

$$\begin{aligned} \mathbf{R}(t) = \mathbf{R}(0) &+ \left[\rho(t) \sin\Omega_B t - \rho_B \cos(\Omega t + \varphi_0) \cos\Omega_B t + \rho_B \cos\varphi_0\right] \mathbf{e}_X - \\ &- \left[\rho(t) \cos\Omega_B t + \rho_B \cos(\Omega t + \varphi_0) \sin\Omega_B t - \rho(0)\right] \mathbf{e}_Y + V_{(\mathrm{K}')} t \mathbf{e}_Z \ , \end{aligned} \tag{60}$$

where $\Omega = \Omega_B\sqrt{1+\eta_B}$, $v_1 = C_1[\Omega_B(1+\eta_B)]^{-1}$, $\eta_B = (\Omega_0^2 + B_Z/\varsigma s_\mathrm{b})\Omega_B^{-2}$,

$$\rho_B = \frac{\varsigma s_\mathrm{b}\Omega v_0}{\varsigma s_\mathrm{b}\Omega_0^2 + B_Z} = \frac{v_0\sqrt{1+\eta_B}}{\Omega_B\eta_B}\ ,$$
$$\rho(t) = \frac{\rho_B}{\sqrt{1+\eta_B}}\left[\frac{v_1\eta_B}{v_0} - \sin(\Omega t + \varphi_0)\right]\ . \tag{61}$$

II.2. If the magnetic field is such that the relation $\Omega_0^2 = -B_Z/\varsigma s_\mathrm{b}$ is valid, i. e. $\eta_B = 0$, then $\Omega = \Omega_B$, $v_1 = C_1/\Omega_B$. Trajectory is presented by radius vector

$$\mathbf{R}(t) = \mathbf{R}(0) + \rho_0\left[k\sin\Omega_B t + 2\Omega_B t\sin\varphi_0 + \cos(\varphi_0 - \cos 2\Omega_B t + \varphi_0)\right]\mathbf{e}_X +$$
$$+\rho_0\left[k(1-\cos\Omega_B t) + 2\Omega_B t\cos\varphi_0 + \sin\varphi_0 - \sin(2\Omega_B t + \varphi_0)\right]\mathbf{e}_Y + V_{(\mathrm{K}')}t\mathbf{e}_Z\ , \tag{62}$$

where $\rho_0 = v_0/4\Omega_B$, $k = 4v_1/v_0$, and it is complicated curve in the center-of-inertia reference frame, which moves with velocity $\mathbf{V}_\mathrm{C} = (v_0/2)(\sin\varphi_0\mathbf{e}_X + \cos\varphi_0\mathbf{e}_Y)$.

II.3.

$$m_0 \neq 0,\ \dot\Phi = \Omega_B,\ s_\tau = s_\mathrm{n} = 0,\ B_X = B_Y = 0,\ \tilde\Omega^2 = -\Omega_0^2 - \Omega_B^2 - B_Z/\varsigma s_\mathrm{b} > 0,$$

i. e. $\eta_B < -1$. It follows from (60) that $v(t) = v_1 + v_0\sinh(\tilde\Omega t + \varphi_0)$,

$$\mathbf{R}(t) = \mathbf{R}(0) - \left[\tilde\rho(t)\sin\Omega_B t + \tilde\rho_B\cosh(\tilde\Omega t + \varphi_0)\cos\Omega_B t - \tilde\rho_B\cosh\varphi_0\right]\mathbf{e}_X +$$
$$+\left[\tilde\rho(t)\cos\Omega_B t - \tilde\rho_B\cosh(\tilde\Omega t + \varphi_0)\sin\Omega_B t - \tilde\rho(0)\right]\mathbf{e}_Y + V_{(\mathrm{K}')}t\mathbf{e}_Z\ , \tag{63}$$

where $\tilde\Omega = \Omega_B\sqrt{-1-\eta_B}$,

$$\tilde\rho_B = \frac{\varsigma s_\mathrm{b}\tilde\Omega v_0}{\varsigma s_\mathrm{b}\Omega_0^2 + B_Z} = \frac{v_0\sqrt{-1-\eta_B}}{\Omega_B\eta_B}\ ,$$
$$\tilde\rho(t) = \frac{\tilde\rho_B}{\sqrt{-1-\eta_B}}\left[-\frac{v_1\eta_B}{v_0} + \sinh(\tilde\Omega t + \varphi_0)\right]. \tag{64}$$

Trajectory (63) in the center-of-inertia r. f. is twisting ($t < 0$), and then untwisting ($t > 0$) helix.

II.4.

$$m_0 \neq 0,\ \dot\Phi = \Omega_B,\ s_\tau = s_\mathrm{n} = 0,\ B_X = B_Y = 0,\ \Omega_0^2 + \Omega_B^2 + B_Z/\varsigma s_\mathrm{b} = 0,$$

i. e. cyclotron frequency is equal to $\Omega_B = \sqrt{-\Omega_0^2 - B_Z/\varsigma s_\mathrm{b}}$. Here (59) gives $v(t) = v_0 + wt + C_1\Omega_B t^2/2$,

$$\mathbf{R}(t) = \mathbf{R}(0) - \frac{1}{\Omega_B^2}\left[(v(t) - C_1)\sin\Omega_B t + \frac{dv(t)}{dt}\cos\Omega_B t - \left.\frac{dv(t)}{dt}\right|_{t=0}\right]\mathbf{e}_X +$$
$$+\frac{1}{\Omega_B^2}\left[(v(t) - C_1)(1-\cos\Omega_B t) + \frac{dv(t)}{dt}\sin\Omega_B t\right]\mathbf{e}_Y + V_{(\mathrm{K}')}t\mathbf{e}_Z\ . \tag{65}$$

The trajectories of this type at $w = 0$, $C_1 = 0$ correspond to the trajectories of classical electrodynamics, where, as it is known, a charged particle, that is flying in uniform magnetic field, moves in a spiral or circle, when its velocity is perpendicular to field, and spin of the particle is not taken into account in no way. As it follows from the solutions obtained above, the spin of a particle that has fallen into magnetic field, has always arranged parallel or antiparallel to the field. This was conclusively proven by experiment of Stern and Gerlach. It is obvious that

condition $B_Z < -\varsigma s_\mathrm{b}\Omega_0^2$ should be satisfied. Assuming for the electron $\Omega_0 = m_e c^2/\hbar \approx 7,77{\cdot}10^{20}$ Hz ([2], Eq. (4.50), or [3], Eq. (89)), $\varsigma = -c^{-2}$, $s_\mathrm{b} = s = \hbar/2$, we find limit value of magnetic field $B_\mathrm{max} = m_e^2 c^2/2\hbar \approx 5,6 \cdot 10^{-11}$ kg/s that corresponds to $B_\mathrm{max} = m_e^2 c^2/2e\hbar \approx 3,5 \cdot 10^8$ T in SI. Large values of magnetic field are occured in magnetars, that are neutron stars with strong magnetic field (up to 10^{11} T), wherein condition $B_Z < -\varsigma s_\mathrm{b}\Omega_0^2$ is not valid. For such fields, apparently can be realized cases II.1, II.2.

Finally, we get the usual solution, assuming $s_\mathrm{b} = 0$. Then (51) looks like

$$\left(m_0 v - \frac{du}{dv}\right)\dot{\Phi} = -vB_Z \ , \tag{66}$$

whence (at $\dot{\Phi} = \Omega_B$)

$$\frac{du}{dv} = (m_0 + B_Z/\Omega_B)v \ , \quad u(v) = \frac{1}{2}(m_0 + B_Z/\Omega_B)v^2 \ . \tag{67}$$

$u(v) = 0$ corresponds to standard dependence of potential function from relative distance. As it follows from (67) this is possible, when $\Omega_B = -B_Z/m_0$, which coincides with standard definition of cyclotron frequency. For constant field (66), (67) give $v = v_0 = \mathrm{const}$. Then trajectory is described by (65), i. e.

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_B}\sin\Omega_B t + \frac{v_0}{\Omega_B}(1 - \cos\Omega_B t) + V_{(\mathrm{K}')} t\mathbf{e}_Z \ . \tag{68}$$

III. Detailed description of solutions for $m_0 = 0$ will be made in a separate article. Here we only note that the equation (51) gives the first integral

$$\frac{[\ddot{v} + (\Omega_0^2 + B_Z/\varsigma s_\mathrm{b})v]^2}{\dot{\Phi}^2} + \dot{v}^2 + (\Omega_0^2 + B_Z/\varsigma s_\mathrm{b})v^2 = D^2 = \mathrm{const} \ . \tag{69}$$

Comparison of (69) with (34) shows that the set of solutions for massless particles in a constant uniform magnetic field at $\dot{\Phi} = \Omega_\mathrm{D}$ may be easily obtained from solutions I-III for free particles, if Ω_0^2 replaced by $\Omega_0^2 + B_Z/\varsigma s_\mathrm{b}$ together with taking into account of equations (23)-(24) instead of (21).

Analyzing the solutions (60) and (63), one can see that they are close to the classical solution (68). Therefore a question arises to what extent the classical solution corresponds to real situation. It should be noted also that the obtained solutions are found in the simplified assumption about dependence of potential function on the velocity. In addition, the motion takes place from the infinite past to the infinite future. Most of the real problems is that a free particle flies into a certain domain of space, where the field is present, and then flies out from it, becoming free again. Therefore, we can assume that solution of such problems can be found by combining the results obtained.

References

[1] Tarakanov A.N. Generalized Dynamics of the Mass Point with Internal Degrees of Freedom. // http://www.arXiv.org/hep-th/1010.4645v1. – 9 pp.

[2] Tarakanov A.N. Zitterbewegung as purely classical phenomenon. // http://www.arXiv.org/physics.class-ph/1201.4965v3. – 15 pp.

[3] Tarakanov A.N. Nonrelativistic Dynamics of the Point with Internal Degrees of Freedom and Spinning Electron. // J. Theor. Phys., 2012, **1**, no. 2, 76-98.

The Dirac equation in parabolic cylindrical coordinates, possible effects of the spinor structures in quantum physics

V. Balan
vladimir.balan@upb.ro; University Politehnica of Bucharest
O.V. Veko
vekoolga@mail.ru; Gymnasium, Kalinkovichi, Belarus
E.M. Ovsiyuk
e.ovsiyuk@mail.ru; Mozyr State Pedagogical University, Belarus
V.M. Red'kov
redkov@dragon.bas-net.by; B.I. Stepanov Institute of Physics

Abstract

Physics with spinor groups significantly differs from the one based on the orthogonal group. The extension procedure to describe a space with spinor structure is performed on the base of cylindrical parabolic coordinates. This is done through expansion of the region for coordinates, accompanied with new identification rules for the boundary points. The solutions of the Dirac equation are constructed for both geometrical models. Solutions of two types (–) and (++) are single-valued in the space with vector structure, whereas the solutions of the types (-+) and(+-) are not single-valued in space with vector structure, so solutions of these two types (-+) and (+-) must be rejected. However, they must be retained in the space with spinor structure.

1 Introduction

]In the literature [1–12], there exist three different approaches, whose intrinsic essence is similar: a space-time spinor structure (Penrose, Rindler et al.); the Hopf bundle and the Kustaanheimo-Stiefel bundles.

In Hopf's technique, one uses only complex 2-spinors (ξ) and their conjugates (ξ^*), instead of real-valued 4-vector (tensor) objects. In the Kustaanheimo-Stiefel approach, there are used four real-valued coordinates, the real and imaginary parts of 2-spinor components.

The formalism developed in the present work exploits as well the possibilities given by spinors to construct 3-vectors; however, the emphasis is put on doubling the set of spacial points, so that we get an extended space model. In such a space, instead of the 2π-rotation, the 4π-rotation is considered - which transfers the space into itself. The procedure of extending the set of manifold points is achieved much easier by using curvilinear coordinates.

Within the framework of applications of spinor theory to Quantum Mechanics, we discuss the role of spinor space structure in classifying solutions of scalar and spinor wave equations specified for cylindric parabolic coordinates.

2 Space with spinor structure and solutions of the Klein–Fock–Gordon equation

Let us start with the parabolic cylindrical coordinates $x = (u^2 - v^2)/2,\ y = uv,\ z = z$. In order to cover the vector space (x, y, z), it suffices to make a choice out of the following four possibilities:

$$
\begin{aligned}
&v = +\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}},\\
&v = -\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}},\\
&v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = +\sqrt{+x + \sqrt{x^2 + y^2}},\\
&v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = -\sqrt{+x + \sqrt{x^2 + y^2}}.
\end{aligned}
$$

For definiteness, let us use the first variant from the above:

$$
v = +\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}.
$$

which is illustrated in Fig. 1.

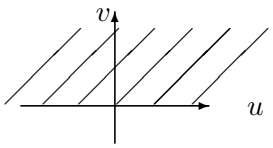


Fig. 1. The region $G(u, v)$, used to parameterize the vector model.

The correspondence between the points (x, y) and (u, v) can be illustrated by the following formulas and Fig. 2:

$$
\begin{aligned}
&u = k\cos\phi, \quad v = k\sin\phi, \quad \phi \in [0, \pi];\\
&x = (k^2/2)\cos 2\phi, \quad y = (k^2/2)\sin 2\phi, \quad 2\phi \in [0, 2\pi].
\end{aligned}
$$

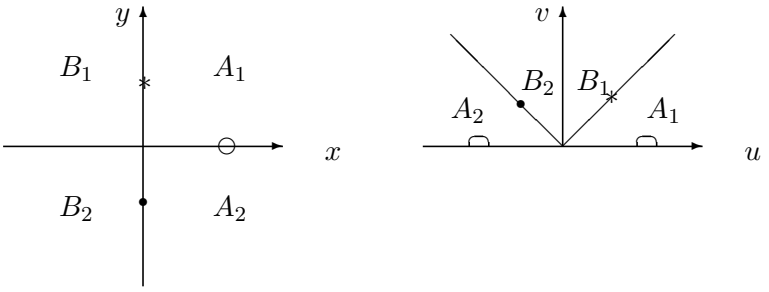


Fig. 2. The mapping $G(x, y) \Longrightarrow G(u, v)$; identification rules

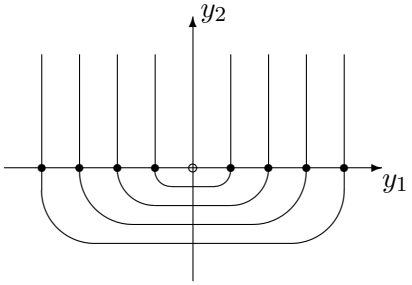


Fig. 3. Parabolic cylindrical coordinates

In Fig. 3, the identified points on the boundary are connected by lines, and the domain $G(y_1, y_2)^{y_3}$ (at arbitrary y_3) ranging in the half-plane (y_1, y_2) covers the whole vector plane $(x_1, x_2)^{x_3}$.

When turning to the case of spinor space, we shall see the complete symmetry between the coordinates u and v; they relate to the Cartesian coordinates of the extended model $(x, y, z) \oplus (x', y', z')$ through the formulas (see Fig. 4.):

$$v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}$$

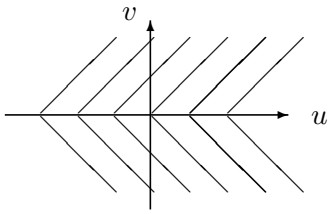


Fig. 4. $\tilde{G}(u, v)$ **covering the spinor space**

The metric of space-time in parabolic cylindrical coordinates has the form

$$dS^2 = c^2 dt^2 - (u^2 + v^2)(du^2 + dv^2) - dz^2.$$

Let us consider the KFG equation

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - \frac{m^2 c^2}{\hbar^2}\right]\Psi = 0. \tag{1}$$

After separating the variables by the substitution $\Psi = e^{-i\epsilon t/\hbar} e^{ipz/\hbar} U(u)V(v)$, one gets

$$\left[\frac{1}{U}\frac{d^2U}{du^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2}\right)u^2\right] + \left[\frac{1}{V}\frac{d^2V}{dv^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2}\right)v^2\right] = 0. \tag{2}$$

In the following, we shall use the notation

$$\lambda^2 = \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2}\right), \quad [\lambda] = \frac{1}{\text{meter}}. \tag{3}$$

By introducing two separation constants, a and b $(a + b = 0)$, we can derive two distinct equations:

$$\frac{d^2U}{du^2} + (\lambda^2 u^2 - a)U = 0, \quad \frac{d^2V}{dv^2} + (\lambda^2 v^2 - b)V = 0. \tag{4}$$

The transition in equations (4) to the canonical form is obtained by using dimensionless variables:

$$\sqrt{2\lambda}u \;\to\; u,\; \frac{a}{2\lambda} \;\to\; a,\; \sqrt{2\lambda}v \;\to\; v,\; \frac{b}{2\lambda} \;\to\; b. \tag{5}$$

The equations will take the form:

$$\frac{d^2U}{du^2} + \left(\frac{u^2}{4} - a\right)U = 0, \quad \frac{d^2V}{dv^2} + \left(\frac{v^2}{4} + a\right)V = 0. \tag{6}$$

The solutions of these similar equations can be found in series form:

$$F(\xi) = c_0 + c_1\xi + c_2\xi^2 + \sum_{k=1,2,...} c_{2k+1}\xi^{2k+1} + \sum_{k=1,2,...} c_{2k+2}\xi^{2k+2}; \tag{7}$$

we note that the terms of even and odd powers of ξ are separated.

After tedious calculation, one derives two independent groups of recurrent relations:

for even powers

$$\begin{aligned} \xi^0 : &\qquad 2c_2 - \alpha c_0 = 0, \\ \xi^2 : &\qquad c_4 4 \times 3 + \tfrac{c_0}{4} - \alpha c_2 = 0, \\ \xi^4 : &\qquad c_6 6 \times 5 + \tfrac{c_2}{4} - \alpha c_4 = 0, \\ n = 3, 4, ..., \xi^{2n} : &\qquad c_{2n+2}(2n+2)(2n+1) + \tfrac{1}{4}c_{2n-2} - \alpha c_{2n} = 0; \end{aligned} \tag{8}$$

for odd powers

$$\begin{aligned} \xi^1 : &\qquad c_3 3 \times 2 - \alpha c_1 = 0, \\ \xi^3 : &\qquad c_5 5 \times 4 + \tfrac{c_1}{4} - \alpha c_3 = 0, \\ n = 3, 4, ..., \xi^{2n-1} : &\qquad c_{2n+1}(2n+1)(2n) + \tfrac{1}{4}c_{2n-3} - \alpha c_{2n-1} = 0. \end{aligned} \tag{9}$$

So one can construct the following two linearly independent solutions

even

$$\begin{aligned} F_1(\xi^2) &= 1 + a_2\frac{\xi^2}{2!} + a_4\frac{\xi^4}{4!} + ..., \\ a_2 = \alpha, \qquad a_4 &= \alpha^2 - \frac{1}{2}, \qquad c_6 = \alpha^3 - \frac{7}{2}\alpha, \\ n = 3, 4, ... : \qquad a_{2n+2} &= \alpha a_{2n} - \frac{(2n)(2n-1)}{4}a_{2n-2}; \end{aligned} \tag{10}$$

odd

$$\begin{aligned} F_2(\xi) &= \xi + a_3\frac{\xi^3}{3!} + a_5\frac{\xi^5}{5!} + ..., \\ a_3 = \alpha, \qquad a_5 &= \alpha^2 - \frac{3}{2}, \\ n = 3, 4, ... : \qquad a_{2n+1} &= \alpha a_{2n-1} - \frac{(2n-1)(2n-2)}{4}a_{2n-3}. \end{aligned} \tag{11}$$

3 Manifestation of vector and spinor space structures

Having combined the two previous solutions F_1 and F_2, we can obtain four types of wave functions[1]

$$\begin{aligned} (\text{even} \otimes \text{even}) : &\qquad \Phi_{++} = E(a, u^2)E(-a, v^2), \\ (\text{odd} \otimes \text{odd}) : &\qquad \Phi_{--} = O(a, u)O(-a, v), \\ (\text{even} \otimes \text{odd}) : &\qquad \Phi_{+-} = E(a, u^2)O(-a, v), \\ (\text{odd} \otimes \text{even}) : &\qquad \Phi_{-+} = O(a, u)E(-a, v^2). \end{aligned} \tag{12}$$

[1]We will change the notation: $F_1 \Longrightarrow E$; $F_2 \Longrightarrow O$.

We note the behavior of the constructed wave functions:

$$\begin{aligned} &\Phi_{++}(x=0,y=0)\neq 0, \quad \Phi_{--}(x=0,y=0)=0,\\ &\Phi_{+-}(x>0,y=0)=0, \quad \Phi_{-+}(x<0,y=0)=0. \end{aligned} \tag{13}$$

Now let us find which restrictions for the wave functions Ψ follow from the requirement of single-valuedness. To this aim, two properties of the parametrization are essential:

$$\underline{v=0}: x=+\frac{u^2}{2}\geq 0, y=0; \quad \underline{u=0}: x=-\frac{v^2}{2}\leq 0, y=0.$$

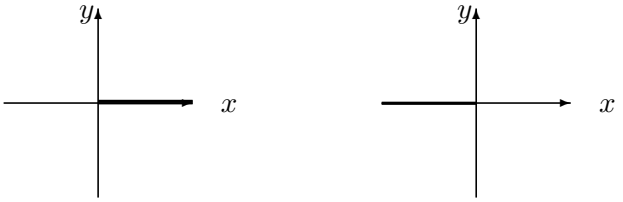


Fig. 5. The peculiarities of the parametrization

The four solutions behave in special regions, as follows:

$$\begin{aligned} &\Phi_{++}(a;u=0,v)=+\Phi_{++}(a;u=0,-v),\\ &\Phi_{++}(a;+u,v=0)=+\Phi_{++}a;-u,v=0),\\ &\Phi_{--}(a;u=0,+v)=+\Phi_{--}(a;u=0,-v)=0,\\ &\Phi_{--}(a;u,v=0)=+\Phi_{--}(a;-u,v=0)=0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\Phi_{+-}(a;u=0,+v)=-\Phi_{+-}(a;u=0,-v),\\ &\Phi_{+-}(a;u,v=0)=\Phi_{+-}(a;-u,v=0)=0,\\ &\Phi_{-+}(a;u=0,+v)=\Phi_{-+}(a;u=0,-v)=0,\\ &\Phi_{-+}(a;+u,v=0)=-\Phi_{-+}(a;-u,v=0). \end{aligned} \tag{15}$$

The boundary properties of the constructed wave functions can be illustrated by the schemes described in Fig. 6.

So we conclude that the solutions Ψ of the types $(++)$ and $(--)$ are single-valued in the space with vector structure, whereas the solutions of the types $(+-)$ and $(-+)$ are not single-valued in such a space, so these latter types $(+-)$ and $(-+)$ must be discarded. However, these solutions ($(+-)$ and $(-+)$) must be retained in the space with *spinor* structure.

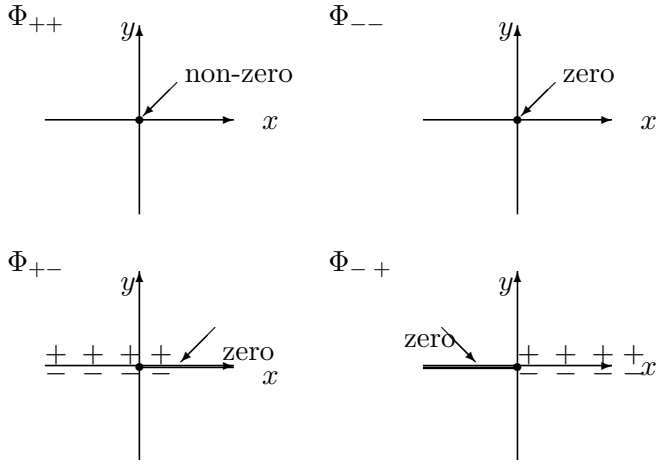


Fig 6. Boundary behavior of the wave functions in the (x,y)-plane

4 Orthogonality and completeness of the bases for vector and spinor spaces

Now let us consider the scalar multiplication $\int \Psi^*_{\mu'}\Psi_{\mu}\sqrt{-g}dtdzdudv$ of the basic constructed wave functions:

$$\Psi_{++}(\epsilon,p,a)=e^{i\epsilon t}e^{ipz}\Phi_{++}(a;u,v),\quad \Psi_{--}(\epsilon,p,a)=e^{i\epsilon t}e^{ipz}\Phi_{--}(a;u,v),$$
$$\Psi_{+-}(\epsilon,p,a)=e^{i\epsilon t}e^{ipz}\Phi_{+-}(a;u,v),\quad \Psi_{-+}(\epsilon,p,a)=e^{i\epsilon t}e^{ipz}\Phi_{-+}(a;u,v). \tag{16}$$

where μ and μ' stand for generalized quantum numbers.

First of all, we note some interesting integrals[2]:

in the vector space

$$I_0=\int_0^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{++}\Phi_{--}(u^2+v^2),$$

in the spinor space

$$I_1=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{++}\Phi_{--}(u^2+v^2),$$
$$I_2=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{+-}\Phi_{-+}(u^2+v^2),$$
$$I_3=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{++}\Phi_{+-}(u^2+v^2),$$
$$I_4=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{++}\Phi_{-+}(u^2+v^2),$$
$$I_5=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{--}\Phi_{+-}(u^2+v^2),$$
$$I_6=\int_{-\infty}^{+\infty}dv\int_{-\infty}^{+\infty}du\Phi^*_{--}\Phi_{-+}(u^2+v^2).$$

All these seven integrals $I_0, I_1 ... I_6$ are equal to zero, which means that the constructed functions provide us with an orthogonal basis for the Hilbert space $\Psi(t,z,u,v)$, where (u,v,z) belong to the extended (spinor) space model.

5 The Dirac particle and the space with spinor structure

We shall apply a tetrad based form of the Dirac equation to construct solutions of the Dirac equation in parabolic cylindric coordinates. The last ones are defined by the formulas

$$x_1=\frac{u^2-v^2}{2},\quad x_2=uv,\quad x_3=z.$$

To parameterize the space with vector structure (x,y,z), it is enough to make use of the following formulas

$$v=+\sqrt{-x_1+\sqrt{x_1^2+x_1^2}},\quad u=\pm\sqrt{+x_1+\sqrt{x_1^2+x_2^2}}.$$

However, to parameterize the space $(x,y,z)\oplus(x,y,z)$ with spinor structure, one must use for u and v more symmetrical, substantially different formulas:

$$v=\pm\sqrt{-x_1+\sqrt{x_1^2+x_2^2}},\quad u=\pm\sqrt{+x_1+\sqrt{x_1^2+x_2^2}}.$$

[2]The arguments $(a;u,v)$ are omitted here.

We will further denote the parabolic cylindric coordinates by $(u,v,z)=(y^1,y^2,y^3)$, and use the explicit formulas

$$dx^i=\frac{\partial x^i}{\partial y^j}dy^j,\ (S^i_{\ j})=\frac{\partial x^i}{\partial y^j}=\left|\begin{array}{ccc} u & -v & 0\\ v & u & 0\\ 0 & 0 & 1\end{array}\right|,$$

$$(S^{-1})^j_{\ k}=\frac{\partial y^j}{\partial x^k}=\frac{1}{u^2+v^2}\left|\begin{array}{ccc} u & v & 0\\ -v & u & 0\\ 0 & 0 & 1\end{array}\right|.$$

The most simple tetrad to use is

$$e_{(k)}^{\ \alpha}=\left|\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & (u^2+v^2)^{-1/2} & 0 & 0\\ 0 & 0 & (u^2+v^2)^{-1/2} & 0\\ 0 & 0 & 0 & 1\end{array}\right|.$$

Let us specify the Dirac equation in this tetrad:

$$\left\{i\gamma^0\frac{\partial}{\partial t}+i\gamma^3\frac{\partial}{\partial z}-M+\frac{i}{\sqrt{u^2+v^2}}\right.$$
$$\left.\times\left[\gamma^1\left(\frac{\partial}{\partial u}+\frac{u}{2(u^2+v^2)}\right)+\gamma^2\left(\frac{\partial}{\partial v}+\frac{v}{2(u^2+v^2)}\right)\right]\right\}\Psi(y)=0.$$

In spinor basis, this is written in 2-block form as follows:

$$\left\{i\frac{\partial}{\partial t}+i\sigma^3\frac{\partial}{\partial z}+\frac{i}{\sqrt{u^2+v^2}}\left[\sigma^1\left(\frac{\partial}{\partial u}+\frac{u}{2(u^2+v^2)}\right)+\sigma^2\left(\frac{\partial}{\partial v}+\frac{v}{2(u^2+v^2)}\right)\right]\right\}E=MH,$$

$$\left\{i\frac{\partial}{\partial t}-i\sigma^3\frac{\partial}{\partial z}-\frac{i}{\sqrt{u^2+v^2}}\left[\sigma^1\left(\frac{\partial}{\partial u}+\frac{u}{2(u^2+v^2)}\right)+\sigma^2\left(\frac{\partial}{\partial v}+\frac{v}{2(u^2+v^2)}\right)\right]\right\}H=ME.$$

By using the substitutions

$$E(t,z,u,v)=e^{-i\epsilon t}e^{ikz}E(u,v),\qquad H(t,z,u,v)=e^{-i\epsilon t}e^{ikz}H(u,v),$$

we further obtain

$$\left\{\epsilon-k\sigma^3+\frac{i}{\sqrt{u^2+v^2}}\left[\sigma^1\left(\frac{\partial}{\partial u}+\frac{u}{2(u^2+v^2)}\right)+\sigma^2\left(\frac{\partial}{\partial v}+\frac{v}{2(u^2+v^2)}\right)\right]\right\}E=MH,$$

$$\left\{\epsilon+k\sigma^3-\frac{i}{\sqrt{u^2+v^2}}\left[\sigma^1\left(\frac{\partial}{\partial u}+\frac{u}{2(u^2+v^2)}\right)+\left(\frac{\partial}{\partial v}+\frac{v}{2(u^2+v^2)}\right)\right]\right\}H=ME.$$

It is convenient to make the following substitutions for the following 2-component entities

$$E(u,v)=(u^2+v^2)^{-1/4}e(u,v),\qquad H(u,v)=(u^2+v^2)^{-1/4}h(u,v).$$

This yields

$$\left[(\epsilon-k\sigma^3)+\frac{i}{\sqrt{u^2+v^2}}\left(\sigma^1\frac{\partial}{\partial u}+\sigma^2\frac{\partial}{\partial v}\right)\right]e(y)=Mh(y)\ ,$$

$$\left[(\epsilon+k\sigma^3)-\frac{i}{\sqrt{u^2+v^2}}\left(\sigma^1\frac{\partial}{\partial u}+\sigma^2\frac{\partial}{\partial v}\right)\right]h(y)=Me(y).\qquad(17)$$

The equations (17) are rather complicated, and to proceed with them we shall diagonalize the helicity operator $(\mathbf{sp})_0$. We shall translate it to cylindric coordinates and then translate it to the cylindric parabolic tetrad (see [12]):

$$(\mathbf{sp}) = S(\mathbf{sp})_0 S^{-1},$$

$$S = \left| \begin{array}{cc} B(y) & 0 \\ 0 & B(y) \end{array} \right|, B(y) = \frac{1}{(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{array} \right|.$$

In this way, we get

$$(\vec{\sigma}\vec{p}) = B(\vec{\sigma}\vec{p})_0 B^{-1} =$$

$$-i\left\{\sigma^3\frac{\partial}{x^3} + \frac{1}{\sqrt{u^2+v^2}}\left[\sigma^1\left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)}\right) + \sigma^2\left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)}\right)\right]\right\}.$$

The explicit form of the eigenvalue equation for the 4-spinor wave function can be simplified by the following substitution

$$\begin{array}{ll} (\vec{\sigma}\vec{p})\, E(y) = \lambda E(y), & E(y) = (u^2+v^2)^{-1/4} e(y), \\ (\vec{\sigma}\vec{p})\, H(y) = \lambda H(y), & H(y) = (u^2+v^2)^{-1/4} h(y); \end{array} \tag{18}$$

so we obtain

$$\begin{array}{l} \left[\sigma^3 k - \frac{i}{\sqrt{u^2+v^2}}\left(\sigma^1\frac{\partial}{\partial u} + \sigma^2\frac{\partial}{\partial v}\right)\right] e(u,v) = \lambda e(u,v), \\ \left[\sigma^3 k - \frac{i}{\sqrt{u^2+v^2}}\left(\sigma^1\frac{\partial}{\partial u} + \sigma^2\frac{\partial}{\partial v}\right)\right] h(u,v) = \lambda h(u,v). \end{array} \tag{19}$$

Let us turn back to eqs. (17). With the help of (19), from (17) we infer the algebraic system

$$(\epsilon - \lambda)e(u,v) = Mh(u,v), \quad (\epsilon + \lambda)h(u,v) = Me(u,v),$$

which leads to the following solutions

$$\begin{array}{c} \lambda = \pm\sqrt{\epsilon^2 - M^2}, \quad \mu = \frac{(\epsilon-\lambda)}{M} = \frac{M}{(\epsilon+\lambda)}, \\ h_j = \mu e_j, \quad \Psi = \frac{e^{-i\epsilon t}e^{ikz}}{\sqrt{u^2+v^2}} \left(\begin{array}{c} e_1(u,v) \\ e_2(u,v) \\ \mu e_1(u,v) \\ \mu e_2(u,v) \end{array} \right). \end{array} \tag{20}$$

Thus, we have only two independent variables, $e_1(u,v)$ and $e_2(u,v)$, which obey the following equations

$$\begin{array}{l} (\lambda - k)e_1 + \frac{i}{\sqrt{u^2+v^2}}(\partial_u - i\partial_v)e_2 = 0, \\ (\lambda + k)e_2 + \frac{i}{\sqrt{u^2+v^2}}(\partial_u + i\partial_v)e_1 = 0. \end{array} \tag{21}$$

Using the substitutions $e_1 = \sqrt{u+iv}F, e_2 = \sqrt{u-iv}G$, and taking into account the identities

$$\begin{array}{l} (\partial_u + i\partial_v)\sqrt{u+iv}F = \sqrt{u+iv}(\partial_u + i\partial_v)F, \\ (\partial_u - i\partial_v)\sqrt{u-iv}G = \sqrt{u-iv}(\partial_u - i\partial_v)G, \end{array}$$

we simplify (21) to the simpler form

$$(\lambda - k)(u+iv)F + i(\partial_u - i\partial_v)G = 0, \quad (\lambda + k)(u-iv)G + i(\partial_u + i\partial_v)F = 0. \tag{22}$$

These equations are associated with the wave functions of the form

$$\Psi = \frac{e^{-i\epsilon t}e^{ikz}}{\sqrt{u^2+v^2}} \begin{pmatrix} \sqrt{u+iv}F(u,v) \\ \sqrt{u-iv}G(u,v) \\ \mu\sqrt{u+iv}F(u,v) \\ \mu\sqrt{u-iv}G(u,v) \end{pmatrix}; \tag{23}$$

the transition of (23) to Cartesian tetrads is made according to the rule

$$\Psi_0(x) = \frac{1}{(u^2+v^2)^{1/4}} \begin{pmatrix} \sqrt{u-iv} & 0 & 0 & 0 \\ 0 & \sqrt{u+iv} & 0 & 0 \\ 0 & 0 & \sqrt{u-iv} & 0 \\ 0 & 0 & 0 & \sqrt{u+iv} \end{pmatrix} \Psi(x),$$

whence it follows

$$\Psi_0(x) = \frac{e^{-i\epsilon t}e^{ikz}}{(u^2+v^2)^{1/4}} \begin{pmatrix} F(u,v) \\ G(u,v) \\ \mu F(u,v) \\ \mu G(u,v) \end{pmatrix}. \tag{24}$$

Let us turn again to the system (22). First, by eliminating the function G from (22), we derive

$$G = -\frac{i}{\lambda+k}\frac{1}{u-iv}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)F,$$
$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + (\lambda^2-k^2)(u^2+v^2)\right]F(u,v) = 0. \tag{25}$$

Alternatively, by eliminating the function F from (22), we obtain

$$F = -\frac{i}{\lambda-k}\frac{1}{u+iv}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)G,$$
$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + (\lambda^2-k^2)(u^2+v^2)\right]G((u,v) = 0. \tag{26}$$

The second order differential equation is the same for both cases, (25) and (26). For definiteness, let us examine the case (25), starting with the equation

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \sigma^2(u^2+v^2)\right]F(u,v) = 0, \sigma^2 = (\lambda^2-k^2) = \epsilon^2 - M^2 - k^2 > 0. \tag{27}$$

Let $F(u,v) = U(u)V(v)$; then, instead of (27), we obtain

$$\left(\frac{1}{U}\frac{d^2U}{du^2} + \sigma^2u^2\right) + \left(\frac{1}{V}\frac{d^2V}{dv^2} + \sigma^2v^2\right) = 0.$$

By introducing a separating constant Λ, we get two equations in variables u and v, respectively:

$$\frac{d^2U}{du^2} + (\sigma^2u^2 + \Lambda)U = 0, \quad \frac{d^2V}{dv^2} + (\sigma^2v^2 - \Lambda)V = 0.$$

For functions U, V we have similar equations, which differ only by the sign of the separation constant Λ. For definiteness, let us consider the function $U(u)$ given by

$$\frac{d^2U}{du^2} + (\sigma^2u^2 + \Lambda)U = 0.$$

Further, rewriting in terms of the variable $bu^2 = z$, this takes the form:

$$\left(z\frac{d^2}{dz^2} + \frac{1}{2}\frac{d}{dz} + \sigma^2\frac{z}{4b^2} + \frac{\Lambda}{4b}\right)U = 0.$$

Separating a simple factor, $U(z) = e^{-z/2}f(z)$, we derive

$$zf'' - zf' + \frac{1}{4}zf + \frac{1}{2}(f' - \frac{1}{2}f) + \sigma^2\frac{z}{4b^2}f + \frac{\Lambda}{4b}f = 0.$$

Let $b^2 = -\sigma^2$ and $b = +i\sigma,\ \sigma > 0$; then get

$$z\frac{d^2f}{dz^2} + (\frac{1}{2} - z)\frac{df}{dz} - \frac{1 + i\Lambda/\sigma}{4}f = 0,$$

which is in fact a confluent hypergeometric equation

$$z\frac{d^2f}{dz^2} + (c - z)\frac{df}{dz} - af = 0, \quad c = \frac{1}{2}, a = \frac{1 + i\Lambda/\sigma}{4}.$$

We further use two independent solutions

$$U_- = e^{-z/2}\Phi(a, \frac{1}{2}; z), \quad U_+ = e^{-z/2}\sqrt{z}\Phi(a + \frac{1}{2}, \frac{3}{2}; z).$$

Analogous results for (**??**) are obtained by the formal change $\Lambda \rightsquigarrow -\Lambda$:

$$\begin{gathered} V(v) = e^{-y/2}g(y), \quad y = +i\sigma v^2, \\ \frac{d^2g}{dy^2} + \left(\frac{1}{2} - y\right)\frac{dg}{dy} - \frac{1 - i\Lambda/\sigma}{4}g = 0, \\ c' = \frac{1}{2} = c, a' = \frac{1 - i\Lambda/\sigma}{4} = \frac{1}{2} - a; \end{gathered} \tag{28}$$

the two independent solutions are written as follows

$$V_-(y) = e^{-y/2}\Phi(\frac{1}{2} - a, \frac{1}{2}; y), \quad V_+(y) = e^{-y/2}\sqrt{y}\Phi(1 - a, \frac{3}{2}; y).$$

Thus, the total functions $F(u, v)$ and $G(u, v)$ will be constructed in terms of the following solutions

$$U_-(u) = e^{-i\sigma u^2/2}\Phi(a, \frac{1}{2}; i\sigma u^2),\ U_+(u) = e^{-i\sigma u^2/2}u\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2),$$

$$V_-(v) = e^{-i\sigma v^2/2}\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2),\ V_+(v) = e^{-i\sigma v^2/2}v\Phi(1 - a, \frac{3}{2}; i\sigma v^2);$$

$$U'_-(u) = e^{-i\sigma u^2/2}\Phi(a', \frac{1}{2}; i\sigma u^2),\ U'_+(u) = e^{-i\sigma u^2/2}u\Phi(a' + \frac{1}{2}, \frac{3}{2}; i\sigma u^2),$$

$$V'_-(v) = e^{-i\sigma v^2/2}\Phi(\frac{1}{2} - a', \frac{1}{2}; i\sigma v^2),\ V'_+(v) = e^{-i\sigma v^2/2}v\Phi(1 - a', \frac{3}{2}; i\sigma v^2).$$

Now, we should turn to the first order equation in (25)

$$i(\lambda + k)G = \frac{1}{u - iv}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)F;$$

by using it and starting with any known $F(u, v)$, we can find an explicit form for the corresponding function $G(u, v)$.

6 Continuity properties of the Dirac solutions and spinor space structure

Let us discuss in more detail some subtleties of the properties of the solutions and of their relations to the stricture of spatial, vector and spinorial models.

While considering the relation between parabolic and Cartesian coordinates:

$$x_1 = \frac{u^2 - v^2}{2}, \qquad x_1 = uv,$$

$$\underline{v = 0 :} x_1 = +\frac{u^2}{2} \geq 0, x_2 = 0, \qquad \underline{u = 0 :} x = -\frac{v^2}{2} \leq 0, y = 0,$$

we noted existence of two special regions (see. Fig. 5).

Now let us consider which restrictions for the Dirac wave functions Ψ follow from the requirement of single-valuedness.

Let us consider the variant $(--)$:

$$F_{--}(u,v) = U_-(u)V_-(v) = e^{-i\sigma u^2/2}\Phi(a, \frac{1}{2}; i\sigma u^2)e^{-i\sigma v^2/2}\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2),$$

$$i(\lambda + k)G = \frac{1}{(u - iv)}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)U_-(u)V_-(v) =$$

$$= \frac{1}{(u - iv)}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}\Phi(a, \frac{1}{2}; i\sigma u^2)e^{-i\sigma v^2/2}\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2) =$$

$$= \frac{1}{(u - iv)}e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\times$$

$$\times\left[\left(-i\sigma u\Phi(a, \frac{1}{2}; i\sigma u^2) + 2i\sigma u\frac{d}{d(i\sigma u^2)}\Phi(a, \frac{1}{2}; i\sigma u^2)\right)\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2) + \right.$$

$$\left. + i\Phi(a, \frac{1}{2}; i\sigma u^2)\left(-i\sigma v\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2) + 2i\sigma v\frac{d}{d(i\sigma v^2)}\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2)\right)\right].$$

We note that related pair of functions, $F_{--} \rightsquigarrow G_{--}(u,v)$ are single-valued in the special region:

$$F_{--}(u = 0, v) = +F_{--}(u = 0, -v) \neq 0,$$

$$F_{--}(u, v = 0) = +G_{--}(-u, v = 0) \neq 0;$$

$$G_{--}(u = 0, v) = +G_{--}(u = 0, -v) \neq 0,$$

$$G_{--}(u, v = 0) = +G_{--}(-u, v = 0) \neq 0.$$

Let us consider the variant $(++)$:

$$F_{++}(u,v) = U_+(u)V_+(v) = e^{-i\sigma u^2/2}u\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2)e^{-i\sigma v^2/2}v\Phi(1 - a, \frac{3}{2}; i\sigma v^2),$$

$$i(\lambda + k)G = \frac{1}{(u - iv)}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)U_+(u)V_+(v) =$$

$$\frac{1}{(u - iv)}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}u\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2)e^{-i\sigma v^2/2}v\Phi(1 - a, \frac{3}{2}; i\sigma v^2) =$$

$$= e^{-i\sigma u^2/2} e^{-i\sigma v^2/2} \frac{1}{(u-iv)} \times$$

$$\times \left[\left((-i\sigma u^2 + 1)\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2) + 2i\sigma u^2 \frac{d}{d(i\sigma u^2)} \Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2) \right) \times \right.$$

$$v\Phi(1-a, \frac{3}{2}; i\sigma v^2) + +iu\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2) \times$$

$$\left. \times \left((-i\sigma v^2 + 1)\Phi(1-a, \frac{3}{2}; i\sigma v^2) + 2i\sigma v^2 \frac{d}{d(i\sigma v^2)} \Phi(1-a, \frac{3}{2}; i\sigma v^2) \right) \right].$$

The related functions, $F_{++} \rightsquigarrow G_{++}(u,v)$ are single-valued in the spacial region:

$$F_{++}(u=0,+v) = +F_{++}(u=0,-v) = 0,$$

$$F_{++}(u,v=0) = +F_{++}(-u,v=0) = 0;$$

$$G_{++}(u=0,+v) = +G_{++}(u=0,-v) \neq 0,$$

$$G_{++}(u,v=0) = +G_{++}(-u,v=0) \neq 0.$$

Let us consider the variant $(-+)$:

$$F_{-+}(u,v) = U_-(u)V_+(v) = e^{-i\sigma u^2/2}\Phi(a, \frac{1}{2}; i\sigma u^2) e^{-i\sigma v^2/2} v\Phi(1-a, \frac{3}{2}; i\sigma v^2),$$

$$i(\lambda + k)G = \frac{1}{(u-iv)} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) U_-(u)V_+(v) =$$

$$= \frac{1}{(u-iv)} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) e^{-i\sigma u^2/2}\Phi(a, \frac{1}{2}; i\sigma u^2) e^{-i\sigma v^2/2} v\Phi(1-a, \frac{3}{2}; i\sigma v^2) =$$

$$= e^{-i\sigma u^2/2} e^{-i\sigma v^2/2} \frac{1}{(u-iv)} \times$$

$$\times \left[\left(-i\sigma u\Phi(a, \frac{1}{2}; i\sigma u^2) + 2i\sigma u \frac{d}{d(i\sigma u^2)} \Phi(a, \frac{1}{2}; i\sigma u^2) \right) v\Phi(1-a, \frac{3}{2}; i\sigma v^2) + \right.$$

$$\left. + i\Phi(a, \frac{1}{2}; i\sigma u^2) \left((-i\sigma v^2 + 1)\Phi(1-a, \frac{3}{2}; i\sigma v^2) + 2i\sigma v^2 \frac{d}{d(i\sigma v^2)} \Phi(1-a, \frac{3}{2}; i\sigma v^2) \right) \right].$$

The related functions, $F_{-+} \rightsquigarrow G_{-+}(u,v)$, are double-valued in the spacial region:

$$F_{-+}(u=0,+v) = -F_{-+}(u=0,-v) \neq 0,$$

$$F_{-+}(u,v=0) = -F_{-+}(-u,v=0) = 0,$$

$$G_{-+}(u=0,+v) = -G_{-+}(u=0,-v) \neq 0,$$

$$G_{-+}(u,v=0) = -G_{-+}(-u,v=0) \neq 0.$$

Let us consider the variant $(+-)$:

$$F_{+-}(u,v) = U_+(u)V_-(v) =$$

$$= e^{-i\sigma u^2/2} u\Phi(a + \frac{1}{2}, \frac{3}{2}; i\sigma u^2) e^{-i\sigma v^2/2}\Phi(\frac{1}{2} - a, \frac{1}{2}; i\sigma v^2),$$

$$i(\lambda + k)G = \frac{1}{(u-iv)} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) U_+(u)V_-(v) =$$

$$\frac{1}{(u-iv)}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}u\Phi(a+\frac{1}{2},\frac{3}{2};i\sigma u^2)e^{-i\sigma v^2/2}\Phi(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2)=$$

$$=e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\frac{1}{(u-iv)}\times$$

$$\times\left[\left((-i\sigma u^2+1)\Phi(a+\frac{1}{2},\frac{3}{2};i\sigma u^2)+2i\sigma u^2\frac{d}{d(i\sigma u^2)}\Phi(a+\frac{1}{2},\frac{3}{2};i\sigma u^2)\right)\times\right.$$

$$\Phi(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2)+iu\Phi(a+\frac{1}{2},\frac{3}{2};i\sigma u^2)\times$$

$$\left.\times\left(-i\sigma v\Phi(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2)+2i\sigma v\frac{d}{d(i\sigma v^2)}\Phi(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2)\right)\right].$$

The related functions, $F_{+-}\rightsquigarrow G_{+-}(u,v)$, are double-valued in the spacial region:

$$F_{+-}(u=0,+v)=-F_{+-}(u=0,-v)=0,$$

$$F_{+-}(+u,v=0)=-F_{+-}(-u,v=0)\neq 0;$$

$$G_{+-}(u=0,+v)=-G_{+-}(u=0,-v)\neq 0,$$

$$G_{+-}(+u,v=0)=-G_{+-}(-u,v=0)\neq 0.$$

When using the spinor space model, the two sets of couples (u,v) and $(-u,v)$ (or, similarly, the sets of couples (u,v) and $(u,-v)$) represent different geometrical points, so the requirement of single valuedness in the case of a spinor space does not imply that the values of the wave functions must be equal at the points (u,v) and $(-u,v)$:

$$\Psi(u,v)=\Psi((x_1,x_2)^{(1)})\neq\Psi(-u,v)=\Psi((x_1,x_2)^{(2)}).$$

Therefore, we conclude that the solutions $F(u,v),G(u,v)$ of the types $(--)$ and $(++)$ are single-valued in the spaces with vector structure, whereas the solutions $F(u,v),G(u,v)$ of the types $(-+)$ and $(+-)$ are not single-valued in spaces with vector structure, so the solutions of these two types $(-+)$ and $(+-)$ must be discarded. However, the types of solutions $(-+)$ and $(+-)$ are valid in the space with spinor structure.

7 Conclusions

The extension procedure which describes a space with spinor structure is performed by relying on cylindrical parabolic coordinates. This is done through the expansion of the region, $G(t,u,v,z)\rightsquigarrow\tilde{G}(t,u,v,z)$, so that instead of the half plane $(u,v>0)$ now the entire plane (u,v) should be used, accompanied with new identification rules for the boundary points. In the Cartesian picture, this procedure corresponds to taking the two-sheet surface $(x',y')\oplus(x'',y'')$ instead of the one-sheet surface (x,y).

The solutions of the Klein–Fock–Gordon and Schrödinger equations are constructed in terms of parabolic cylindric functions. Four types of solutions are possible: $\Psi_{++},\Psi_{--};\Psi_{+-},\Psi_{-+}$. The first two ones, Ψ_{++} and Ψ_{--}, provide us with single-valued functions of the vector space points, whereas the last two, Ψ_{+-} and Ψ_{-+}, have discontinuities in the framework of vector spaces, and therefore they must be discarded in this model. All the four types of functions are continuous while regarded in the spinor space. It is established that all solutions $\Psi_{++},\Psi_{--},\Psi_{+-}$ and Ψ_{-+}, are orthogonal to each other, provided that integration is done over the extended region of integration which covers (corresponds to) the spinor space. Similar results are obtained for the

Dirac equation. The solutions of the type $(--), (++)$ are single-valued in the space with vector structure, whereas the solutions of the types $(-+), (+-)$ are not single-valued in the space with vector structure, so the solutions of types $(-+)$ and $(+-)$ must be discarded. However, they must are valid solutions in the space with spinor structure.

Acknowledgments. The present work was developed under the auspices of the Grant No. 11962012-BRFFR-RAF14RA-006, within the cooperation framework between the Romanian Academy and the Belarussian Republican Foundation for Fundamental Research.

References

[1] R. Penrose, W. Rindler. Spinors and Space-Time. Volume I: Two-spinor calculus and relativistic fields. Cambridge University Press, 1984.

[2] V.M. Red'kov. On spinor structure of pseudo-Riemannian space-time and global continuity property for fermion wave functions. Vesti AN BSSR. ser. fiz.-mat. **3**. 49–55 (1994).

[3] V.M. Red'kov. On spinor P-oriented model of a 3-dimensional Euclidean space. Vesti AN BSSR. ser. fiz.-mat. **3**. 56–62 (1995).

[4] V.M. Red'kov. On connection between assumption of spinor geometry and the concept of a fermion intrinsic parity. Vesti AN BSSR. ser. fiz.-mat. **1**. 36–43 (1996).

[5] V.M. Red'kov. P-orientation and spacial spinors. Vesti AN BSSR. Ser. fiz.-mat. **2**. 76–80, (2000).

[6] V.M. Red'kov. Linear representations of spinor coverings of the Lorentz group and intrinsic space-time parity of a fermion. Vesti AN BSSR. ser. fiz.-mat. **2**. 70–76 (2003).

[7] V.M. Red'kov. Geometry of 3-spaces with spinor structure. Nonlinear Phenomena in Complex Systems. **7**. 106–128 (2004).

[8] V.M. Red'kov. Spinor structure of P-oriented space, Kustaanheimo-Stiefel and Hopf bundle – connection between formalisms. Nonlinear Phenomena in Complex Systems **8**. 222–239 (2005).

[9] V.M. Red'kov. Space with spinor structure and analytical properties of the solutions of Klein–Fock and Schrödinger equations in cylindric parabolic coordinates. Proceedings of 13th International School & Conference "Foundation & Advances in Nonlinear Science", Eds.: Kuvshinov V. I., Krylov G. G.. – Minsk, 2006. – P. 22–42.

[10] E. Ovsiyuk, O. Veko, M. Neagu, V. Balan, V. Red'kov. On possible effects of the spinor structures in quantum physics. Hypercomplex Number in geometry and Physics. **10**, no 2(20). 290–314 2013.

[11] V.M. Red'kov, E.M. Ovsiyuk. Space with spinor structure and analytical properties of the solutions of Klein–Fock equation in cylindric parabolic coordinates. P. 143–155 in: Theoretical Physics and its new Applications. Selected contributions at the 3rd International Conference on Theoretical Physics. June 24–28, 2013; ed. T.F. Kamalov. – Moscow, Russia. M.: MIPT, 2014. – 193 p.

[12] O.V. Veko, E.M. Ovsiyuk, A. Oana, M. Neagu, V. Balan, V.M. Red'kov. Spinor Structures in Geometry and Physics. Nova Sci. Publ. Inc.: New York, 2015.

Ортогональные системы функций в пространстве Лобачевского и их приложение в квазипотенциальном подходе

В.Н. Капшай, С.И. Фиалка
ГГУ им. Ф. Скорины, Беларусь

Аннотация

На основе парциальных волн, описывающих свободные состояния релятивистских частиц с произвольным значением орбитального момента, определён явный вид парциальных функций Грина в релятивистском конфигурационном представлении. В случае локальных в импульсном пространстве Лобачевского квазипотенциалов, получены в явном виде парциальные уравнения для волновых функций состояний рассеяния, связанных состояний и резонансных состояний. Описан метод приближённого решения задач о рассеянии. Ключевые слова: квазипотенциальный подход, функция Грина, дельта-потенциалы, сечение рассеяния.

1 Введение

В данной работе исследуются квазипотенциальные уравнения квантовой теории поля для состояний рассеяния двух бесспиновых частиц массы m [1, 2]. Эти уравнения можно записывать в импульсном представлении и релятивистском конфигурационном представлении. Переход между представлениями осуществляется посредством разложения по полной и ортогональной системе функций в пространстве Лобачевского. Использование того или иного представления часто определяет принципиальную возможность решить задачу. Вместе с тем в релятивистской теории число аналитически решаемых задач очень мало, в то время как очень важны именно точно решаемые задачи.

В нерелятивистской теории большое внимание привлекают потенциалы задаваемые с помощью δ-функций [3, 4, 5, 6, 7], так как задачи с такими потенциалами решаются точно. В течение последних лет развивались подходы, использующие δ-функции в качестве потенциала и в релятивистской теории, так решения одномерных релятивистских двухчастичных уравнений с δ-потенциалами были получены в работах [8, 9]. Трехмерные релятивистские уравнения, содержащие суперпозицию двух δ-потенциалов для двух бесспиновых частиц, находящихся в s-состояниях ($\ell = 0$), подробно исследованы в работе [10].

Здесь нами изложен метод приближенного решения релятивистских двухчастичных задач о рассеянии с квазипотенциалами достаточно общего вида, для произвольного значения орбитального момента частиц ℓ. Метод основан на точном решении задачи, где в качестве потенциала используется суперпозиция N δ-потенциалов. Широкое применение при этом находят системы ортогональных функций - релятивистские парциальные волны.

2 Релятивистские парциальные уравнения в конфигурационном представлении

Рассмотрим трехмерное релятивистское уравнение квазипотенциального типа для состояний рассеяния систем двух бесспиновых частиц массы m. В импульсном представлении уравнение имеет вид [11]:

$$\psi(\vec{q},\vec{p}) = (2\pi)^3 \frac{E_p}{m}\delta(\vec{p}-\vec{q}) + \frac{1}{(2\pi)^3} G_0(E_q,E_p) \int V(E_q;\vec{p},\vec{k})\,\psi(\vec{q},\vec{k}) \frac{m}{E_k}\, d\vec{k}. \tag{1}$$

Для перехода к релятивистскому конфигурационному представлению применяется разложение по ортогональным функциям, которые реализуют унитарные неприводимые представления группы Лоренца [11]:

$$\xi(\vec{p},\vec{r}) = \left(\frac{E_p - \vec{p}\,\vec{n}}{m}\right)^{-1-i\,m\,r}. \tag{2}$$

Разложение по функциям (2) называется преобразованием Шапиро, оно играет роль преобразования Фурье в импульсном пространстве Лобачевского, отображая последнее на релятивистское конфигурационное пространство.

Рассмотрим случай, когда квазипотенциал в импульсном пространстве Лобачевского локален и сферически-симметричен [11]. Применяя преобразование Шапиро, уравнение (1) получим в виде:

$$\psi(\vec{q},\vec{r}) = \xi(\vec{q},\vec{r}) + \int G_0(E_q;\ \vec{r},\vec{r}')\ V(E_q;\ r')\,\psi(\vec{q},\vec{r}')\, d\vec{r}\,'\,. \tag{3}$$

Связь между функциями Грина в релятивистском конфигурационном представлении и импульсном представлении определяется уравнением вида [11]:

$$G_0(E_q;\ \vec{r},\vec{r}') = \frac{1}{(2\pi)^3}\int \xi(\vec{p},\vec{r}) G_0(E_q,E_p)\,\xi^*(\vec{p},\vec{r}')\ \frac{m}{E_p}\, d\vec{p}. \tag{4}$$

Процесс решения уравнений типа (3), (4) можно значительно упростить с помощью парциального анализа, для чего удобно перейти к использованию быстрот (χ):

$$\vec{p} = m\,\mathrm{sh}\chi_p\,\vec{n}_p; \qquad E_p = m\,\mathrm{ch}\chi_p; \qquad \frac{m}{E_p}\,dp = m\,d\chi_p.$$

Применим к (2), (3), (4) разложение по базису сферических гармоник:

$$\xi(\vec{p},\vec{r}) = \frac{4\pi}{p\,r}\sum_{\ell\,\mu} i^\ell\, s_\ell(\chi_p,r)\, Y_{\ell\,\mu}(\vec{n}_r)\, Y^*_{\ell\,\mu}(\vec{n}_p), \tag{5}$$

$$\psi(\vec{q},\vec{r}) = \frac{4\pi}{q\,r}\sum_{\ell\,\mu} i^\ell\, \psi_\ell(\chi_q,r)\, Y_{\ell\,\mu}(\vec{n}_r)\, Y^*_{\ell\,\mu}(\vec{n}_q), \tag{6}$$

$$G_0(E_q;\ \vec{r},\vec{r}') = \frac{1}{r\,r'}\sum_{\ell\,\mu} G_0^{(\ell)}(\chi_q;r,r')\, Y_{\ell\,\mu}(\vec{n}_r)\, Y^*_{\ell\,\mu}(\vec{n}_{r'}). \tag{7}$$

Подставив в уравнение (3) разложения (5), (6), (7), находим одномерные парциальные уравнения в r-представлении:

$$\psi_\ell(\chi_q,r) = s_\ell(\chi_q,r) + \int_0^\infty G_0^{(\ell)}(\chi_q;r,r')\, V(E_q;\ r')\,\psi_\ell(\chi_q,r')\, dr'\,. \tag{8}$$

Аналогично, подставив в (4) парциальные разложения (5) и (7), получим формулу, связывающую парциальные функции Грина в релятивистском конфигурационном представлении с функциями Грина в импульсном пространстве:

$$G_0^{(\ell)}(\chi_q; r, r') = \frac{2}{\pi} m \int_0^\infty s_\ell(\chi_p, r)\, G_0(E_q, E_p)\, s_\ell^*(\chi_p, r')\, d\chi_p. \tag{9}$$

В квазипотенциальном подходе было предложено несколько вариантов уравнений (1). Они отличаются видом функций Грина, которые в нерелятивистском пределе совпадают и переходят в функцию Грина уравнения Шрёдингера. Здесь мы рассмотрим четыре вида функции Грина $G_{0,j}(E_q, E_p)$, которые в импульсном пространстве имеют вид

$$\begin{aligned} &G_{0,1}(E_q, E_p) = \tfrac{1}{m^2\,\mathrm{ch}^2\chi_q - m^2\,\mathrm{ch}^2\chi_p + i\,0}; \quad G_{0,2}(E_q, E_p) = \tfrac{1}{m\,\mathrm{ch}\chi_p (2m\,\mathrm{ch}\chi_q - 2m\,\mathrm{ch}\chi_p + i\,0)}; \\ &G_{0,3}(E_q, E_p) = \tfrac{\mathrm{ch}\chi_p}{m^2\,\mathrm{ch}^2\chi_q - m^2\,\mathrm{ch}^2\chi_p + i\,0}; \qquad G_{0,4}(E_q, E_p) = \tfrac{1}{m\,(2m\,\mathrm{ch}\chi_q - 2m\,\mathrm{ch}\chi_p + i\,0)}\,. \end{aligned} \tag{10}$$

Здесь принято соответствие: функция Грина уравнения Логунова-Тавхелидзе - $j = 1$, Кадышевского - $j = 2$ и их модифицированные версии - $j = 3$, $j = 4$ [10].

3 Парциальные функции Грина в конфигурационном представлении

Как известно, парциальные плоские волны выражаются через присоединённые функции Лежандра первого рода [11]:

$$s_\ell(\chi, r) = (-i)^{\ell+1} \sqrt{\frac{\pi\,\mathrm{sh}\chi}{2}}\, \frac{\Gamma(imr + \ell + 1)}{\Gamma(imr)} P^{-1/2-\ell}_{-1/2+imr}(\mathrm{ch}\chi). \tag{11}$$

Нами найден более компактный и удобный для расчётов явный вид парциальных волн на основе присоединённые функций Лежандра второго рода:

$$s_\ell(\chi, r) = (-i)^{\ell+1}\, \frac{\exp(m\,r\,\pi)}{\Gamma(imr)}\, Q_\ell^{i\,m\,r}(\mathrm{cth}\chi). \tag{12}$$

Для упрощения вычислений введём функции

$$\tilde{h}_\ell^{\pm}(\chi, r) = (-1)^\ell\, s_{-\ell-1}(\chi, r) \pm i\, s_\ell(\chi, r)\,. \tag{13}$$

Подставив (12) в (9), можно показать, что парциальные функции Грина могут быть представлены в следующем, удобном для расчетов виде:

$$G_0^{(\ell)}(\chi_q; r, r') = \;\; \dot{G}^{(\ell)}(\chi_q; r - r', r + r') - \ddot{G}^{(\ell)}(\chi_q; r + r', r - r') \tag{14}$$

где

$$\begin{aligned} &\dot{G}_0^{(\ell)}(\chi_q; r - r', r + r') = \\ &\qquad = \tfrac{m}{2\pi} \int_0^\infty \left[\tilde{h}_\ell^+(\chi_k, r)\tilde{h}_\ell^{+*}(\chi_k, r') + \tilde{h}_\ell^-(\chi_k, r)\tilde{h}_\ell^{-*}(\chi_k, r')\right] G_0(E_q, E_k)\, d\chi_k\,; \end{aligned} \tag{15}$$

$$\begin{aligned} &\ddot{G}_0^{(\ell)}(\chi_q; r + r', r - r') = \\ &\qquad = \tfrac{m}{2\pi} \int_0^\infty \left[\tilde{h}_\ell^+(\chi_k, r)\tilde{h}_\ell^{-*}(\chi_k, r') + \tilde{h}_\ell^-(\chi_k, r)\tilde{h}_\ell^{+*}(\chi_k, r')\right] G_0(E_q, E_k)\, d\chi_k\,. \end{aligned} \tag{16}$$

Используя свойства функций (13), нами получено соотношение для величин (15) и (16):

$$\frac{\Gamma(-imr'-\ell)}{\Gamma(-imr')}\;\dot{G}_0^{(\ell)}(\chi_q;r-r',r+r')=\frac{\Gamma(imr'-\ell)}{\Gamma(imr')}\;\ddot{G}_0^{(\ell)}(\chi_q;r-r',r+r')\,. \tag{17}$$

То есть, чтобы получить парциальную функцию Грина (14), достаточно вычислить одну из величин (15) или (16) и воспользоваться формулой (17).

Величины (15), (16) можно вычислить, пользуясь методами контурного интегрирования [12]. Производя вычисления при $\ell=0$, получим известные выражения ($\rho=r-r'$; $\rho'=r+r'$) [8, 9, 10, 13]:

$$\dot{G}_1^{(0)}(\chi_q;\rho,\rho')=\tfrac{-i}{m}\cdot\tfrac{\mathrm{sh}[m\rho\,(\pi/2+i\,\chi_q)]}{\mathrm{sh}(m\rho\,\pi/2)\,\mathrm{sh}(2\,\chi_q)};\quad \dot{G}_2^{(0)}(\chi_q;\rho,\rho')=\tfrac{-i}{m}\cdot\tfrac{\mathrm{sh}[m\rho\,(\pi+i\,\chi_q)]}{\mathrm{sh}(m\rho\,\pi)\,\mathrm{sh}(2\,\chi_q)}+\tfrac{(4\,m\,\mathrm{ch}\,\chi_q)^{-1}}{\mathrm{ch}(m\rho\,\pi/2)};$$

$$\dot{G}_3^{(0)}(\chi_q;\rho,\rho')=\tfrac{-i}{2\,m}\cdot\tfrac{\mathrm{ch}[m\rho\,(\pi/2+i\,\chi_q)]}{\mathrm{ch}(m\rho\,\pi/2)\,\mathrm{sh}\,\chi_q};\qquad \dot{G}_4^{(0)}(\chi_q;\rho,\rho')=\tfrac{-i}{2\,m}\cdot\tfrac{\mathrm{sh}[m\rho\,(\pi+i\,\chi_q)]}{\mathrm{sh}(m\rho\,\pi)\,\mathrm{sh}\,\chi_q}\,. \tag{18}$$

Для случая произвольного орбитального момента, нами получен явный вид парциальных функций Грина в следующем виде:

$$\begin{aligned}
&G_{0,1}^{(\ell)}(\chi_q;r,r')=\tfrac{1}{4i}\cdot\tfrac{1}{m\,\mathrm{sh}\,\chi_q\,\mathrm{ch}\,\chi_q}\times\\
&\times\Big[\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho\,\tfrac{\pi}{2}))-\\
&-\Big(\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho'\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho'\,\tfrac{\pi}{2}))\Big)\Big];\\
&G_{0,2}^{(\ell)}(\chi_q;r,r')=\tfrac{1}{4i}\cdot\tfrac{1}{m\,\mathrm{sh}\,\chi_q\,\mathrm{ch}\,\chi_q}\times\\
&\times\Big[\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho\,\pi))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho\,\pi))-\\
&-\Big(\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho'\,\pi))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho'\,\pi))\Big)\Big]-\\
&-\tfrac{1}{8}\cdot\tfrac{(-1)^\ell}{m\,\mathrm{ch}\,\chi_q}\times\\
&\times\Big[\tilde h_\ell^{+}(\tfrac{\pi}{2}\,i,r)\,\tilde h_\ell^{+*}(\tfrac{\pi}{2}\,i,r')\,(1+\mathrm{th}(m\,\rho'\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\tfrac{\pi}{2}\,i,r)\,\tilde h_\ell^{-*}(\tfrac{\pi}{2}\,i,r')\,(1-\mathrm{th}(m\,\rho'\,\tfrac{\pi}{2}))-\\
&-\Big(\tilde h_\ell^{+}(\tfrac{\pi}{2}\,i,r)\,\tilde h_\ell^{-*}(\tfrac{\pi}{2}\,i,r')\,(1+\mathrm{th}(m\,\rho\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\tfrac{\pi}{2}\,i,r)\,\tilde h_\ell^{+*}(\tfrac{\pi}{2}\,i,r')\,(1-\mathrm{th}(m\,\rho\,\tfrac{\pi}{2}))\Big)\Big];\\
&G_{0,3}^{(\ell)}(\chi_q;r,r')=\tfrac{1}{4i}\cdot\tfrac{1}{m\,\mathrm{sh}\chi_q}\times\\
&\times\Big[\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1+\mathrm{th}(m\,\rho\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1-\mathrm{th}(m\,\rho\,\tfrac{\pi}{2}))-\\
&-\Big(\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1+\mathrm{th}(m\,\rho'\,\tfrac{\pi}{2}))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1-\mathrm{th}(m\,\rho'\,\tfrac{\pi}{2}))\Big)\Big];\\
&G_{0,4}^{(\ell)}(\chi_q;r,r')=\tfrac{1}{4i}\cdot\tfrac{1}{m\,\mathrm{sh}\,\chi_q}\times\\
&\times\Big[\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho\,\pi))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho\,\pi))-\\
&-\Big(\tilde h_\ell^{+}(\chi_q,r)\,\tilde h_\ell^{-*}(\chi_q,r')\,(1+\mathrm{cth}(m\,\rho'\,\pi))+\tilde h_\ell^{-}(\chi_q,r)\,\tilde h_\ell^{+*}(\chi_q,r')\,(1-\mathrm{cth}(m\,\rho'\,\pi))\Big)\Big].
\end{aligned} \tag{19}$$

Таким образом, подставив (19) в (8) можно в явном виде записать парциальные уравнения для волновых функций состояний рассеяния, связанных состояний и резонансных состояний, в случае любых локальных в импульсном пространстве Лобачевского квазипотенциалов.

4 Метод приближенного решения

Тот факт, что уравнение (8) является непосредственным релятивистским обобщением интегрального уравнения Шрёдингера, дало нам возможность расширить популярный в нерелятивистской теории приближённый метод решения [3, 4, 5, 6, 7]. Этот метод основан на аппроксимации гладкого потенциала суперпозицией из N δ-потенциалов, локализованных на сферах конечного радиуса $r_k > 0$ (delta-shell potentials):

$$V(r) = \sum_{k=1}^{N} V_k \, \delta(r - r_k). \tag{20}$$

Как и в случае уравнения Шрёдингера, релятивистское уравнение (8) с δ-потенциалами решается аналитически точно. Подставляя (20) в (8) и интегрируя по r', получим

$$\psi_\ell(\chi_q, r) = s_\ell(\chi_q, r) + \sum_{k=1}^{N} G_0^{(\ell)}(\chi_q; r, r_k) V_k \, \psi_\ell(\chi_q, r_k) \; . \tag{21}$$

В пределе, когда $r \to \infty$, уравнение (21) примет вид:

$$\psi_\ell(\chi_q, r)|_{r\to\infty} = s_\ell(\chi_q, r) + \left(\sum_{k=1}^{N} V_k \, \frac{-s_\ell^*(\chi_q, r_k)}{m \, \mathrm{sh}\, \chi_q \, \mathrm{ch}\, \tilde{\chi}_{(j)}} \; \psi_\ell(\chi_q, r_k) \right) \; e^{i \, (m\, r\, \chi_q - \, \pi/2\, \ell)} \, . \tag{22}$$

В теории рассеяния асимптотическое выражение волновой функции при $r \to \infty$ представляется суммой падающей плоской волны и расходящейся сферической волны. Аналогично нерелятивистской теории, определим релятивистскую парциальную амплитуду рассеяния как коэффициент при рассеянной волне делённый на импульс [8, 9, 10, 13]. Тогда из (22) получим ($\tilde{\chi} = (\chi_q, \; \chi_q, \; 0, \; 0)$):

$$f_\ell(\chi_q) = \frac{-1}{m^2 \, \mathrm{sh}^2 \chi_q \, \mathrm{ch}\, \tilde{\chi}_{(j)}} \; \cdot \; \sum_{k=1}^{N} s_\ell^*(\chi_q, r_k) \; V_k \, \psi_\ell(\chi_q, r_k) \, . \tag{23}$$

Неизвестные $\psi_\ell(\chi_q, r_k)$ являются решением системы линейных алгебраических уравнений, которая получается из уравнения (21):

$$\begin{pmatrix} 1 - G_0^{(\ell)}(\chi_q; r_1, r_1) V_1 & \cdots & - G_0^{(\ell)}(\chi_q; r_1, r_N) V_N \\ - G_0^{(\ell)}(\chi_q; r_2, r_1) V_1 & \cdots & - G_0^{(\ell)}(\chi_q; r_2, r_N) V_N \\ \vdots & \ddots & \vdots \\ - G_0^{(\ell)}(\chi_q; r_N, r_1) V_1 & \cdots & 1 - G_0^{(\ell)}(\chi_q; r_N, r_N) V_N \end{pmatrix} \begin{pmatrix} \psi_\ell(\chi_q, r_1) \\ \psi_\ell(\chi_q, r_2) \\ \vdots \\ \psi_\ell(\chi_q, r_N) \end{pmatrix} = \begin{pmatrix} s_\ell(\chi_q, r_1) \\ s_\ell(\chi_q, r_2) \\ \vdots \\ s_\ell(\chi_q, r_N) \end{pmatrix} .$$

По аналогии с нерелятивистской теорией, на основе амплитуд (23) можно построить полное сечение рассеяния [13]:

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1) |f_\ell(\chi_q)|^2 \tag{24}$$

Таким образом, подставляя параметры потенциала (20) и явный вид парциальных плоских волн (11), (12) и функций Грина (19) в (22), можно рассчитать амплитуду рассеяния (23). Учитывая найденную амплитуду в (24), получим приближенное решение задачи о рассеянии с аналитическим потенциалом.

5 Метод приближенного решения

В работе предложен новый вариант определения явного вида релятивистских парциальных волн. Установлен явный вид парциальных функций Грина в релятивистском конфигурационном представлении для четырёх вариантов уравнений квазипотенциального подхода. Парциальные функции Грина не зависят от используемого потенциала и могут быть использованы как при решении задач о рассеянии, так и при решении задач о связанных состояниях и резонансных состояниях.

Описан метод приближённого решения релятивистских двухчастичных уравнений с локальными в импульсном пространстве Лобачевского квазипотенциалами. Метод основан на замене гладкого потенциала суперпозицией из N δ-потенциалов. Представлены формулы для расчёта парциальных релятивистских амплитуд рассеяния и сечения рассеяния. С применением данного метода можно находить приближенные решения задач о рассеянии частиц в полях, характеризующихся различными несингулярными быстроубывающими потенциалами и вычислять различные характеристики рассеянных частиц. Важно отметить, что даже при небольшом количестве δ-потенциалов, используя зависимость сечения рассеяния от энергии двухчастичной системы, можно качественно правильно описать физический процесс взаимодействия частиц.

Полученные результаты могут быть использованы при решении широкого круга задач в рамках квазипотенциального подхода.

Список литературы

[1] Logunov A.A., Tavkhelidze A.N. Quasi-optical approach in quantum field theory // NuovoCimento. - 1963. - Vol. 29, №2. - P. 380 - 399.

[2] Kadyshevsky V.G. Quasipotential type equation for the relativistic scattering amplitude // Nucl. Phys. - 1968. - Vol. B6, N1. - P. 125 - 148.

[3] Aviles J.B. δ-Function Model for the Two-Nucleon Interaction // Physical Review C. - 1972. - T. 6, № 5. - P. 1467 - 1484.

[4] Perez R.N., Amaro J.E., Arriola E.R. Partial-wave analysis of nucleon-nucleon scattering below the pion-production threshold // Physical Review C. - 2013. - T. 88, №. 2. - P. 024002.

[5] Yanetka I. On Resonant Tunnelling in the Biased Double Delta-Barrier // Acta Physica Polonica, A. - 2009. - T. 116, №. 6. - P. 1059 - 1066.

[6] Kraynov V.P., Ishkhanyan H.A. Resonant reflection of Bose-Einstein condensate by a double delta-function barrier within the Gross-Pitaevskii equation // Laser physics. - 2010. - T. 20, №. 9. - P. 1856 - 1860.

[7] Kapshai V.N., Alferova T.A., Elander N. Numerical determination of complex resonance energies by using a superposition of δ-potentials // International journal of quantum chemistry. - 2003. - T. 91, № 4. - P. 533 - 540.

[8] Kapshai V.N., Alferova T.A. Relativistic two-particle one-dimensional scattering problem for superposition of delta-potentials // Journal of Physics A: Mathematical and General. - 1999. - T. 32, №. 28. - P. 5329.

[9] Kapshai V.N., Alferova T.A. One-Dimensional Relativistic Problems on the Bound States and Degradation for a Superposition of Two δ Potentials // Russian physics journal. - 2002. - T. 45, №. 1. - P. 1 - 9.

[10] Kapshai V.N., Grishechkin Y.A. Relativistic two-particle equations with superposition of delta-shell potentials: scattering and bound states // arXiv preprint arXiv:1312.1902. - 2013.

[11] Кадышевский В.Г., Мир-Касимов Р.М., Скачков Н.Б. Трехмерная формулировка релятивистской проблемы двух тел // ЭЧАЯ. - 1972. - Т. 2, №. 3. - С. 635 - 690.

[12] Howie J.M. Complex analysis / J.M. Howie. - London: Springer-Verlag, 2003. - 276 c.

[13] Kapshai V.N., Fialka S.I. Partial two-particle relativistic scattering problems and superpositions of δ-shell potentials // Proceedings of the F. Scorina Gomel State University. - 2015. - 3(90). - P. 140-146.

Approximate radial wave functions for a scalar particle with polarizability in the presence of the Coulomb field

V. V. Kudryashov
kudryash@dragon.bas-net.by; Institute of Physics,
National Academy of Sciences of Belarus

Abstract

The approximate radial wave functions and energy levels are obtained by means of the modified WKB method for a scalar particle with polarizability in the presence of the Coulomb field. The extraordinarily weak dependence of energy on a polarizability parameter is established.

1 Scalar particle with polarizability in the Coulomb field

The possibility to describe polarizability of a scalar particle in the presence of electromagnetic field was established in [1] within the framework of theory of the relativistic first order wave equations with extended representations of the proper Lorentz group. The system of 15 radial equations for a particle with polarizability in the Coulomb field was reduced in [2] to a single second order differential equation

$$\frac{d^2R(r)}{dr^2} = \left(\frac{sk^2\hbar^2}{m^2c^2r^4} + \frac{l(l+1)-k^2}{r^2} - \frac{2kE}{\hbar cr} + \frac{m^2c^4-E^2}{\hbar^2c^2} \right) R(r) \tag{1}$$

containing the Coulomb parameter k and polarizability parameter s. Here $l = 0, 1, 2, \ldots$, m is a particle mass and E is energy.

Introducing the dimensionless quantities

$$x = \frac{mcr}{\hbar}, \quad e = \frac{E}{mc^2} \tag{2}$$

it is convenient to rewrite Eq.(1) in the form

$$\frac{d^2R(x)}{dx^2} = F(x)R(x) \tag{3}$$

with the following function

$$F(x) = \frac{sk^2}{x^4} + \frac{l(l+1)-k^2}{x^2} - \frac{2ke}{x} + (1-e^2). \tag{4}$$

For $k > 0$, $s > 0$, $1 > e > 0$, the function $F(x)$ tends to infinity at $x \to 0$, tends to asymptotics $F_{as} = 1 - e^2$ at $x \to \infty$ and has minimum F_{min} at the point x_m. There are two turning points $x_\pm$, where $F(x) = 0$, if $F_{min} < 0$. In this case the existence of bound states is allowed.

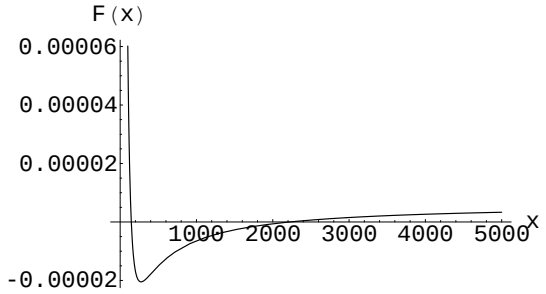


Figure 1: Function $F(x)$

Fig. 1 shows behavior of the function $F(x)$ for $l = 1$, $s = 0.1$, $k = 1/137$, $e = 0.999996918$. In this case

$$x_- = 145.997,\ x_m = 273.994,\ F_{min} = -0.000020476,\ x_+ = 2222.34,\ F_{as} = 6.16403 \cdot 10^{-6}.$$

Note that the extraordinary smallness of F_{min} and F_{as} in comparison with the distance between the turning points is typical for the considered problem.

If the function $F(x)$ is replaced by the the dominating term $F_0(x) = sk^2/x^4$ at $x \to 0$ then the exact decreasing solution of Eq.(3) is described by formula

$$R_0(x) = x \exp(-\sqrt{sk^2}/x), \tag{5}$$

which gives behavior of a desired function $R(x)$ in the neighborhood of point $x = 0$. Note that, the function $F_0(x)$ is the exclusive case, when the usual WKB approximation reproduces the exact solution. However, it is impossible to employ the conventional WKB approximation in the whole range of the variable x since this approximation has the known nonphysical singularities in the vicinity of the turning points.

2 Modified WKB approximation

In order to overcome difficulties of the standard semiclassical method the essential improvement of the conventional WKB approximation was given in [3, 4] with the help of explicit summation of the partial WKB series. In [5, 6], the initial proposed approximation was modified by analogy with the phase-integral approximation [7]. This modification leads to the considerable simplification of approach conserving advantages of initial approximation. In the present paper, we apply new modified WKB approximation to Eq.(1).

The modified WKB approximation for the continuous radial wave function is of the form

$$R_{app}(x) = \begin{cases} \dfrac{c}{2} \dfrac{\exp\left(-\int_x^{x_-} P_o(x')dx'\right)}{\sqrt{P_o(x)}}, & 0 < x < x_-, \\ c\sqrt[4]{\frac{3}{4}} \dfrac{\cos\left(\int_{x_-}^{x} P_i(x')dx' - \pi/3\right)}{\sqrt{P_i(x)}}, & x_- < x < x_+, \\ \dfrac{c}{2}(-1)^n \dfrac{\exp\left(-\int_{x_+}^{x} P_o(x')dx'\right)}{\sqrt{P_o(x)}}, & x_+ < x < \infty. \end{cases} \tag{6}$$

Here c is a normalization constant and

$$P_o(x) = \frac{b(x)}{2\pi \mathrm{Ai}(a(x))\mathrm{Bi}(a(x))}, \quad P_i(x) = \frac{b(x)}{\pi\left(\mathrm{Ai}^2(a(x)) + \mathrm{Bi}^2(a(x))\right)} \tag{7}$$

are simply presented via well-studied Airy functions $Ai(a)$ and $Bi(a)$ [8]. The specific character of the considered problem appears when we substitute the explicit expression (4) into the general formulas

$$a(x) = F(x) \left| \frac{F(x)}{dx} \right|^{-2/3}, \quad b(q) = \left| \frac{dF(x)}{dx} \right|^{1/3}. \tag{8}$$

Moreover we have the new quantization condition

$$\int_{x_-}^{x_+} P_i(x)\, dx = 2\pi/3 + n\pi, \quad n = 0, 1, 2 \ldots, \tag{9}$$

determining the approximate eigenvalues of energy e. This condition ensures continuity of the first derivative $dR_{app}(x)/dx$ at the turning points x_- and x_+.

It is not hard to show that $P_o(x) \to \sqrt{F(x)}$ if $x \to 0$ and $x \to \infty$ while $P_i(x) \to \sqrt{-F(x)}$ if $x \to x_m$, where $F(x) < 0$, $dF(x)/dx = 0$. The modified WKB approximation reproduces the limiting behavior $R_0(x)$ of the exact wave function at the origin. At the same time our approximation reproduces the known satisfactory approximation [9] in the neighborhood of the turning points.

3 Numerical results

The numerical results are presented in the special case when the Coulomb parameter is equal to the fine structure constant $k = 1/137$.

Tables 1 and 2 show that the values of energy are very close to 1. The distance between the energy levels decreases rapidly with the growth of n. The energy increases in the case of transition from $l = 0$ to $l = 1$ for the same n. It should be stressed that the peculiarity of the considered problem is the extraordinarily weak dependence of energy e on a polarizability parameter s. This property becomes more distinct if n and l increase.

Table 1: Energy levels for $l = 0$

n	$e, s = 0.1$	$e, s = 100$
0	0.9998933474	0.9999192989
1	0.9999869710	0.9999883467
2	0.9999954022	0.9999957063
3	0.9999976920	0.9999978026

Table 2: Energy levels for $l = 1$

n	$e, s = 0.1$	$e, s = 1000000$
0	0.9999931366	0.9999931388
1	0.9999969180	0.9999969187
2	0.9999982742	0.9999982746
3	0.9999989012	0.9999989013

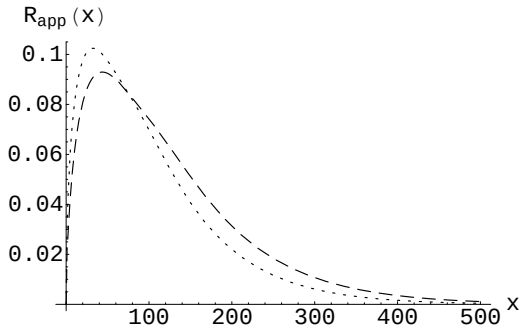


Figure 2: Radial wave functions for $l = 0$, $n = 0$

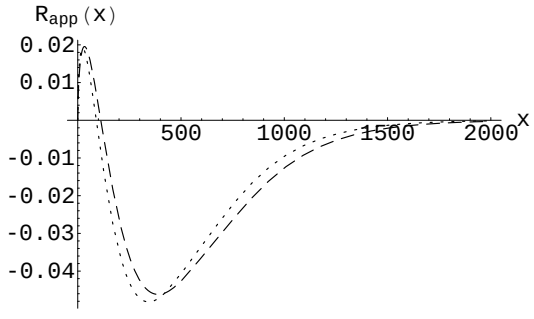


Figure 3: Radial wave functions for $l = 0$, $n = 1$

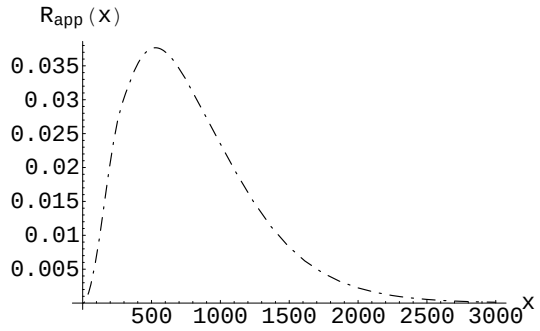


Figure 4: Radial wave functions for $l = 1$, $n = 0$

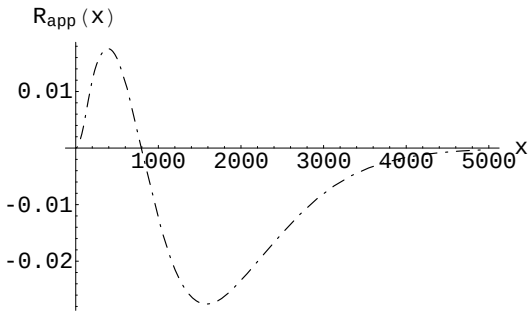


Figure 5: Radial wave functions for $l = 1$, $n = 1$

Figures 2– 5 demonstrate the continuous behavior of the approximate radial wave functions. These functions are normalized by condition

$$\int_0^\infty R_{app}^2(x)\,dx = 1.$$

The dotted lines correspond to the case $s = 0.1$ and the dashed lines correspond to the cases $s = 100$ for $l = 0$ and $s = 1000000$ for $l = 1$. Two lines become indistinguishable when $l = 1$. It is seen that the functions $R_{app}(x)$ decrease very slowly.

This work is supported by Belarusian Republican Foundation for Fundamental Research, Grant F14ARM-021.

References

[1] A. A. Bogush, V. V. Kisel, M. I. Levchuk, L. G. Moroz. Covariant Methods in Theoretical Physics, Minsk, 1981, P. 81-90 (Russian).

[2] V. Kisel, E. Ovsiyuk, G. Krylov, M. Amirfachrian, V. Red'kov. Nonlinear Dynamics and Applications, Vol. 18, Minsk, 2011, P. 168-179.

[3] V. V. Kudryashov. Dokl. Nats. Akad. Nauk Belarusi, Vol. 42, no. 6, P. 45-49 (1998) (Russian).

[4] V. V. Kudryashov, Yu. V. Vanne. J. Appl. Math., Vol. 2, no. 6, P. 265-275 (2002)).

[5] V. V. Kudryashov. Nonlinear Phenomena in Complex Systems (NPCS'98), Vol. 7, Minsk, 2000, P. 129-133.

[6] V. V. Kudryashov, A. V. Baran. Nonlinear Dynamics and Applications, Vol. 16, Minsk, 2009, P. 76-82.

[7] N. Fröman. Ann. Phys. (N.Y.),Vol. 61, P. 451-464 (1970).

[8] M. Abramovitz, I. A. Stegun. *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).

[9] C. M. Bender, S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

Spin 1/2 particle with anomalous magnetic moment in presence of external magnetic field, exact solutions

E.M. Ovsiyuk
e.ovsiyuk@mail.ru; Mozyr State Pedagogical University, Belarus
V.V. Kisel
Belarusian State University of Informatics and Radioelectronics, Minsk
V.M. Red'kov
redkov@dragon.bas-net.by; B.I. Stepanov Institute of Physics

Abstract

We examine a generalize Dirac equation for spin 1/2 particle with anomalous magnetic moment in presence of the external uniform magnetic field. After separation of the variables, the problem is reduced to a 4-order ordinary differential equation, which is solved exactly with the use of the factorization method. A generalized formula for Landau energy levels is found. Solutions are expressed in terms of confluent hypergeometric functions.

1 Ordinary Dirac equation in cylindrical coordinates, separation of the variables

We use the known representation for the uniform magnetic field: $\mathbf{A} = \frac{1}{2}\, c\mathbf{B} \times \mathbf{r},\ \mathbf{B} = (0, 0, B)$. After translating to cylindrical coordinates we get

$$A_t = 0\,, \qquad A_r = 0\,, \qquad A_z = 0\,, \qquad A_\phi = -Br^2/2\,. \tag{1}$$

a non-vanishing component of the electromagnetic tensor is $F_{\phi r} = Br$. We consider the Dirac equation in the magnetic field (1), using the tetrad formalism [1] for cylindrical coordinates $x^\alpha = (t, r, \phi, z)$:

$$dS^2 = dt^2 - dr^2 - r^2\, d\phi^2 - dz^2\,, \qquad e^\beta_{(a)}(x) = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{array}\right|\,. \tag{2}$$

Generally covariant tetrad Dirac equation [1] is

$$\left\{\gamma^c[\ i\hbar\ (\ e^\beta_{(c)}\partial_\beta + \frac{1}{2}\sigma^{ab}\gamma_{abc}\ )\ -\ \frac{e}{c}A_c\ ]\ -\ mc\right\}\Psi = 0\,, \tag{3}$$

where γ_{abc} are the Ricci rotation symbols rotation: $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha}\, e^\beta_{(a)} e^\alpha_{(c)}$, $A_a = e^\beta_{(a)} A_\beta$ is the tetrad components of the 4-vector A_β; $\sigma^{ab} = 1/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$ are generators for bispinor representation of the Lorentz group.

We will use the shortening notation: $e/c\hbar \Longrightarrow e,\ mc/\hbar \Longrightarrow M$. The Dirac equation takes the form (let $\Psi = \varphi/\sqrt{r}$):

$$\left[\ i\gamma^0\frac{\partial}{\partial t} + i\gamma^1\frac{\partial}{\partial r} + \gamma^2\ \left(\frac{i\partial_\phi}{r} + \frac{eBr}{2}\right) + i\gamma^3\frac{\partial}{\partial z} - M\right]\varphi = 0\,. \tag{4}$$

We search solutions in the form

$$\varphi = e^{-i\epsilon t}e^{im\phi}e^{ikz}\left|\begin{array}{c} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{array}\right|, \quad \left[+\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \gamma^2\,\mu(r) - k\gamma^3 - M\right]\left|\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array}\right| = 0\,,$$

where $\mu(r) = m/r - eBr/2$; further we will use the shortening notation $eB \Longrightarrow B$. When choosing Dirac matrices in the spinor basis, we find equations for the four functions $f_a(t,z)$:

$$\begin{aligned}
&(\frac{d}{dr}+\mu)f_4 + ikf_3 + i(\epsilon f_3 - Mf_1) = 0,\ (\frac{d}{dr}-\mu)f_3 - ikf_4 + i(\epsilon f_4 - Mf_2) = 0, \\
&(\frac{d}{dr}+\mu)f_2 + ikf_1 - i(\epsilon f_1 - Mf_3) = 0,\ (\frac{d}{dr}-\mu)f_1 - ikf_2 - i(\epsilon f_2 - Mf_4) = 0.
\end{aligned} \tag{5}$$

The equations are consistent with the linear constraint $f_3 = Af_1,\ f_4 = Af_2$, if the following condition is imposed

$$\epsilon - \frac{M}{A} = -\epsilon + MA \qquad \Longrightarrow \qquad A = A_{1,2} = \frac{\epsilon \pm \sqrt{\epsilon^2 - M^2}}{M}\,. \tag{6}$$

As a result, the problem is reduced to the system of two equations

$$\left(\frac{d}{dr}+\mu\right)f_2 + i(k - \epsilon + MA)f_1 = 0,\ \left(\frac{d}{dr}-\mu\right)f_1 + i(-k - \epsilon + MA)f_2 = 0. \tag{7}$$

In accordance with (6), we have two types of states:

$AM = \epsilon + \sqrt{\epsilon^2 - M^2}\,, \qquad (\ \sqrt{\epsilon^2 - M^2} = p\ )$

$$\left(\frac{d}{dr} + \mu\right)\,f_2 + i\,(k+p)\,f_1 = 0\,,\ \left(\frac{d}{dr} - \mu\right)\,f_1 - i\,(k-p)\,f_2 = 0\,; \tag{8}$$

$AM = \epsilon - \sqrt{\epsilon^2 - M^2}\,, \qquad (\ \sqrt{\epsilon^2 - M^2} = p\ )$

$$\left(\frac{d}{dr} + \mu\right)\,f_2 + i\,(k-p)\,f_1 = 0\,,\ \left(\frac{d}{dr} - \mu\right)\,f_1 - i\,(k+p)\,f_2 = 0\,. \tag{9}$$

For definiteness, we follow the variant (8).

2 Solving equations for r-variable

From (8) we obtain the second order equation for R_1

$$\frac{d^2R_1}{dr^2} + \left[\frac{m}{r^2} + \frac{B}{2} - \left(\frac{m}{r} - \frac{Br}{2}\right)^2 + \lambda^2\right]R_1 = 0\,. \tag{10}$$

where $\lambda^2 = \epsilon^2 - m^2 - k^2$. Parameter λ^2 describes the contribution of the electron transversal motion to the total energy, this part of the energy is quantized. Note that we diagonalize the operator

$$-i\frac{\partial}{\partial\phi}\Psi = m\ \Psi\ , \tag{11}$$

which represents the third projection of the total angular momentum of the Dirac particle in cylindrical tetrad basis:

$$\hat{J}_3\Psi_{Cart} = (-i\frac{\partial}{\partial\phi} + \Sigma_3)\ \Psi_{Cart} = m\ \Psi = m\ \Psi_{Cart} \tag{12}$$

therefore for m are permitted only half-integer values $m = \pm 1/2,\ \pm 3/2, ...$

We turn to eq. (10) and introduce variable $x = Br^2/2$, equation becomes[1]

$$4x\frac{d^2R_1}{dx^2} + 2\frac{dR_1}{dx} + \left(\frac{m(1-m)}{x} - x + 1 + 2m + \frac{2\lambda^2}{B}\right) R_1 = 0\,. \tag{13}$$

We seek solutions in the form $R_1(x) = x^A e^{-Cx} R(x)$. If A, C are chosen according $A = m/2\,,\ (1-m)/2\,,\ C = \pm 1/2$, the equation for R reads

$$x\frac{d^2R}{dx^2} + \left(2A + \frac{1}{2} - x\right)\frac{dR}{dx} - \left(A - \frac{m}{2} - \frac{\lambda^2}{2eB}\right) R = 0\,,$$

which is the confluent hypergeometric equation

$$x\ Y'' + (\gamma - x)Y' - \alpha Y = 0\,, \qquad \alpha = A - \frac{m}{2} - \frac{\lambda^2}{2B}\,, \qquad \gamma = 2A + \frac{1}{2}\,.$$

To obtain solutions that vanish at the origin $r \to 0$ and infinity $r \to \infty$, we must take positive values for A and C:

$$C = +1/2\,, \qquad A = \begin{cases} m = +1/2, +3/2, ..., & A = m/2\,, \\ m = -1/2, -3/2, ..., & A = (1-m)/2\,. \end{cases}$$

To obtain polynomials, we impose the know restriction $\alpha = -n,\ n = 0, 1, 2, ...$ This leads to the following rule for quantization of the parameter λ^2:

$$\frac{\lambda^2}{2eB} = A - \frac{m}{2} + n\,.$$

Depending on the sign of the quantum number m we get two formulas for $\lambda^2 = \epsilon^2 - M^2 - k^2$:

$$\begin{aligned} m > 0\,, \qquad & \lambda^2 = 2eB\ n\,, \qquad n = 0, 1, 2, ...; \\ m < 0\,, \qquad & \lambda^2 = 2eB\ \left(n - m + \frac{1}{2}\right), \quad n = 0, 1, 2, ... \end{aligned} \tag{14}$$

3 Accounting of the anomalous magnetic moment

The Dirac equation for a particle with spin 1/2 with anomalous magnetic moment in the Riemannian space-time (using the tetrad formalism) can be represented in the form [2, 3]

$$\left\{\gamma^c[\ i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2}\sigma^{ab}\gamma_{abc}\ )\ -\ \frac{e}{\hbar c}A_c\ ] - i\lambda\frac{2e}{Mc^2}\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) - \frac{Mc}{\hbar}\right\}\Psi = 0\,. \tag{15}$$

Dimensions of the parameters in the equation are

$$[\frac{Mc}{\hbar}] = l^{-1}, \quad [\frac{e}{\hbar c}A] = l^{-1}, \quad [\frac{e}{\hbar c}F] = l^{-2}, \quad [\frac{eF}{Mc^2}] = l^{-1}\,;$$

[1] Without loss of generality, we assume that the parameter B positive.

free parameter λ is dimensionless. Consider this equation in the uniform magnetic field. In view of

$$\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) = 2\sigma^{\phi r}F_{\phi r} = i\gamma^2\gamma^1 B = iB\Sigma_3$$

instead of (4) we get a more general equation

$$\left[i\gamma^0\frac{\partial}{\partial t} + i\gamma^1\frac{\partial}{\partial r} + \gamma^2\ \left(\frac{i\partial_\phi}{r} + \frac{eBr}{2}\right) + i\gamma^3\frac{\partial}{\partial z} + \lambda\frac{2eB}{Mc^2}\Sigma_3 - \frac{Mc}{\hbar}\right]\varphi = 0\,. \tag{16}$$

Substitution for the wave function is the same as above

$$\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \gamma^2\ \mu(r) - k\gamma^3 + \Gamma\Sigma_3 - M\right]\left|\begin{array}{c} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{array}\right| = 0\,, \tag{17}$$

where we use the notation

$$\frac{m}{r} - \frac{eBr}{2} \Longrightarrow \mu(r)\,,\quad \lambda\frac{2eB}{Mc^2} \Longrightarrow \Gamma\,,\qquad \frac{Mc}{\hbar} \Longrightarrow M\,,\quad \frac{\epsilon}{\hbar c} \Longrightarrow \epsilon\,. \tag{18}$$

Further we get four radial equations

$$\begin{aligned}
&-i\left(\frac{d}{dr} + \mu\right) f_4 + (\epsilon + k)f_3 + (\Gamma - M)f_1 = 0,\\
&-i\left(\frac{d}{dr} - \mu\right) f_3 + (\epsilon - k)f_4 - (\Gamma + M)f_2 = 0,\\
&+i\left(\frac{d}{dr} + \mu\right) f_2 + (\epsilon - k)f_1 + (\Gamma - M)f_3 = 0,\\
&+i\left(\frac{d}{dr} - \mu\right) f_1 + (\epsilon + k)f_2 - (\Gamma + M)f_4 = 0.
\end{aligned}$$

Let us try to impose a linear constraint (see the case of ordinary Dirac particle): $f_3 = Af_1,\ f_4 = Af_2$, equations take the form

$$\begin{aligned}
&-i\left(\frac{d}{dr} + \mu\right) f_2 + \left[\epsilon + k + \frac{(\Gamma - M)}{A}\right] f_1 = 0\,,\\
&-i\left(\frac{d}{dr} - \mu\right) f_1 + \left[\epsilon - k - \frac{(\Gamma + M)}{A}\right] f_2 = 0\,,\\
&+i\left(\frac{d}{dr} + \mu\right) f_2 + \left[\epsilon - k + (\Gamma - M)A\right] f_1 = 0\,,\\
&+i\left(\frac{d}{dr} - \mu\right) f_1 + \left[\epsilon + k - (\Gamma + M)A\right] f_2 = 0\,.
\end{aligned} \tag{19}$$

In the system (19) equations 1 and 3, as well as equations 2 and 4 are the same, only if the ratio

$$\begin{aligned}
\epsilon + k + \frac{(\Gamma - M)}{A} &= -\left[\epsilon - k + (\Gamma - M)A\right],\\
\epsilon - k - \frac{(\Gamma + M)}{A} &= -\left[\epsilon + k - (\Gamma + M)A\right];
\end{aligned}$$

they can be rewritten as:

$$\begin{aligned}
\epsilon + \frac{(\Gamma - M)}{A} = -\epsilon - (\Gamma - M)A \quad &\Longrightarrow \quad 2\epsilon = (M - \Gamma)(A + \frac{1}{A})\,,\\
\epsilon - \frac{(\Gamma + M)}{A} = -\epsilon + (\Gamma + M)A \quad &\Longrightarrow \quad 2\epsilon = (M + \Gamma)(A + \frac{1}{A})\,.
\end{aligned} \tag{20}$$

Obviously, this system is not consistent. Thus, we are to examine the system of four equation in a different way.

4 Solving the radial equations

We turn again to the system of four equations. They can be represented in the form of two linear systems

$$
\begin{aligned}
(\Gamma - M) f_1 + (\epsilon + k) f_3 = i \left(\frac{d}{dr} + \mu \right) f_4 = i D_+ f_4 \,, \\
+(\epsilon - k) f_1 + (\Gamma - M) f_3 = -i \left(\frac{d}{dr} + \mu \right) f_2 = -i D_+ f_2 \,; \qquad (21) \\
-(\Gamma + M) f_2 + (\epsilon - k) f_4 = i \left(\frac{d}{dr} - \mu \right) f_3 = i D_- f_3 \,, \\
+(\epsilon + k) f_2 - (\Gamma + M) f_4 = -i \left(\frac{d}{dr} - \mu \right) f_1 = -i D_- f_1 \,. \qquad (22)
\end{aligned}
$$

Their solutions are as follows:

$$
f_1 = +i \frac{(\epsilon + k) D_+ f_2 + (\Gamma - M) D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}, \quad f_3 = -i \frac{(\Gamma - M) D_+ f_2 + (\epsilon - k) D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}; \qquad (23)
$$

$$
f_2 = +i \frac{(\epsilon - k) D_- f_1 - (\Gamma + M) D_- f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}, \quad f_4 = -i \frac{-(\Gamma + M) D_- f_1 + (\epsilon + k) D_- f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}. \qquad (24)
$$

We note two identities

$$
\begin{aligned}
D_+ D_- = (\frac{d}{dr} + \mu)(\frac{d}{dr} - \mu) = \frac{d^2}{dr^2} - \mu' - \mu^2 \,, \\
D_- D_+ = (\frac{d}{dr} - \mu)(\frac{d}{dr} + \mu) = \frac{d^2}{dr^2} + \mu' - \mu^2 \,. \qquad (25)
\end{aligned}
$$

Substituting expression (23) into equation (22), we get

$$
\begin{aligned}
-(\Gamma + M) f_2 + (\epsilon - k) f_4 = \frac{(\Gamma - M) D_- D_+ f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\epsilon - k) D_- D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \,, \\
+(\epsilon + k) f_2 - (\Gamma + M) f_4 = \frac{(\epsilon + k) D_- D_+ f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\Gamma - M) D_- D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \,. \qquad (26)
\end{aligned}
$$

By combining equations (26), we obtain

$$
f_2 = \frac{1}{2\Gamma\,(\epsilon + k)} \left(\Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} + \mu' - \mu^2 \right) f_4 \,, \qquad (27)
$$

and then we derive the 4-th order equation for the function f_4

$$
\begin{aligned}
-\frac{d^4 f_4}{dr^4} + \left[\frac{B^2}{2} r^2 - B\,(2\,m - 1) - 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2) + 2\,\frac{m\,(m+1)}{r^2} \right] \frac{d^2 f_4}{dr^2} + \\
+ \left[B^2 r - 4\,\frac{m\,(m+1)}{r^3} \right] \frac{df_4}{dr} + \\
+ \left[-\frac{B^4}{16} r^4 + \frac{B^2}{4} \Big(B\,(2m - 1) + 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2) \Big) r^2 - \right. \\
\left. -B\,(2m - 1)\,(\Gamma^2 - M^2 - k^2 + \epsilon^2) - (\Gamma^2 + M^2 + k^2 - \epsilon^2)^2 + 4\,\Gamma^2 M^2 - \frac{B^2}{4} \Big(6\,m^2 - 2\,m - 1 \Big) + \right.
\end{aligned}
$$

$$+\frac{m\,(m+1)\,(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2))}{r^2}-\frac{m\,(m-2)\,(m+3)\,(m+1)}{r^4}\Bigg]\,f_4=0\,.$$

(28)

Similarly, we can find the expression for f_4

$$f_4=\frac{1}{2\Gamma\,(\epsilon-k)}\left(\Gamma^2-M^2-k^2+\epsilon^2+\frac{d^2}{dr^2}+\mu'-\mu^2\right)f_2\,,\tag{29}$$

and then derive the 4-th order equation for the function f_2:

$$\frac{d^4f_2}{dr^4}+\left[-\frac{B^2}{2}\,r^2+B\,(2\,m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2)-2\,\frac{m\,(m+1)}{r^2}\right]\frac{d^2f_2}{dr^2}+$$

$$+\left[-B^2r+4\,\frac{m\,(m+1)}{r^3}\right]\frac{df_2}{dr}+$$

$$+\left[\frac{B^4}{16}\,r^4-\frac{B^2}{4}\left(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2)\right)r^2+\right.$$

$$+B\,(2m-1)\,(\Gamma^2-M^2-k^2+\epsilon^2)+(\Gamma^2+M^2+k^2-\epsilon^2)^2-4\,\Gamma^2M^2+\frac{e^2B^2}{4}\,\Big(6\,m^2-2\,m-1\Big)-$$

$$\left.-\frac{m\,(m+1)\,(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2))}{r^2}+\frac{m\,(m-2)\,(m+3)\,(m+1)}{r^4}\right]f_2=0\,.$$

(30)

We note that the equation for f_2 and f_4 are the same. Therefore it is sufficient to consider one of them. To study the arising equations (28) and (30) we will use the factorization method:

$$\hat{F}_4(r)\;f(r)=\hat{f}_2(r)\hat{g}_2(r)\;f(r)=0\,,$$

$$\hat{f}_2(r)=\frac{d^2}{dr^2}+P_0r^2+P_1+\frac{P_2}{r^2}\,,\quad \hat{g}_2(r)=\frac{d^2}{dr^2}+Q_0r^2+Q_1+\frac{Q_2}{r^2}\,.\tag{31}$$

Calculating the operator $\hat{F}_4$ and comparing with (30), we find two sets of numerical coefficients:

$$1)\qquad P_0=-\frac{1}{4}\,B^2\,,\quad P_2=-m\,(m+1)\,,$$

$$P_1=B\,\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2+2\,\Gamma\sqrt{\epsilon^2-k^2}\,,$$

$$Q_0=-\frac{1}{4}\,B^2\,,\quad Q_2=-m\,(m+1)\,,$$

$$Q_1=B\,\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2-2\,\Gamma\sqrt{\epsilon^2-k^2}\,;$$

and

$$2)\qquad P_0=-\frac{1}{4}\,B^2\,,\quad P_2=-m\,(m+1)\,,$$

$$P_1 = B\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2-2\,\Gamma\sqrt{\epsilon^2-k^2}\,,$$

$$Q_0 = -\frac{1}{4}\,B^2\,,\quad Q_2 = -m\,(m+1)\,,$$

$$Q_1 = B\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2+2\,\Gamma\sqrt{\epsilon^2-k^2}\,.$$

Thus, we are to solve two equations (they differ only in the sign of Γ)

$$\left(\frac{d^2}{dr^2}-\frac{B^2r^2}{4}+B\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2+2\,\Gamma\sqrt{\epsilon^2-k^2}-\frac{m\,(m+1)}{r^2}\right)f=0\,, \tag{32}$$

$$\left(\frac{d^2}{dr^2}-\frac{B^2r^2}{4}+B\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2-2\,\Gamma\sqrt{\epsilon^2-k^2}-\frac{m\,(m+1)}{r^2}\right)g=0\,. \tag{33}$$

Consider the first equation (32). We turn it to the variable $x = Br^2/2$:

$$x\,\frac{d^2f}{dx^2}+\frac{1}{2}\,\frac{df}{dx}+$$

$$+\left[-\frac{x}{4}+\frac{4\,\Gamma\sqrt{\epsilon^2-k^2}+(2\,m-1)\,B+2\,(\Gamma^2-M^2-k^2+\epsilon^2)}{4B}-\frac{1}{4}\,\frac{m(m+1)}{x}\right]f=0\,.$$

We will build solutions in the form:

$$f = x^a e^{bx}F\,,$$

$$x\,\frac{d^2F}{dx^2}+\left(\frac{1}{2}+2a+2bx\right)\,\frac{dF}{dx}+\left[\left(b^2-\frac{1}{4}\right)\,x+\right.$$

$$+\frac{2\,Bb(4\,a+1)+4\,\Gamma\sqrt{\epsilon^2-k^2}+(2\,m-1)\,B+2\,(\Gamma^2-M^2-k^2+\epsilon^2)}{4B}+$$

$$\left.+\frac{1}{4}\,\frac{(2a+m)\,(2a-m-1)}{x}\right]F=0\,.$$

If a, b are chosen as

$$a=-\frac{m}{2},\ \frac{1}{2}+\frac{m}{2},\quad b=-\frac{1}{2}\,, \tag{34}$$

the equation is simplified

$$x\,\frac{d^2F}{dx^2}+\left(\frac{1}{2}+2a-x\right)\,\frac{dF}{dx}-$$

$$-\frac{B(4\,a+1)-4\,\Gamma\sqrt{\epsilon^2-k^2}-(2\,m-1)\,B-2\,(\Gamma^2-M^2-k^2+\epsilon^2)}{4B}\,F=0\,,$$

and it is the equation for a confluent hypergeometric function with parameters:

$$\alpha=\frac{B(4\,a+1)-4\,\Gamma\sqrt{\epsilon^2-k^2}-(2\,m-1)\,eB-2\,(\Gamma^2-M^2-k^2+\epsilon^2)}{4B}\,,\ \gamma=\frac{1}{2}+2a\,. \tag{35}$$

To get solutions that meet the bound states, we should use the positive value of the parameter a and the negative values of the parameter b (for definiteness assume that $B > 0$):

$$a = -\frac{m}{2}\,, \quad (m < 0)\,; \qquad a = \frac{m}{2} + \frac{1}{2} > 0 \quad (m \geq 0)\,. \tag{36}$$

Conditions of terminating the hypergeometric series to polynomials $\alpha = -n$ (introduce the notation $\epsilon^2 - k^2 = \lambda$):

$$\frac{B(4\,a+1) - 4\,\Gamma\sqrt{\lambda} - (2\,m-1)\,B - 2\,(\Gamma^2 - M^2) - 2\lambda}{4B} = -n \tag{37}$$

gives quantization rule for the energy values:

$$a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} + n = \frac{\Gamma\sqrt{\lambda}}{B} + \frac{\lambda}{2B}\,,$$

we obtain

$$(\sqrt{\lambda} + \Gamma)^2 = N,\ N = M^2 + 2B(a + \frac{1}{2} - \frac{m}{2} + n) \implies \lambda = (\sqrt{N} - \Gamma)^2 > 0\,. \tag{38}$$

From (38) we find the formula for the allowed values of λ

$$\epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(a + \frac{1}{2} - \frac{m}{2} + n)} - \Gamma\right)^2. \tag{39}$$

Depending on the sign of m, we obtain two formulas:

$$m < 0, \quad a = -\frac{m}{2}\,, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(\frac{1}{2} - m + n)} - \Gamma\right)^2; \tag{40}$$

$$m \geq 0, \quad a = \frac{m}{2} + \frac{1}{2}\,, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(1+n)} - \Gamma\right)^2. \tag{41}$$

This, we have two possibilities for quantization

$$I, \quad \lambda = (\sqrt{N} - \Gamma)^2\,, \qquad II, \quad \lambda = (\sqrt{N} + \Gamma)^2\,. \tag{42}$$

that is the particle with anomalous magnetic moment has two series of energy levels, formally differing in sign of the parameter Γ.

5 Further analysis of solutions

Let us consider the function $f_2(r)$ as the primary. Obtained above ratios allow us to calculate other three functions. We start from the explicit form of the function f_2, the solution of equation (32):

$$f_2 = x^a e^{-x/2} F(\alpha, \gamma, x)\,, \tag{43}$$

where the parameters are given by

$$\alpha = a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = \frac{1}{2} + 2a\,. \tag{44}$$

The function f_4 can be found according to the following relationship:

$$f_4 = \frac{1}{2\Gamma\,(\epsilon - k)} \left(\Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} - \frac{m(m+1)}{r^2} + \frac{B(2m-1)}{2} - \frac{B^2}{4} r^2 \right) f_2 \,;$$

from where after translating to the variable x we find

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x \frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + \frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} - \frac{m(m+1)}{4x} - \frac{1}{4} x \right) f_2 \,. \qquad (45)$$

Given the identities

$$+\frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} = n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B}$$

the above relation can be written as

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x \frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{1}{4} x \right) f_2 \,. \qquad (46)$$

Depending on the values of the parameter m, we have two different cases:

$$A) \quad m < 0, \qquad a = -\frac{m}{2}, \quad f_2 = x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x) \,,$$

$$\alpha = m + \frac{1}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = -m + \frac{1}{2} \,,$$

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x \frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n - \frac{m}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2 \,,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(\frac{1}{2} - m + n)} - \Gamma \,; \qquad (47)$$

$$B) \quad m \geq 0, \qquad a = \frac{m}{2} + \frac{1}{2}, \quad f_2 = x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x) \,,$$

$$\alpha = 1 + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = m + \frac{3}{2} \,,$$

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x \frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n + \frac{m+1}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2 \,,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(1+n)} - \Gamma \,. \qquad (48)$$

Using the relations (47) and (48) we find the explicit expression for the function f_4:
for the variant A,

$$m < 0, \qquad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k}\, x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k}\, f_2(x) \,, \qquad (49)$$

for the variant B,

$$m \geq 0, \qquad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k}\, x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k} f_2(x) \,. \qquad (50)$$

Changing in these formulas the parameter Γ on $-\Gamma$, we obtain the relations describing the second series of states. It is not difficult to calculate the explicit form of the other two functions f_1, f_3; corresponding calculated are made, we will not detail on this.

References

[1] V.M. Red'kov Fields of particles in Riemannian space and the Lorentz group. – Belarusian Science: Minsk, 2009. – 486 pages.

[2] V.V. Kisel, N.G. Tokarevskaya, V.M Red'kov. Petrash theory for particles with spin 1/2 in curved space-time. Preprint 737. – Institute of Physics of NAS of Belarus: Minsk, 2002.

[3] A.A. Bogush, V.V. Kisel, N.G. Tokarevskaya, V.M. Red'kov Petrasha theory for particles with spin 1/2 in curved space-time // Vesti of NASB. Ser. fiz.-mat. nauk. 2002. no 1. P. 63–68.

Hydrogen atom in de Sitter spaces

E.M. Ovsiyuk
e.ovsiyuk@mail.ru; Mozyr State Pedagogical University, Belarus
K.V. Dashuk
kristinash2@mail.ru; Mozyr State Pedagogical University
O.V. Veko
vekoolga@mail.ru; Gymnasium, Kalinkovichi, Belarus
V.M. Red'kov
redkov@dragon.bas-net.by; B.I. Stepanov Institute of Physics
A.M. Ishkhanyan
aishkhanyan@gmail.com; Institute for Physical Research of NAS, Armenia

Abstract

The hydrogen atom theory is developed for the de Sitter and anti de Sitter spaces on the basis of the Klein-Gordon-Fock wave equation in static coordinates. In both models, after separation of the variables, the problem is reduced to the general Heun equation, a second order linear differential equation having four regular singular points. A qualitative examination shows that the energy spectrum for the hydrogen atom in the de Sitter space should be quasi-stationary, and the atom should be unstable. We derive an approximate expression for energy levels within the quasi-classical approach and estimate the probability of decay of the atom. A similar analysis shows that in the anti de Sitter model the hydrogen atom should be stable in the quantum-mechanical sense. Using the quasi-classical approach, we derive approximate formulas for energy levels for this case as well. Finally, we present the extension to the case of a spin 1/2 particle for both de Sitter models. This extension leads to complicated differential equations with 8 singular points.

1 Introduction

Starting from the pioneering work by Dirac [1], a steady attention is devoted to the de Sitter geometrical models in the context of quantum theory for curved space-time (see, for instance, [2, 3]). In particular, the problem of description of the particles with different spins on such curved backgrounds has a long history (see [1]–[42]). The most of the work done, however, concerns the field theory on the background of such geometrical models. In the present paper, we address the quantum mechanics and examine the influence of the de Sitter geometries on the behavior of the hydrogen atom. Since for the de Sitter spaces, when they are parameterized by static coordinates, no nonrelativistic Schrödiner equation is known, we have to employ the relativistic quantum mechanical approach, the Klein-Gordon-Fock scalar equation and the Dirac spinor equation. We consider these equations for both de Sitter (dS) and anti de Sitter (AdS) geometries.

In dS space, we first analyze the expression for the classical squared radial momentum $p_r^2(r)$. We show that the equation $p_r^2 = 0$ may have three positive and one negative roots, which correspond to a particle moving in a potential well with a domain forbidden for classical motion and a domain permitted for classical motion. In other words, in the de Sitter geometry the hydrogen atom should be unstable in the quantum-mechanical sense. The corresponding radial

equation is reduced to the general Heun equation having four regular singular points [45], [46]. Studying the solutions of this equation, we are led to a fourth order polynomial equation the roots of which allow approximate calculation of the energy levels using the WKB-method. The probability of decay of the atom is then estimated.

A similar treatment is presented for the case of the anti de Sitter model. The classical equation $p_r^2 = 0$ is again reduced to a fourth order polynomial equation, which, however, this time may have two positive and two negative roots. This configuration corresponds to a particle moving in a perfect potential well without tunneling. In other words, this means that in the AdS-model the hydrogen atom is a stable quantum-mechanical system. The corresponding radial equation is again reduced to the general Heun equation having four singular points. Discussing the solutions of this equation, we again consider a relevant fourth order polynomial equation, the roots of which provide approximate expressions for energy levels using the WKB-method.

When treating the hydrogen atom using the Dirac equation, the radial differential equations arising after separation of the variables are much more complicated compared with the ones given by the scalar theory. Here, the problem is reduced to a differential equation with 8 singular points (note that the Dirac equation with Coulomb potential in hyperbolic Lobachevsky and spherical Riemann models is reduced to differential equations with 5 singular points [47]). However, the physical situation here is expected to be qualitatively similar to that arising in scalar models, because the influence of any special space-time geometry is universal and, in principle, it should not depend on the spin of a quantum-mechanical particle. The dependence on the value of the spin should reveal only technical differences.

2 Separation of the variables in dS-space

Electric field of a point charge in static coordinates of dS-model is

$$dS^2 = (1 - \frac{r^2}{\rho^2})dt^2 - (1 - \frac{r^2}{\rho^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad A_0 = \frac{e}{r}. \tag{1}$$

from the scalar eq., with the substitution $\Phi = e^{-i\epsilon t/\hbar}Y_{lm}(\theta\phi)f(r)$ we derive

$$\frac{d^2}{dr^2}\, f \;+\; \frac{2(1 - 2r^2/\rho^2)}{r(1 - r^2/\rho^2)}\, \frac{d}{dr}\, f \;+$$

$$+ \left[\frac{(\epsilon + e^2/r)^2}{c^2\hbar^2}\frac{1}{(1 - r^2/\rho^2)^2} - (\frac{M^2c^2}{\hbar^2} + \frac{l(l+1)}{r^2})\frac{1}{1 - r^2/\rho^2}\right] f = 0. \tag{2}$$

The squared momentum is

$$p_r^2 = \frac{(\epsilon + e^2/r)^2}{c^2}\frac{1}{(1 - r^2/\rho^2)^2} - (M^2c^2 + \frac{L^2}{r^2})\frac{1}{1 - r^2/\rho^2}. \tag{3}$$

3 Qualitative treatment

In Minkowski space, the momentum

$$p_r^2 = \frac{(\epsilon + e^2/r)^2}{c^2} - (M^2c^2 + \frac{L^2}{r^2})\,. \tag{4}$$

two positive turning points of p_r (2-nd order polynomial) exist. When $(L^2 - e^4/c^2) > 0$ and $\epsilon < Mc^2$, The graph of p_r^2 schematically looks as shown below

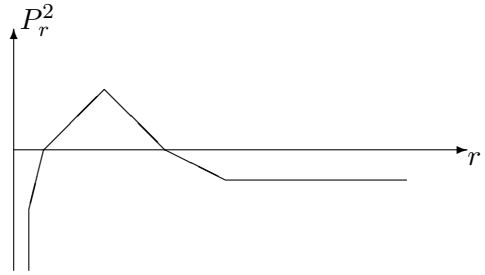


Fig. 1. Coulomb problem in Minkowski space

In de Sitter space

$$p_r^2 = \frac{1}{r^2(1-\frac{r^2}{\rho^2})^2}\left[\frac{M^2c^2}{\rho^2}r^4 + (\frac{\epsilon^2}{c^2} - M^2c^2 + \frac{L^2}{\rho^2})r^2 + \frac{2e^2\epsilon}{c^2}r + (\frac{e^4}{c^2} - L^2)\right] \tag{5}$$

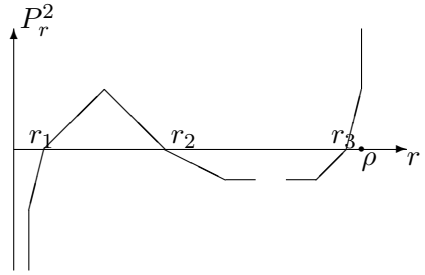


Fig. 2. Coulomb problem in dS space

One should perform the following change of the variable

$$r \Longrightarrow r^* \ , \ \frac{d}{dr} = \frac{1}{1-\frac{r^2}{\rho^2}}\frac{d}{dr^*} \ , \quad r^* = \frac{\rho}{2}\ln\frac{1+r/\rho}{1-r/\rho} \in [0,\infty)$$

The graph for $p_{r^*}^2$ looks (in the coordinate r^*)

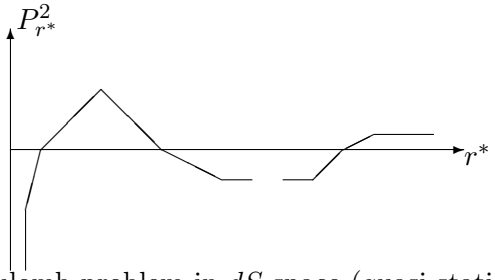


Fig. 3. Coulomb problem in dS space (quasi-stationary states)

4 The turning points

It will be more convenient to use dimensionless quantities

$$\frac{p_r^2}{M^2c^2} \Longrightarrow p^2 \ , \quad \frac{r}{\rho} \Longrightarrow r \ , \quad \frac{L^2}{M^2c^2\rho^2} \Longrightarrow L^2 \ , \quad \frac{e^2}{\rho}\frac{1}{Mc^2} = q \ .$$

Physical domain for the variable r is the interval $r \in [0,+1)$. From eq. $p^2 = 0$ we get a 4-order equation

$$r^4 + Ar^2 + Br + C = 0 \ ,$$

$$A = \epsilon^2 + L^2 - 1\,, \qquad B = +2\epsilon q\,, \quad C = -(L^2 - q^2) < 0\,. \tag{6}$$

The explicit form of the roots is practically useless. Instead let us examine qualitatively the possible location of these roots. Starting with the identity

$$r^4 + Ar^2 + Br + C = [\,(r - r_1)(r - r_2]\,[\,(r - r_3)(r - r_4)\,]$$

we get

$$\begin{gathered} r_1 + r_2 + r_3 + r_4 = 0\,, \\ A = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4\,, \\ B = -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)\,, \quad C = r_1r_2r_3r_4 \end{gathered} \tag{7}$$

where

$$A = \epsilon^2 + L^2 - 1\,, \quad B = +2\epsilon q > 0\,, \quad C = -(L^2 - q^2) < 0\,.$$

First, we conclude that three next variants:

$$(r_1, r_2, r_3, r_4) \sim (-,-,+,+)\,, \quad (r_1, r_2, r_3, r_4) \sim (-,-,-,-)\,, \quad (r_1, r_2, r_3, r_4) \sim (+,+,+,+)\,,$$

cannot be realized. Assuming two real-valued roots, and two conjugate ones, we get the variant $(r_1, r_2, r_3, r_4) \sim (+,-,z,z^*)$, it is not of physical interest. Relation $r_1r_2r_3r_4 = C < 0$ says, that there are possible only two variants:

$$(r_1, r_2, r_3, r_4) \sim (-,-,-,+)\,, \quad (r_1, r_2, r_3, r_4) \sim (+,+,+,-)\,; \tag{8}$$

The most interesting physically is the second variant in (8) which corresponds to the graph in Fig. 3.

5 Reducing the radial equation to the Heun type

In dimensionless quantities

$$x = \frac{r}{\rho}\,, \qquad \frac{\epsilon\rho}{c\hbar} = E\,, \qquad \frac{e^2}{c\hbar} = \alpha\,, \qquad \frac{M^2c^2\rho^2}{\hbar^2} \Longrightarrow M^2$$

radial equation takes the form

$$\left[\frac{d^2}{dx^2} + \frac{2(1-2x^2)}{x(1-x^2)}\frac{d}{dx} + (E + \frac{\alpha}{x})^2\frac{1}{(1-x^2)^2} - (M^2 + \frac{l(l+1)}{x^2})\frac{1}{1-x^2}\right] f = 0\,. \tag{9}$$

Making the substitution $f = x^A\,(1-x)^B\,(1+x)^C\,F$, at the following A, B, C

$$A = -\frac{1}{2} \pm \sqrt{(l+1/2)^2 - \alpha^2}\,,\; B = \pm\frac{i}{2}\,(E+\alpha)\,,\; C = \pm\frac{i}{2}\,(E-\alpha)$$

we arrive at general Heun equation for $G(-1; q, \lambda, \beta, \gamma, \delta; z)$

$$\frac{d^2G}{dx^2} + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x+1}\right]\frac{dG}{dx} + \frac{\lambda\beta x - q}{x(x-1)(x+1)}\,G = 0, \quad \epsilon = \lambda + \beta - \gamma - \delta + 1 \tag{10}$$

with relations determining its parameters

$$\gamma = 2(A+1)\,, \qquad \delta = 2B + 1\,, \quad q = 2\,(E\alpha - B + C - AB + AC)\,,$$

$$\lambda,\beta = \frac{3}{2} + A + B + C \pm \sqrt{B^2 + C^2 + \frac{E^2+\alpha^2}{2} - M^2 + \frac{9}{4}}\,. \tag{11}$$

To have solutions vanishing in the origin $r = 0$, one takes $A = -1/2 + \sqrt{(l+1/2)^2 - \alpha^2}$. Depending on the choice for B and C, we have four possibilities:

$$\begin{aligned}
(+,+)\ q &= 2\alpha[E - i(A+1)], \quad \lambda,\beta = \frac{1}{2} + A + 1 + iE \pm i\sqrt{M^2 - \frac{9}{4}}\,,\\
(-,-)\ q &= 2\alpha[E + i(A+1)]\,, \quad \lambda,\beta = \frac{1}{2} + A + 1 - iE \pm i\sqrt{M^2 - \frac{9}{4}}\,,\\
(+,-)\ q &= 2E[\alpha - i(A+1)], \quad \lambda,\beta = \frac{1}{2} + A + 1 + i\alpha \pm i\sqrt{M^2 - \frac{9}{4}}\,,\\
(-,+) \qquad q &= 2E[\,\alpha + i(A+1)]\,, \quad \lambda,\beta = \frac{1}{2} + A + 1 - i\alpha \pm i\sqrt{M^2 - \frac{9}{4}}\,.
\end{aligned} \tag{12}$$

6 Approximate roots r_i of equation $p_r^2 = 0$

Approximate expression for r_1, r_2 are

$$r_i = r_{0i} + \Delta_i/\rho^2\,, \qquad i = 1,2\,,$$

$$r_{01} = \frac{c^2}{m^2c^4 - \epsilon^2}\left[\frac{e^2\epsilon}{c^2} + m\sqrt{\frac{e^4\epsilon^2}{c^4} - (L^2 - \frac{e^4}{c^2})\frac{M^2c^4 - \epsilon^2}{c^2}}\right] > 0\,,$$

$$r_{02} = \frac{c^2}{m^2c^4 - \epsilon^2}\left[\frac{e^2\epsilon}{c^2} - \sqrt{\frac{e^4\epsilon^2}{c^4} - (L^2 - \frac{e^4}{c^2})\frac{M^2c^4 - \epsilon^2}{c^2}}\right] > 0\,,$$

$$\Delta_i = -r_{0i}^2\,(L^2 + M^2c^2r_{0i}^2)\left[\frac{2e^2\epsilon}{c^2} + 2r_{0i}(\frac{\epsilon^2}{c^2} - M^2c^2)\right]^{-1}. \tag{13}$$

approximate expression for $r_{3,4}$ are

$$r_3 = \rho(b_{03} + \frac{b_1}{\rho})\,, \qquad r_4 = \rho(b_{04} + \frac{b_1}{\rho})\,,$$

$$b_{03} = +\sqrt{1 - \frac{\epsilon^2}{m^2c^4}} > 0\,, \quad b_{04} = -\sqrt{1 - \frac{\epsilon^2}{m^2c^4}} < 0\,, \quad b_1 = -\frac{e^2\epsilon}{M^2c^4 - \epsilon^2} \tag{14}$$

At the large curvature radius ρ, the domain between r_1 and r_2 lays far from $r_3 > 0$ and $r_4 < 0$.

7 Semi-classical analysis

Let us study the problem within semi-classical method. We obtain approximate form for p_r in the domain $r << \rho$ where the potential well is located:

$$P_r \sim mc\frac{\sqrt{(r-r_1)((r-r_2)}}{r}(\frac{\epsilon^2}{m^2c^4} - 1)^{1/2}\left(1 + A\,\frac{r^2}{\rho^2}\right), \quad A = 1 - \frac{1}{2(1-\epsilon^2/m^2c^4)}\,.$$

Bohr-Zommerfeld quantization rule is

$$\int_L P_r dr = 2\pi\hbar(n + 1/2)\,.$$

From this, choosing the path of integration surrounding the turning points r_1 and r_2, we reduce the problem to calculating residues in two points

$$\sum_{0,\infty} \text{res } P_r = i\hbar(n+1/2) \; .$$

Taking into account residues in 0 and $r=\infty$

$$\text{res}_{r=0} P_r = \sqrt{(\frac{\epsilon^2}{c^2} - m^2c^2) r_1 r_2} \; , \quad \text{res}_{r=\infty} P_r = \frac{1}{2}\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2}\left[(r_1+r_2) - \frac{A}{\rho^2} r_1 r_2 (r_1+r_2)\right].$$

we arrive at the formula

$$\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2}\left[(\sqrt{r_1 r_2} + \frac{r_1+r_2}{2}) - \frac{A}{\rho^2} r_1 r_2 \frac{r_1+r_2}{2}\right] = i\hbar(n+1/2). \tag{15}$$

From this it follows an approximate formula for energy levels in dS-space

$$\epsilon = \epsilon_0 + \frac{\Delta}{\rho^2}, \quad \epsilon_0 = mc^2\left[1 + \frac{e^4/c^2}{[\hbar(n+1/2) + \sqrt{\hbar^2(l+1/2)^2 - e^4/c^2}]^2}\right]^{-1/2},$$

$$\Delta = \frac{(m^2c^4 - \epsilon_0^2)^2}{4e^2m^2c^4}\left[\Delta_1 + \Delta_2 + \Delta_1\sqrt{\frac{r_{02}}{r_{01}}} + \Delta_2\sqrt{\frac{r_{01}}{r_{02}}} - A r_{01} r_{02}(r_{01}+r_{02})\right]. \tag{16}$$

In accordance with behavior of the function P_r^2, the particle can go from the domain between r_1 and r_2 to the domain of horizon $r \sim \rho$ due to the quantum-mechanical tunneling effect. Probability of this process is given by the formula

$$W = \exp\left(-\frac{2}{\hbar}\int_{r_2}^{r_3} \sqrt{-P_r^2}\, dr\right), \tag{17}$$

from whence we can obtain a rough estimation $W \sim e^{-2\rho/\lambda}$, where λ stands for the Compton wave length.

8 The hydrogen atom in AdS space

Now, let us consider the Coulomb problem in AdS space

$$dS^2 = (1+\frac{r^2}{\rho^2})dt^2 - (1+\frac{r^2}{\rho^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r \in [0, +\infty) \tag{18}$$

radial equation is

$$\frac{d^2}{dr^2} f \; + \; \frac{2(1+2r^2/\rho^2)}{r(1+r^2/\rho^2)} \; \frac{d}{dr} f \; +$$

$$+ \left[\frac{(\epsilon + e^2/r)^2}{c^2\hbar^2}\frac{1}{(1+r^2/\rho^2)^2} - \left(\frac{M^2c^2}{\hbar^2} + \frac{l(l+1)}{r^2}\right)\frac{1}{1+r^2/\rho^2}\right] f = 0 \; . \tag{19}$$

It is convenient to use dimensionless quantities

$$\frac{p_r^2}{M^2c^2} \Longrightarrow p^2, \; \frac{r}{\rho} \Longrightarrow r, \; \frac{L^2}{M^2c^2\rho^2} \Longrightarrow L^2, \frac{e^2}{\rho}\frac{1}{Mc^2} = q \; ; ,$$

$$p^2(r) = \left(\epsilon + \frac{q}{r}\right)^2 \frac{1}{(1+r^2)^2} - \left(1 + \frac{L^2}{r^2}\right) \frac{1}{1+r^2} \, .$$

In the origin and the infinity behavior is as shown

$$r \to 0 \, , \quad p^2(r) \sim -\frac{L^2 - q^2}{r^2} < 0 \, , \quad r \to +\infty \, , \quad p^2(r) = -\frac{1}{r^2} < 0 \, ;$$

therefore both these domain are forbidden for classical motion. Let us examine the location of the turning points

$$r^4 - r^2(\epsilon^2 - L^2 - 1) - 2\epsilon q \, r + (L^2 - q^2) = 0$$

or

$$r^4 + Ar^2 + Br + C = (r - r_1)(r - r_2 \;\; (r - r_3)(r - r_4),$$

$$r_1 + r_2 + r_3 + r_4 = 0 \, , \quad A = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 \, ,$$

$$B = -(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4) \, , \quad C = r_1 r_2 r_3 r_4 > 0 \, . \tag{20}$$

The relation $r_1 r_2 r_3 r_4 = C > 0$ says that variants below are forbidden:

$$(r_1, r_2, r_3, r_4) \sim (-, -, -, +) \, , \qquad (r_1, r_2, r_3, r_4) \sim (+, +, +, -) \, .$$

the following variants are not possible as well:

$$(r_1, r_2, r_3, r_4) \sim (-, -, -, -) \, , \qquad (r_1, r_2, r_3, r_4) \sim (+, +, +, +) \, ,$$

due to the identity $r_1 + r_2 + r_3 + r_4 = 0$. So we conclude that there exist only two possibilities of the roots location:

$$(r_1, r_2, r_3, r_4) \sim (-, -, +, +) \, , \qquad r_4 > r_3 > 0 \, ;$$

$$(r_1, r_2, r_3, r_4) \sim (z, z^* +, +) \, , \qquad r_4 > r_3 > 0 \, ;) \tag{21}$$

where r_4, r_3 stand for physical turning points.

Thus, qualitative study shows that in AdS space we can expect for the Coulomb problem the situation of a complicated potential well with normal (real) energy spectrum, so the hydrogen atom seems to be a stable system in the sense of the quantum-mechanical sense.

9 The hydrogen atom in AdS space and the Heun functions

In dimensionless form

$$x = \frac{ir}{\rho} \, , \qquad \frac{\epsilon\rho}{c\hbar} = E \, , \qquad \frac{e^2}{c\hbar} = \alpha \, , \qquad \frac{M^2 c^2 \rho^2}{\hbar^2} \Longrightarrow M^2 \, ,$$

the radial equation will read

$$\frac{d^2 f}{dx^2} + \left[\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1} \right] \frac{df}{dx} + \left[\frac{\alpha^2 - l(l+1)}{x^2} - \frac{1}{4} \frac{(E + i\alpha)^2}{(x-1)^2} - \frac{1}{4} \frac{(E - i\alpha)^2}{(x+1)^2} - \right.$$

$$\left. - \frac{2Ei\alpha}{x} + \frac{1}{4} \frac{(E + 2i\alpha)^2 + \alpha^2 - 2[M^2 - l(l+1)]}{x-1} + \frac{1}{4} \frac{-(E - 2i\alpha)^2 - \alpha^2 + 2[M^2 - l(l+1)]}{x+1} \right] f = 0 \, .$$

With the substitution $f = x^A \, (x-1)^B \, (x+1)^C \, F$, at

$$A = -\frac{1}{2} \pm \sqrt{(l + 1/2)^2 - \alpha^2} \, , B = \pm \frac{1}{2} \, (E + i\alpha) \, , C = \pm \frac{1}{2} \, (E - i\alpha) \tag{22}$$

the function F obeys the generale Heun equation

$$\frac{d^2G}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\epsilon}{x+1}\right)\frac{dG}{dx}+\frac{\lambda\beta x-q}{x(x-1)(x+1)}G=0;$$

$$\gamma=2A+2\,,\qquad \delta=2B+1\,,\quad q=-2\left(iE\alpha+B-C+AB-AC\right),$$

$$\lambda,\beta=\frac{3}{2}+A+B+C\pm\sqrt{B^2+C^2+\frac{\alpha^2-E^2}{2}+M^2+\frac{9}{4}}\,. \tag{23}$$

To have solutions finite at the infinity, we should use the negative value of A:

$$A=-\frac{1}{2}+\sqrt{(l+1/2)^2-\alpha^2}>0\;;.$$

Depending on the signs in B and C we have four different possibilities:

$$(+,+)\quad q=-2i\alpha\,[\,E+(A+1)\,]\,,\quad \lambda,\beta=\frac{3}{2}+A+E\pm\sqrt{M^2+\frac{9}{4}}\,;$$

$$(-,-)\quad q=-2i\alpha\,[\,E-(A+1)\,]\,,\quad \lambda,\beta=\frac{3}{2}+A-E\pm\sqrt{M^2+\frac{9}{4}}\,;$$

$$(+,-)\quad q=-2E\,[\,i\alpha+(A+1)\,]\,,\quad \lambda,\beta=\frac{3}{2}+A+i\alpha\pm\sqrt{M^2+\frac{9}{4}}\,;$$

$$(-,+)\quad q=-2E\,[\,i\alpha-(A+1)\,]\,,\lambda,\beta=\frac{3}{2}+A-i\alpha\pm\sqrt{M^2+\frac{9}{4}}\,. \tag{24}$$

10. Semi-classical consideration. Let us make use of formal method of analytical continuation

$$dS\Longrightarrow AdS\qquad \rho\Longrightarrow i\rho$$

Approximate roots of $P^2(r)=0$:

$$r_i=r_{0i}-\Delta_i/\rho^2\,,\qquad i=1,2\,,$$

$$r_{01}=\frac{c^2}{m^2c^4-\epsilon^2}\left[\frac{e^2\epsilon}{c^2}+m\sqrt{\frac{e^4\epsilon^2}{c^4}-(L^2-\frac{e^4}{c^2})\frac{M^2c^4-\epsilon^2}{c^2}}\right]>0\,,$$

$$r_{02}=\frac{c^2}{m^2c^4-\epsilon^2}\left[\frac{e^2\epsilon}{c^2}-\sqrt{\frac{e^4\epsilon^2}{c^4}-(L^2-\frac{e^4}{c^2})\frac{M^2c^4-\epsilon^2}{c^2}}\right]>0.$$

$$\Delta_i=-r_{0i}^2\,(L^2+M^2c^2r_{0i}^2)\,\left[\frac{2e^2\epsilon}{c^2}+2r_{0i}(\frac{\epsilon^2}{c^2}-M^2c^2)\right]^{-1}. \tag{25}$$

Approximate roots $r_{3,4}$

$$r_3=i\rho(b_{03}+\frac{b_1}{i\rho})\,,\qquad r_4=i\rho(b_{04}+\frac{b_1}{i\rho})\,,$$

$$b_{03}=+\sqrt{1-\frac{\epsilon^2}{m^2c^4}}>0\,,\quad b_{04}=-\sqrt{1-\frac{\epsilon^2}{m^2c^4}}<0\,,\quad b_1=-\frac{e^2\epsilon}{M^2c^4-\epsilon^2}\,. \tag{26}$$

Approximate form of P_r in the region $r<<\rho$ (where the potential well is located) is

$$P_r\sim mc\frac{\sqrt{(r-r_1)((r-r_2)}}{r}(\frac{\epsilon^2}{m^2c^4}-1)^{1/2}\left(1-A\,\frac{r^2}{\rho^2}\right),\quad A=1-\frac{1}{2(1-\epsilon^2/m^2c^4)}\,.$$

Quantization condition has the form:

$$\int_L P_r dr = 2\pi\hbar(n+1/2) \quad \longrightarrow \quad \sum_{0,\infty} \text{res}\ P_r = i\hbar(n+1/2)\ .$$

Allowing for these two residues

$$\text{res}_{r=0}\ P_r = \sqrt{(\frac{\epsilon^2}{c^2} - m^2c^2)r_1r_2}\ .$$

$$\text{res}_{r=\infty} P_r = \frac{1}{2}\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2}\ \left[\ (r_1+r_2) + \frac{A}{\rho^2}\ r_1r_2(r_1+r_2)\ \right]\ .$$

we arrive at

$$\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2}\left[(\sqrt{r_1r_2} + \frac{r_1+r_2}{2}) + \frac{A}{\rho^2}r_1r_2\frac{r_1+r_2}{2}\right] = i\hbar(n+1/2). \tag{27}$$

From whence it follows an approximate energy spectrum in AdS space:

$$\epsilon = \epsilon_0 - \frac{\Delta}{\rho^2}, \quad \epsilon_0 = mc^2\left[1 + \frac{e^4/c^2}{[\hbar(n+1/2) + \sqrt{\hbar^2(l+1/2)^2 - e^4/c^2}]^2}\right]^{-1/2},$$

$$\Delta = \frac{(m^2c^4 - \epsilon_0^2)^2}{4e^2m^2c^4}\left[\Delta_1 + \Delta_2 + \Delta_1\sqrt{\frac{r_{02}}{r_{01}}} + \Delta_2\sqrt{\frac{r_{01}}{r_{02}}} + Ar_{01}r_{02}(r_{01}+r_{02})\right]. \tag{28}$$

10 Spin 1/2 particle in the Coulomb field

Let start with the Dirac free wave equation in dS space

$$\left(i\ \frac{\gamma^0}{\sqrt{1-r^2}}\partial_t + i\sqrt{1-r^2}\gamma^3\partial_r + \frac{1}{r}\Sigma_{\theta,\phi} - M\right)\Psi(x) = 0\ , \quad \Psi_{\epsilon jm}(x) = \frac{e^{-i\epsilon t}}{r}\ \left|\begin{array}{c} f_1(r)D_{-1/2} \\ f_2(r)D_{+1/2} \\ f_3(r)D_{-1/2} \\ f_4(r)D_{+1/2}\end{array}\right|;$$

where $D_\sigma = D^j_{-m,\sigma}(\phi,\theta,0)$ stands for Wigner functions [48]. To simplify the system, let us diagonalize the space reflection operator:

$$\hat{\Pi}_{sph.}\ \Psi_{jm} = \ \Pi\ \Psi_{jm}, \quad \Pi = \delta\ (-1)^{j+1},\ \delta = \pm 1, \quad \Psi(x)_{\epsilon jm\delta}\ =\ \frac{e^{-i\epsilon t}}{r}\ \left|\begin{array}{r} f_1(r)\ D_{-1/2} \\ f_2(r)\ D_{+1/2} \\ \delta\ f_2(r)\ D_{-1/2} \\ \delta\ f_1(r)\ D_{+1/2}\end{array}\right|.$$

For definiteness, let us consider the case $\delta = +1$

$$\left(\sqrt{\Phi}\frac{d}{dr} + \frac{\nu}{r}\right)f + \left(\frac{\epsilon}{\sqrt{\Phi}} + M\right)g = 0\ , \quad \left(\sqrt{\Phi}\frac{d}{dr} - \frac{\nu}{r}\right)g - \left(\frac{\epsilon}{\sqrt{\Phi}} - M\right)f = 0\ ,$$

where new functions $f = (f_1+f_2)/\sqrt{2}$, $g = (f_1-f_2)/i\sqrt{2}$ are used instead of f_1 and f_2. (formally the second case $\delta = -1$ corresponds to the change $M \Longrightarrow -M$): To take into account the presence of external Coulomb field it suffices to make one formal change $\epsilon \implies (\epsilon + e^2/r)$:

$$\left(\sqrt{1-r^2}\frac{d}{dr} + \frac{\nu}{r}\right)f + \left(\frac{1}{\sqrt{1-r^2}}(\epsilon + \frac{e^2}{r}) + M\right)g = 0\ ,$$

$$\left(\sqrt{1-r^2}\frac{d}{dr}-\frac{\nu}{r}\right)g-\left(\frac{1}{\sqrt{1-r^2}}(\epsilon+\frac{e^2}{r})-M\right)f=0\,. \tag{29}$$

In the variable $r=\sin\rho$, introducing two new functions

$$f+g=e^{-i\rho/2}(F+G)\,,\qquad f-g=e^{+i\rho/2}(F-G)\,, \tag{30}$$

one transforms eqs. into

$$\left(\frac{d}{d\rho}+\nu\frac{\cos\rho}{\sin\rho}-i\,\frac{\sin\rho}{\cos\rho}(\epsilon+\frac{e^2}{\sin\rho})\right)F+\left(\epsilon+\frac{e^2}{\sin\rho}+M-i\nu-\frac{i}{2}\right)G=0\,,$$

$$\left(\frac{d}{d\rho}-\nu\frac{\cos\rho}{\sin\rho}+i\,\frac{\sin\rho}{\cos\rho}(\epsilon+\frac{e^2}{\sin\rho})\right)G+\left(-\epsilon-\frac{e^2}{\sin\rho}+M+i\nu-\frac{i}{2}\right)F=0\,. \tag{31}$$

To have coefficients as rational functions, one should use a new variable (we will use one from two possible):

$$y=\tan\frac{\rho}{2}\,,\quad r=\sin\rho=\frac{2y}{1+y^2},\qquad yY=1\,,$$

$$y=\frac{1+\sqrt{1-r^2}}{r},\ Y=\frac{1-\sqrt{1-r^2}}{r},\quad y\in(1,+\infty),\ \ Y\in(0,1)\,; \tag{32}$$

so we derive

$$\left[(1+y^2)\,\frac{d}{dy}-\nu y+\frac{\nu}{y}-\frac{i(2\epsilon+2e^2)}{1-y}+\frac{i(2\epsilon-2e^2)}{1+y}+i2e^2\right]F+$$

$$+\left(e^2y+\frac{e^2}{y}+2\epsilon+2M-2i\nu-i\right)G=0\,,$$

$$\left[(1+y^2)\,\frac{d}{dy}+\nu\,y-\frac{\nu}{y}+\frac{i(2\epsilon+2e^2)}{1-y}-\frac{i(2\epsilon-2e^2)}{1+y}-i2e^2\right]G-$$

$$-\left(e^2y+\frac{e^2}{y}+2\epsilon-2M-2i\nu+i\right)F=0\,. \tag{33}$$

With notation (note that $y_1,y_2,y_3,y_4\in C$)

$$\left(e^2y+\frac{e^2}{y}+2\epsilon+2M-2i\nu-i\right)=\frac{e^2}{y}(y-y_1)(y-y_2)\,,$$

$$\left(e^2y+\frac{e^2}{y}+2\epsilon-2M-2i\nu+i\right)=\frac{e^2}{y}(y-y_3)(y-y_4)\,, \tag{34}$$

the system takes the form

$$\left[(1+y^2)\,\frac{d}{dy}-\nu y+\frac{\nu}{y}-\frac{a}{1-y}+\frac{b}{1+y}+\frac{a-b}{2}\right]F+\frac{e^2}{y}(y-y_1)(y-y_2)\,G=0\,,$$

$$\left[(1+y^2)\,\frac{d}{dy}+\nu\,y-\frac{\nu}{y}+\frac{a}{1-y}-\frac{b}{1+y}-\frac{a-b}{2}\right]G-\frac{e^2}{y}(y-y_3)(y-y_4)\,F=0\,. \tag{35}$$

Eq. for $F(y)$ contains singular points $0,\infty,\pm1,\pm i,y_1,y_2$

A similar situation is faced for the Dirac equation in the Coulomb potential on the background of the anti de Sitter space-time. The analysis here is much similar to that for the above de Sitter model. Using the variable $r = \sinh\rho$, now we have the equations

$$\left(\frac{d}{d\rho} + \frac{\nu}{\sinh\rho}\right) f + \left(\frac{1}{\cosh\rho}(\epsilon + \frac{e^2}{\sinh\rho}) + M\right) g = 0\,,$$

$$\left(\frac{d}{d\rho} - \frac{\nu}{\sinh\rho}\right) g - \left(\frac{1}{\cosh\rho}(\epsilon + \frac{e^2}{\sinh\rho}) - M\right) f = 0\,. \tag{36}$$

Introducing now new functions F and G:

$$f + g = e^{-\rho/2}(F + G)\,, \qquad f - g = e^{+\rho/2}(F - G)\,, \tag{37}$$

and a new variable $y = \tanh(\rho/2)$, this system is rewritten as

$$\left[(1-y^2)\,\frac{d}{dy} + \nu\,y + \frac{\nu}{y} - \frac{4(\epsilon y + e^2)}{1+y^2} + 2e^2\right] F + \frac{e^2}{y}(y - \bar{y}_1)(y - \bar{y}_2)\,G = 0\,,$$

$$\left[(1-y^2)\,\frac{d}{dy} - \nu\,y - \frac{\nu}{y} + \frac{4(\epsilon y + e^2)}{1+y^2} - 2e^2\right] G + \frac{e^2}{y}(y - \bar{Y}_1)(y - \bar{Y}_2)\,F = 0\,. \tag{38}$$

where

$$\bar{y}_{1,2} = \frac{(\epsilon + M - \nu - 1/2) \pm \sqrt{(\epsilon + M - \nu - 1/2)^2 + e^4}}{e^2}\,,$$

$$\bar{Y}_{1,2} = \frac{(\epsilon - M - \nu + 1/2) \pm \sqrt{(\epsilon - M - \nu + 1/2)^2 + e^4}}{e^2}\,. \tag{39}$$

Again, as in the case of the de Sitter space, the second order differential equations for $F(y)$ and $G(y)$ have eight singularities: $0, \pm 1, \pm i, \bar{y}_1, \bar{y}_2, \infty$ and $0, \pm 1, \pm i, \bar{Y}_1, \bar{Y}_2, \infty$, respectively.

11 Conclusion

Thus, we have developed the theory of the hydrogen atom in the de Sitter and anti de Sitter spaces on the basis of the Klein–Gordon–Fock wave equation in static coordinates. In both models, after separation of the variables, the problem is reduced to the general Heun equation, which is a second order Fuchsian linear differential equation having four regular singular points. The appearance of the Heun functions in treating the simplest and basic quantum mechanical system, the hydrogen atom, on the de Sitter geometrical backgrounds seems to be a useful point because of the progress in the theory of these functions observed during the last years. Conversely, the observed frequent appearance of the Heun equations in relativistic quantum-mechanical problems on the curved space-time background can serve as an additional stimulus to elaborate the mathematical theory of these functions.

A qualitative examination shows that the energy spectrum for the hydrogen atom in the de Sitter space is quasi-stationary, and the hydrogen atom is unstable in the sense of the quantum mechanics. We have derived an approximate expression for the energy levels within the quasi-classical method, and have estimated the probability of decay of the hydrogen atom. In contrast to this, the analysis shows that in the anti de Sitter model the hydrogen atom is stable. Approximate formulas for energy levels within the quasi-classical approach are also derived for this case.

Finally, we have extended the developments to the case of a spin 1/2 particle for both de Sitter space models. This extension results in complicated second order differential equations having 8 regular singular points.

We may expect (because of the universal character of the influence of the gravitational forces) that the behavior of the hydrogen atom model is typical for all quantum mechanical systems in the de Sitter geometries: any system, being stationary in the Minkowski space, will become quasi-stationary on the background of the de Sitter geometry, whereas in the case of the anti de Sitter model it preserves the stability property in the quantum-mechanical sense.

Going beyond, we further conjecture that the influence of any fixed type of a space-time geometry is in a sense universal and is much the same for various quantum mechanical systems, irrespective of the spin of the involved particles (various examples supporting this supposition are presented in [44, 47, 38]).

Finally, it should be mentioned that the de Sitter cosmological models can be parameterized by non-static coordinates [2], in which the metrical coefficient $g_{00} = 1$. For such cases, the Schrödinger-type non-relativistic wave equations do exist, and one can develop a traditional quantum-mechanics influenced by a non-trivial geometry for the space-time in cosmological scales.

12 Acknowledgements

This work was supported by the Fund for Basic Researches of Belarus (Grant No. F14ARM-021) and by the Armenian State Committee of Science (Grant No. 13RB-052), within the cooperation framework between Belarus and Armenia. The research has been partially conducted within the scope of the International Associated Laboratory (CNRS-France & SCS-Armenia) IRMAS. The work by A. Ishkhanyan has received funding from the European Union Seventh Framework Programme, grant No. 295025 - IPERA).

References

[1] P.A.M. Dirac. The electron wave equation in the de Sitter space. Ann. Math. **36**. 657–669 (1935).

[2] S.W. Hawking, G.F.R. Ellis. The large scale structure of space-time. Cambridge University Press, 1973.

[3] G.W. Gibbons. Anti-de-Sitter spacetime and its uses, in Mathematical and quantum aspects of relativity and cosmology (Pythagoreon, 1998). Lecture Notes in Phys. **537**. Springer-Verlag. P. 102–142 (2000).

[4] P.A.M. Dirac. Wave equations in conformal space. Ann. of Math. **37**. 429–442 (1936).

[5] E. Schrödinger. The proper vibrations of the expanding universe. Physica. **6**. 899–912 (1939).

[6] E. Schrödinger. General theory of relativity and wave mechanics. Wiss. en Natuurkund. **10**. 2–9 (1940).

[7] K. Goto. Wave equations in de Sitter space. Progr. Theor. Phys. **6**. 1013–1014 (1951).

[8] O. Nachtmann. Quantum theory in de-Sitter space. Commun. Math. Phys. **6**. 1–16 (1967).

[9] N.A. Chernikov, E.A. Tagirov. Quantum theory of scalar field in de Sitter space-time. Ann. Inst. Henri Poincare. **IX**. 109–141 (1968).

[10] G. Börner, H.P. Dürr. Classical and quantum theory in de Sitter space. Nuovo Cim. A. **64**. 669–713 (1969).

[11] W.L. Fushchych, I.Yu. Krivsky. On representations of the inhomogeneous de Sitter group and equations in five-dimensional Minkowski space. Nucl. Phys. B. **14**. 573–585 (1969).

[12] G. Börner, H.P. Dürr. Classical and Quantum Fields in de Sitter space. Nuovo Cim. **LXIV**. 669 (1969).

[13] M. Castagnino. Champs spinoriels en Relativité générale; le cas particulier de l'éspace-temps de De Sitter et les équations d'ond pour les spins éléves. Ann. Inst. Henri Poincaré. A. **16**. 293–341 (1972).

[14] E.A. Tagirov. Consequences of field quantization in de Sitter type cosmological models. Ann. Phys. **76**. 561–579 (1973).

[15] F. Riordan. Solutions of the Dirac equation in finite de Sitter space. Nuovo Cim. B. **20**. 309–325 (1974).

[16] P. Candelas, D.J. Raine. General-relativistic quantum field theory: an exactly soluble model. Phys. Rev. D. **12**. 965–974 (1975).

[17] Ch. Schomblond, P. Spindel P. Propagateurs des champs spinoriels et vectoriels dans l'univers de de Sitter. Bull. Cl. Sci., V. Ser., Acad. R. Belg. **LXII**. 124 (1976).

[18] S.W. Hawking, G.W. Gibbons. Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D. **15**. 2738–2751 (1977).

[19] S.J. Avis, C.J. Isham, D. Storey. Quantum Field Theory In Anti-de Sitter Space-Time. Phys. Rev. D. **18**. 3565 (1978).

[20] S.J. Avis, C.J. Isham, D. Storey. Quantum field theory in anti-de Sitter space-time. Phys. Rev. D. **18** . 3565–3576 (1978).

[21] D. Lohiya, N. Panchapakesan. Massless scalar field in a de Sitter universe and its thermal flux. J. Phys. A. **11**. 1963–1968 (1978).

[22] D. Lohiya, N. Panchapakesan. Particle emission in the de Sitter universe for massless fields with spin. J. Phys. A. **12**. 533–539 (1979).

[23] S. Hawking, D. Page. Thermodynamics Of Black Holes In Anti-de Sitter Space. Commun. Math. Phys. **87**. 577–588 (1983).

[24] V.S. Otchik. On the Hawking radiation of spin 1/2 particles in the de Sitter space-time. Class. Quantum Crav. **2**. 539–543 (1985).

[25] F. Motolla. Particle creation in de Sitter space. Phys. Rev. D. **31**. 754–766 (1985).

[26] Takashi Mishima, Akihiro Nakayama. Particle production in de Sitter spacetime. Progr. Theor. Phys. **77**. 218–222 (1987).

[27] D. Polarski. The scalar wave equation on static de Sitter and anti-de Sitter spaces. Class. Quantum Grav. **6**. 893–900 (1989).

[28] J. Bros, J.P. Gazeau, U. Moschella. Quantum Field Theory in the de Sitter Universe. Phys. Rev. Lett. **73**. 1746 (1994).

[29] H. Suzuki, E. Takasugi. Absorption Probability of De Sitter Horizon for Massless Fields with Spin. Mod. Phys. Lett. A. **11**. 431–436 (1996).

[30] S.A.: Pol'shin. Group Theoretical Examination of the Relativistic Wave Equations on Curved Spaces. I. Basic Principles. http://arxiv.org/abs/gr-qc/9803091; II. De Sitter and Anti-de Sitter Spaces. http://arxiv.org/abs/gr-qc/9803092; III. Real reducible spaces. http://arxiv.org/abs/gr-qc/9809011.

[31] I.I. Cotaescu. Normalized energy eigenspinors of the Dirac field on anti-de Sitter spacetime. Phys. Rev. D(3) **60**. 124006, 4pp (1999).

[32] T. Garidi, E. Huguet, J. Renaud. De Sitter Waves and the Zero Curvature Limit Comments. Phys. Rev. D. **67**. 124028 (2003).

[33] S. Moradi, S. Rouhani, M.V. Takook. Discrete Symmetries for Spinor Field in de Sitter Space. Phys. Lett. B. **613**. 74–82 (2005).

[34] A. Bachelot. The Dirac equation on the Anti-de-Sitter Universe. L'équation de Dirac sur l'univers Anti-de Sitter Comptes Rendus Mathematique. **345**. Issue 8, 435–440 (2007).

[35] E.M. Ovsiyuk, V.M. Red'kov. Spherical waves of spin 1 particle in anti de Sitter space-time. Acta Physica Polonica. B. **41**, no 6. 1247–1276 (2010).

[36] V.M. Red'kov, E.M. Ovsiyuk. On exact solutions for quantum particles with spin S= 0,1/2, 1 and de Sitter event horizon. Ricerche di matematica. **60**, no 1. 57–88 (2011).

[37] V.M. Red'kov, E.M. Ovsiyuk, G.G. Krylov. Hawking Radiation in de Sitter Space: Calculation of the Reflection Coefficient for Quantum Particles. Vestrnik RUDN. ser. Matemika-Informatika-Fizika. no 4. 153–169 (2012).

[38] E.M. Ovsiyuk, V.V. Kisel, V.M. Red'kov. Maxwell Electrodynamics and Boson Fields in Spaces of Constant Curvature. Nova Science Publishers Inc., New York, 2014. 486 pages.

[39] E.M. Ovsiyuk, O.V. Veko. Spin 1/2 particle in presence ob thre Abelian monopole on thebackground of anti de Sitter space-time. The case $j = j_{min}$. Doklady NAN of Belarus. **55**, no 6. 49–55 (2011).

[40] E.M. Ovsiyuk, O.V. Veko. Spin 1/2 particle in presence ob thre Abelian monopole on thebackground of anti de Sitter space-time. The case $j > j_{min}$. Doklady NAN of Belarus. **56**, no 1. 43–49 (2011).

[41] V. Red'kov, E. Ovsiyuk, O. Veko. Spin 1/2 particle in the field of Dirac string on the background of de Sitter space–time. Uzhhorod University Scientific Herald. Series Physics. **32**. 141–150 (2012).

[42] V.S. Otchik, V.M. Red'kov. Quantum-mechanival Kepler problem in spaces of constant curvature. Preprint no 298, Institut of Physics. AN BSSR, 1986. 49 pages (in Russian).

[43] V.M. Red'kov. Fields in Riemannian space and the Lorentz group. Publ. House "Belarusian Science": Minsk, 496 pages, 2009 (in Russian).

[44] V.M. Red'kov. Tetrad formalism, spherical symmetry and Schrödinger basis. Publishing House "Belarusian Science": Minsk, 339 pages, 2011 (in Russian).

[45] A. Ronveaux. Heun's Differential Equations. Oxford: Oxford Univ.Press, 1995.

[46] S.Yu. Slavyanov, W. Lay. Special functions. A unified theory based on singularities. Oxford: Oxford Univ.Press, 2000.

[47] V.M. Red'kov, E.M. Ovsiyuk. Quantum mechanics in spaces of constant curvature. Nova Science Publishers. Inc., New York, 2012.

[48] Varshalovich, D.A., Moskalev, A.N., Hersonskiy, V.K.: Quantum theory of angular moment. Nauka, Leningrad, 1975 (in Russian).

Quantum Matheu oscillator with cubic force and ohmic friction

A.L. Sanin
andreylsanin@yandex.ru
Peter the Great St.Petersburg Polytechnic University, Russia
A.A. Smirnovsky
smirnovskysaha@gmail.com; Ioffe Physics Institute, Russia

Abstract

The quantum Matheu oscillator as analog of classical one with cubic force and friction is proposed and investigated in context of the Schrödinger–Langevin method. Temporal evolution of the dynamical means and standard deviations, the frequency spectra were investigated numerically.

Keywords: wave packet dynamics, Schrödinger–Langevin–Kostin equation, Matheu oscillator, driving oscillations

1 Basic equations and numerical integration

The classical Matheu equation with cubic nonlinearity and dissipative term was widely applied to study the various dynamical systems. In simple dimensionless form it can be written as

$$\ddot{\xi} + k\dot{\xi} + (1 + \epsilon\cos\tau)\xi^3 = 0, \tag{1}$$

where ξ is a position, $\dot{\xi}$ is a time derivative and k, ϵ are the parameters characterizing the friction and the external force. This equation was analyzed numerically and analytically in [1].

It is known that at very low temperatures the classical systems can transit in the quantum regime of motion, for instance see [2] for the Duffing oscillator. The classical equation (1) can be generalized to the quantum area of motion through the Schrödinger–Langevin approach described in [3]. Here, it is formulated as the equation of motion including the ohmic friction, external potential and Langevin stochastic source as follows

$$i\frac{\partial\psi}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\psi}{\partial\zeta^2} + U_\Sigma\psi - \frac{ik}{2}\left(\ln\frac{\psi}{\psi^*} - \left\langle\ln\frac{\psi}{\psi^*}\right\rangle\right)\psi + F_{rand}\psi. \tag{2}$$

The equation (2) is dimensionless (invariant about the choice of measure units). It can be named as the Schrödinger–Langevin–Kostin equation (SchLK) and was firstly introduced in [4, 5]. This equation was analyzed both with and without stochastic Langevin source F_{rand} for the quantum quadratic oscillators and for the Duffing oscillators in many papers [6, 7, 8, 9, 10, 11, 12]. The equation (2) defines the time evolution of the wave function $\psi(\zeta,\tau)$ (where ζ,τ are position and time, respectively) which satisfies to the normalization condition

$$\int_{-\zeta_L}^{\zeta_L}\psi^*\psi d\zeta = 1.$$

The quantum system is confined by impenetrable walls in the points $\pm\zeta_L$ (ζ_l is a half-width of the system). The symbol $\langle\rangle$ denotes averaging in space as follows

$$\left\langle \ln \frac{\psi}{\psi^*} \right\rangle = \int \psi^* \ln \frac{\psi}{\psi^*} \psi d\zeta.$$

The total potential U_Σ consists of the two terms

$$U_\Sigma = \zeta^4 + U_{ext},\ U_{ext} = U_0 \zeta^4 \sin(\Omega_{ext}\tau), \tag{3}$$

if $U_0 \leq 1$. In the regime when $U_0 > 1$ the next potential is used to conserve sign before time term:

$$U_\Sigma = \left(1 + \frac{U_0}{2}\zeta^4\right) - \frac{U_0}{2}\zeta^4 \cos(2\Omega_{ext}\tau), \tag{4}$$

Here U_0, Ω_{ext} are the amplitude and frequency of the external potential. The dissipative properties of the system depend on the friction coefficient k.

If $k = 0$ and temporal terms in U_Σ are absent then we can derive from (2) the equation for eigenfunctions ϕ_n and eigenvalues ϵ_n

$$\left(-\frac{1}{2}\frac{d^2}{d\zeta^2} + \zeta^4\right)\phi_n = \epsilon_n \phi_n, \tag{5}$$

n is a state index.

The boundary conditions on the system walls and the initial wave function are specified as

$$\psi(\pm\zeta_L, \tau) = 0,\ \psi(\zeta, \tau = 0) = \psi_0(\zeta). \tag{6}$$

Here, $\psi_0(\zeta)$ is a solution of (5) for the ground state and the oscillations are excited by single short-time impulse potential

$$U_0 = -F_0\zeta,\ \tau \in (0; \Delta\tau), \tag{7}$$

where F_0 is a force that acts for a time $\Delta\tau$. The quantity $\Delta\tau$ is much less than the characteristic oscillation period.

The numerical integration of non-stationary equation (2) was performed at specified conditions (6), (7) by using the iterative finite-difference method. The discretization in space has the second order of accuracy, the discretization in time was performed by Crank-Nicolson scheme also with second order. It was checked that during calculations the normalization conditions persisted with errors less than 10^{-6}. The numerical method has been tested in the different cases (e.g. for the harmonic oscillator). The wave function ψ_0 of the ground state can be found by variation method or other. In given article it was determined numerically.

The quantum wave packets dynamics was investigated using the distributions of probability density $N = \psi^*\psi$, the mean values of the position $\langle\zeta\rangle$ and velocity $\langle V\rangle$, and standard deviations $\sigma_\zeta = \sqrt{\langle(\Delta\zeta)^2\rangle}$, $\sigma_V = \sqrt{\langle(\Delta V)^2\rangle}$. The product $\sigma_\zeta\sigma_V$ defines the uncertainty relation, it was analyzed as a function of time. The frequency spectra for the time evolution of $\langle\zeta\rangle$ were studied by the fast Fourier transform method where the quantity of modulus square $F_{\langle\zeta\rangle}(\Omega) \equiv \left|\Phi_{\langle\zeta\rangle}(\Omega)\right|^2$ as a function of frequency Ω has been investigated.

n	1	2	3	4	5	6	7	8
ϵ_n	0.668	2.394	4.697	7.336	10.244	13.379	16.712	20.221
$\Delta\epsilon_n$	1.726	2.303	2.639	2.907	3.135	3.333	3.509	3.669

Table 1: Energy spectrum and differences between levels (transition frequency)

2 Transition frequencies and free oscillations

To interpret the dynamical behavior it is necessary to know the energy spectra of the stationary Schrödinger equation for spatially confined quantum system without external force and friction. The calculations of energy spectra and transition frequencies between neighboring levels were performed for two well semi-widths: $\zeta_L = 3$ and $\zeta_L = 7$. The corresponding curves are presented in Fig. 1. There is no fracture in plot the for the greatest well width and this curve is well

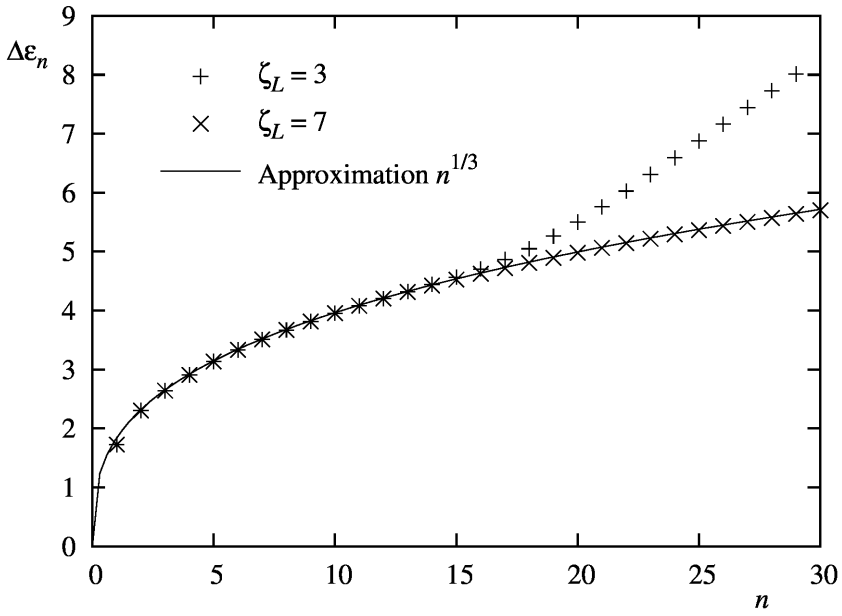


Figure 1: Difference between energy levels $\Delta\epsilon_n$ as a function of state number n.

approximated by the function $1.84n^{1/3}$. For the smaller potential well width $\zeta_L = 3$ the curve $\Delta\epsilon_n$ has a fracture and consists of two pieces. Before the fracture two curves for different well width coincide whereas after the fracture they are diverging and function $\Delta\epsilon_n$ from n becomes straight. It corresponds to the energy levels of empty potential well (without quartic potential). Let us consider the energy spectra and differences of energies between the neighboring levels $\Delta\epsilon_n$ at $\zeta_L = 3$ in details (see Table 1).

Since the quantities ϵ_n, $\Delta\epsilon_n$ are dimensionless the transition frequency between neighboring levels Ω_n satisfies to the relation $\Omega_n = \Delta\epsilon_n$. The frequencies Ω_n are detected in the dynamical regimes which will be discussed below. The number of excited frequencies Ω_n depends on the initial conditions and external action. We can notice (see Table 1) that the growth of state number tends to increase Ω_n but the difference $\Omega_{n+1} - \Omega_n$ is decreasing. According to the Fig. 1 the fracture takes place at some critical value $n > 16$ but at $n > 20$ the dependence $\Delta\epsilon_n = f(n)$ is almost linear.

As a first step we investigated the free oscillations generated under action of single impulse on the ground state without external harmonic action and friction. The duration of single pulse

was equal to 0.5 and impulse amplitude F_0 was varied. The total calculation time $T \geq 1000$, in separate cases the calculations were carried out at $T = 5000$ to decrease the quantity $\Delta\Omega = 2\pi/T$. It is necessary to fulfill the condition $\Delta\Omega \ll \Omega_{n+1} - \Omega_n$ for Fourier spectrum calculation. Most simple behavior and oscillatory patterns are realized at small F_0, for example at $F_0 = -1$. In this case the map of probability density N on the plane (ζ, τ) takes form like a fir (see Fig. 2a, dark color corresponds to large values of N). The time realization of $\langle\zeta\rangle$ and its Fourier

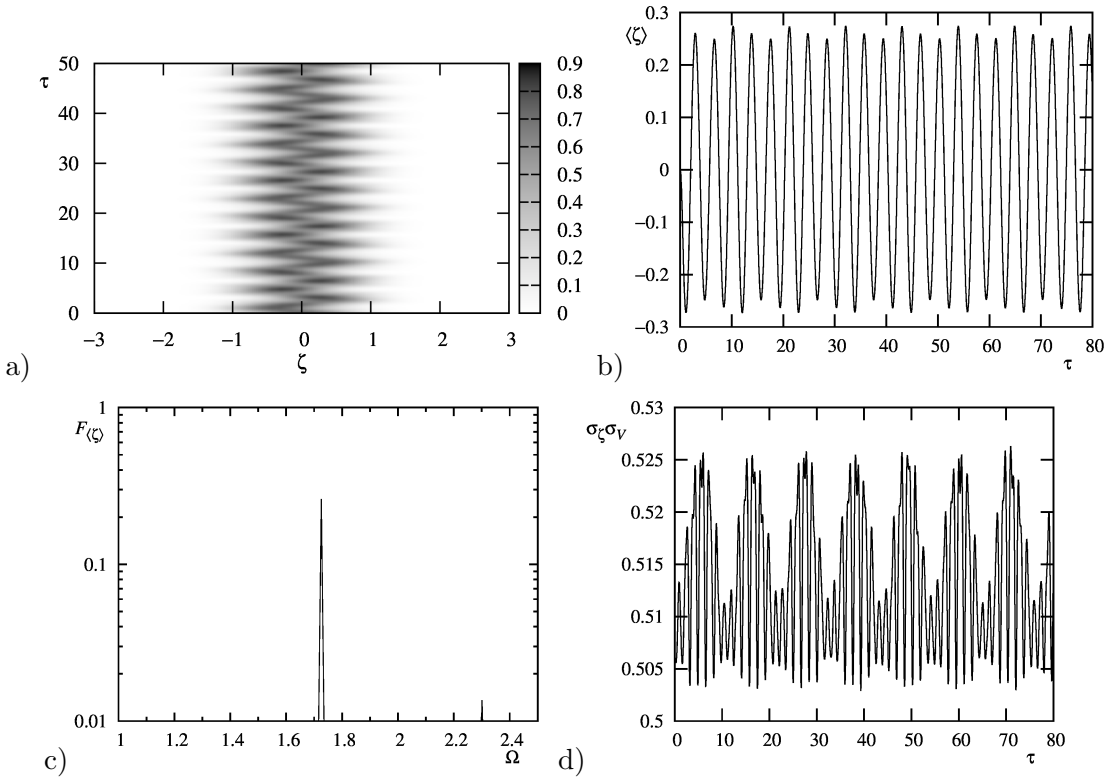


Figure 2: Free oscillations in quartic oscillator at $F_0 = -1$: a) the map of probability density N; b) time realization of $\langle\zeta\rangle$; c) Fourier transform $F_{\langle\zeta\rangle}(\Omega)$; d) the uncertainty product $\sigma_\zeta\sigma_V$

transform $F_{\langle\zeta\rangle}(\Omega)$ are presented in Fig. 2b, c. Here, two spectral components on $\Omega_1 = 1.7257$ and $\Omega_2 = 2.3032$ are effectively excited. It corresponds to excitement of the two energy levels ϵ_2, ϵ_3. The difference $\Omega_2 - \Omega_1$ determines a shallow amplitude modulation which is well seen in time realization $\langle\zeta\rangle$. Most extensive spectral component is realized at Ω_1 with $F_{\langle\zeta\rangle}(\Omega_1) \approx 0.3$, component on frequency Ω_2 is relatively small, its value $F_{\langle\zeta\rangle}(\Omega_2) \approx 0.06 F_{\langle\zeta\rangle}(\Omega_1)$. In Fig. 2d the standard deviations product as function of time is presented. The values of product do not greatly exceed the minimized value of 0.5.

The oscillation patterns become more complicated with increasing of F_0 due to generation of new Fourier components. As an example the time realization for mean position $\langle\zeta\rangle$ and its Fourier spectrum for impulse amplitude $F_0 = -5$ are presented in Fig. 3. Comparing with previous regime, one can see that the Fourier component Ω_1 is not practically changed and $F_{\langle\zeta\rangle}(\Omega_1) \approx 0.3$, but amplitude for Ω_2 is sharply increased so $F_{\langle\zeta\rangle}(\Omega_2) \approx 0.5$. In addition there are the components on frequencies Ω_3 and Ω_4 with $F_{\langle\zeta\rangle}(\Omega_3) \approx 0.3$ and $F_{\langle\zeta\rangle}(\Omega_4) \approx 0.07$. The spectral components with more high frequencies are very weak. With further increasing of F_0 the energy states with higher number n are excited and the energy redistribution on frequency spectra occurs.

The attenuation of quantum oscillator considered were also investigated in case when external

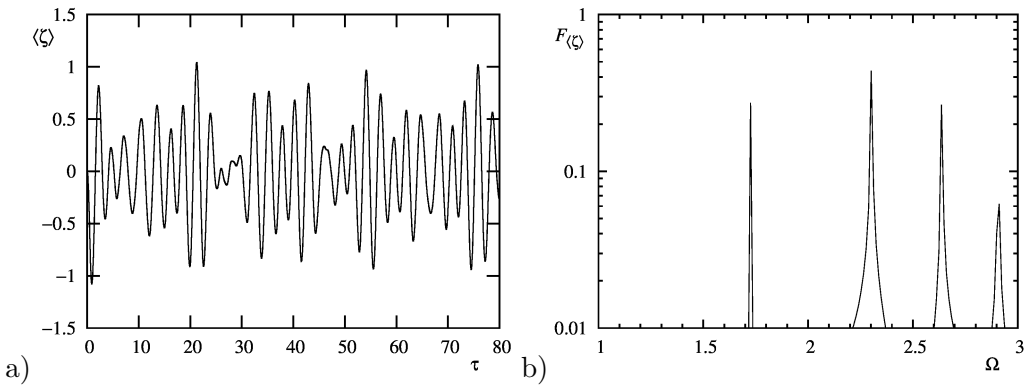


Figure 3: Fragment of time realization of $\langle\zeta\rangle$ in interval $[0, 80]$ (a) and its Fourier spectrum (b) at $F_0 = -5$

temporal term U_{ext} is absent. For this simple case we present the results of numerical simulations. If $U_{ext} = 0$, $F_0 = -1$, $k = 0.1$ we have the slow process of oscillation attenuation. In Fig. 4 the mean position $\langle\zeta\rangle$ tends to equilibrium over prolonged time. If the friction becomes

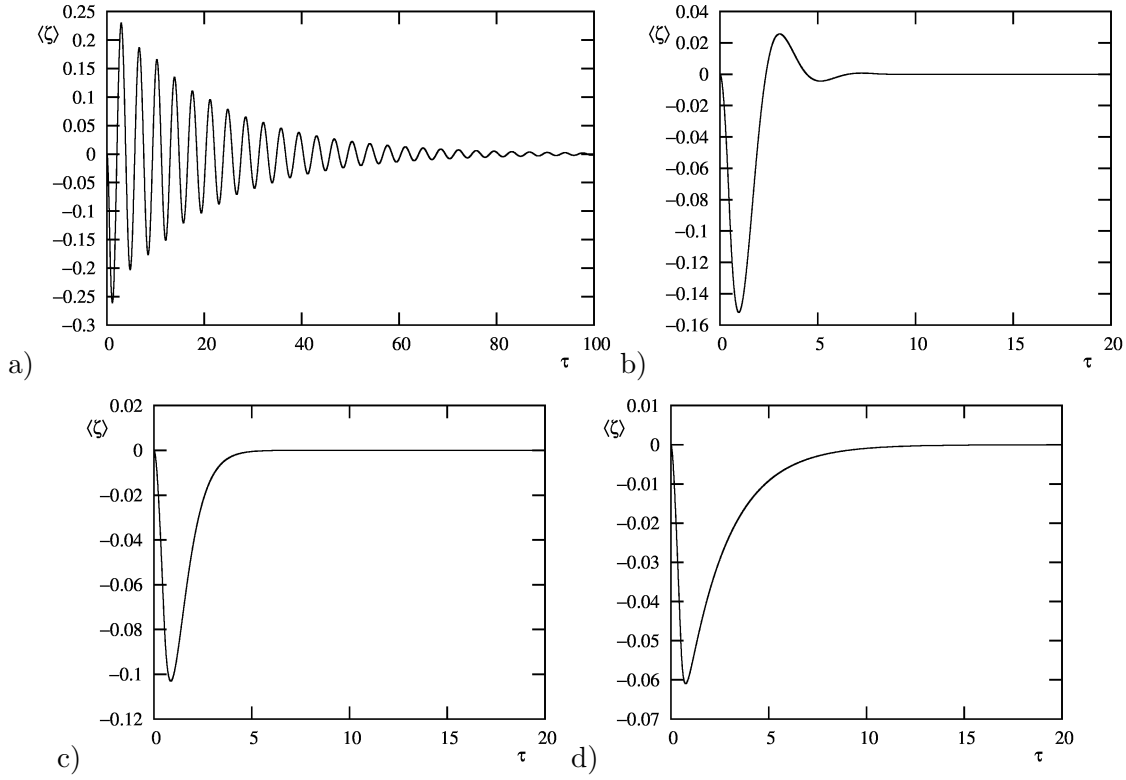


Figure 4: Time evolution of $\langle\zeta\rangle$ at $F_0 = -1$ and different k: 0.1 (a), 1.7 (b), 3.4 (c), 6.9 (d)

dominated the aperiodic attenuation of oscillations takes place. The quantum quartic oscillator transforms fast into ground state. The numerical calculations performed at $F_0 = -1$ and $k \in [1.7, 10]$ validate this properties. It is appropriate to note the analogy to over-damping oscillator with quadratic potential. Under the friction force with $k = 1.7$ the process of transition into equilibrium ground state occurs during the short time. At $k \geq 3.4$ we have one positive maximum, with the following grow of k the maximum is decreased.

3 Quantum Matheu oscillator with quartic potential

The generalized quantum equation of motion (2) describes quantum Matheu oscillator with the parametric cubic force and specified external frequency Ω_{ext}. The more simple regimes of oscillations are realized at $U_0 \leq 1$ and the weak single pulses F_0. The simple regime of oscillations takes place for $U_0 = 0.1$, $F_0 = -0.1$, $\Omega_{ext} = \Omega_1 = 1.7256$, $k = 0$ (Fig.5).

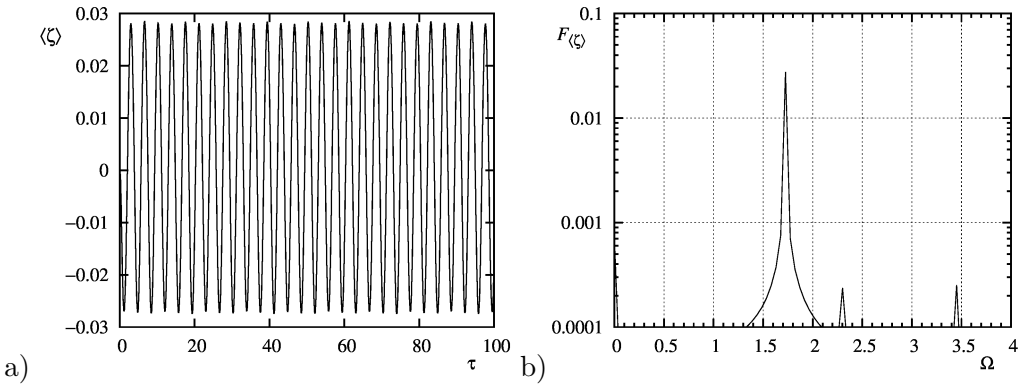


Figure 5: Mean position (a) and frequency spectra (b) for small values of F_0, U_0

As illustrated in Fig. 5, the mean position $\langle\zeta\rangle$ is characterized by the main temporal scale $2\pi/\Omega_{ext}$, the small amplitude modulation depends on difference of frequencies $\Omega_2 - \Omega_1$. The more effective Fourier component is determined by the transition frequency Ω_1, the Fourier component at Ω_2 is very weak. Here, there is also the Fourier component at $\Omega = 2\Omega_{ext}$. The formula for frequencies can be written as $\Omega = \Omega_1 + 2\Omega_{ext}$, where Ω is combined frequency.

If the dissipative force is included, the oscillations become attenuated. At $k = 0.1$ the mean position $\langle\zeta\rangle$ tends to ground equilibrium state as for regime without external force (see Fig. 4).

In the paper [1] the periodical oscillatory regimes were studied without friction. Similar nature of the oscillatory processes have been also found only at very weak friction and in special case. At other parameters for the classical Matheu oscillator the oscillatory regimes are attenuated. In our calculations for the quantum Matheu oscillator with cubic force the inclusion of friction causes only an attenuation. The analogy with classical Matheu oscillator is confirmed. Comparison studies of two regimes both with and without friction were performed for $F_0 = -0.1$, $U_0 = 1$, $\Omega_{ext} = 1.7257$, $k = 0, 0.1$ (Fig. 6). In both cases the oscillations go on above some

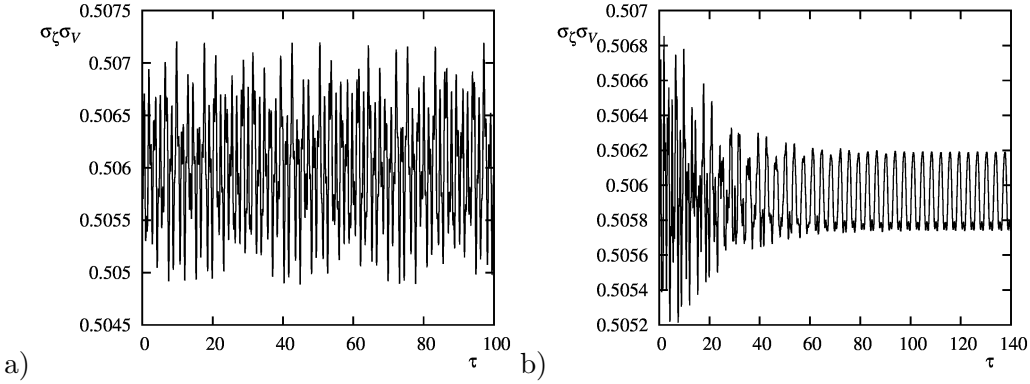


Figure 6: Standard deviations product: a) $k = 0$, b) $k = 0.1$

level which is caused by the uncertainty relation. A comparison between graphs a and b in Fig.

6 demonstrates the distinction in behavior. In system with friction the transition stage and sustained oscillations are present. As one would expect, the quantum Matheu oscillator with the external temporal force is also controlled by uncertainty relation.

Now, we discuss the oscillatory regimes at variables Ω_{ext}, but again for small values of F_0, U_0. We consider the cases with $F_0 = -0.1$, $U_0 = 0.1$, $k = 0$ and $\Omega_{ext} = 0.13 + 0.1n$ where $n = 1, 2, ..., 8$. The combined frequencies of the mean position spectrum satisfy to the formula $\Omega = \Omega_1 + l\Omega_{ext}$, l is positive and negative numbers. The same extensive spectral component takes place at $\Omega_1 = 1.7256$ (this is the transition frequency from the ground state to the first excited). If $n = 1$ then we have three spectral components on frequencies Ω with $l = -2, -1, 1$. At subsequent increasing n the pictures of the frequency spectra are conserved at whole, and the separate component on Ω_2 occurs Influence of the quantity $l\Omega_{ext}$ on the main component $F_{\langle\zeta\rangle}(\Omega_!)$ is not essential. Here we see the simple regimes of the oscillations. The complexity of the oscillatory processes depends on the amplitudes U_0, Ω_{ext} and other parameters.

Next step is a study of oscillations for $\Omega_{ext} = 0.25\Omega_1$ and variable U_0. As previously, we discuss the temporal evolution of the Matheu oscillator at the weak pulse of the excitement with $F_0 = -0.1$; for the regime at $U_0 = 0.1$, $k = 0$ we have simple oscillatory picture (Fig. 7). The probability density N on a plane (ζ, τ) describes the localization of a quantum wave packet

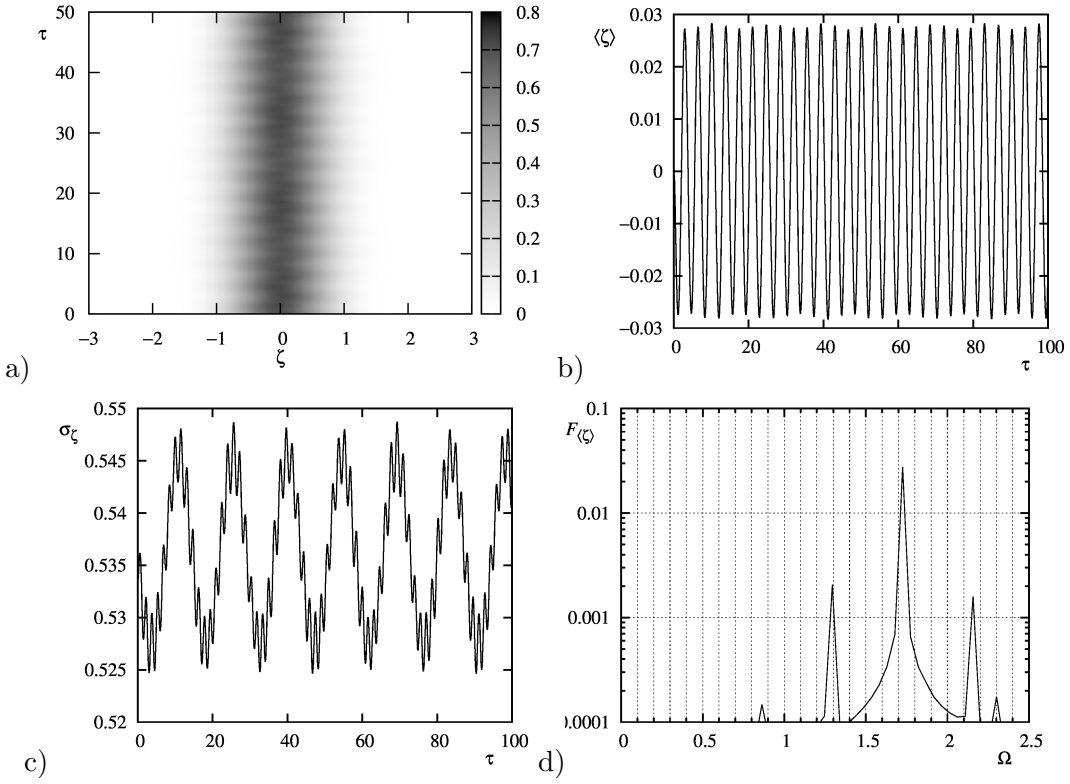


Figure 7: Dynamical properties of Matheu oscillator at small U_0: a) probability density N on plane(ζ, τ), b) mean position $\langle\zeta(\tau)\rangle$, c) standard deviation σ_ζ, d) frequency response $F_{\langle\zeta\rangle}$

which is approximately distributed in an interval $\zeta \in [-1, 1]$ (dark color in Fig. 7), it forms the dependence $\langle\zeta(\tau)\rangle$ and σ_ζ (Fig. 7b, c).

It can be seen from Fig. 7c that the large scaling beating is repeated through interval $T \approx 14.286$ with frequency $\Omega = 0.4396$. This frequency is close to the value $\Omega_{ext} = 0.4386$. The Fourier spectrum of temporal realization for $\langle\zeta(\tau)\rangle$ is given in Fig. 7d, where $F_{\langle\zeta\rangle}(\Omega_1)$ is more powerful. In addition to the main spectral component, there are the other components

n	1	2	3	4
ϵ_n	3.6343	7.1311	11.1378	15.5533
$\Delta\epsilon_n$	4.0067	4.4161	4.7599	5.0598

Table 2: Energy spectrum and differences between levels for potential (4)

on the combined frequencies $F_{\langle\zeta\rangle}(\Omega_1 \pm \Omega_{ext})$ and also $F_{\langle\zeta\rangle}(\Omega_2)$ on transition frequency Ω_2. When the friction is switched on the dissipative process takes place and the quantum Matheu oscillator tends to the background state. In classical limit, the attenuation process occurs for different parameters values. As noted above only under very weak friction in special case periodic regime is possible. Our dissipative picture is consistent with the classical Matheu oscillator for the wide parameters range. The oscillatory regime (without attenuation) of the quantum Matheu oscillator with combined frequencies can be explained due to the Floquet states which are generated by periodic in time potential.

As the amplitude of U_0 is increased the temporal realizations $\langle\zeta(\tau)\rangle$ and its frequency spectra undergo transformations. Even though the amplitude U_0 will be doubled (equal to 0.2) the components $F_{\langle\zeta\rangle}(\Omega_1)$, $F_{\langle\zeta\rangle}(\Omega_1 \pm \Omega_{ext})$ are growing and widening to base, it is also formed "pedestal". In addition the new spectral components on the combined frequencies are generated. This process is illustrated in Fig. 8 for $U_0 = 0.5, 1$.

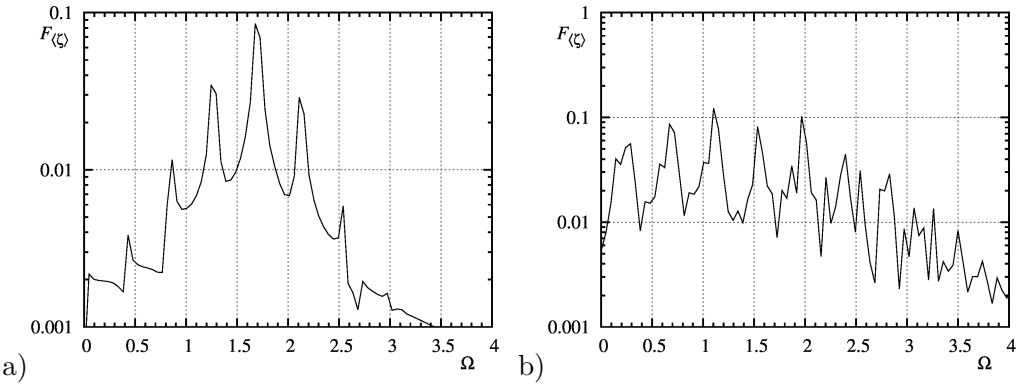


Figure 8: Frequency response: a) $U_0 = 0.5$, b) $U_0 = 1$

For $U_0 = 0.5$ the more power spectral component $F_{\langle\zeta\rangle}(\Omega^*)$ takes place where $\Omega^* < \Omega_1$. The other spectral peaks satisfy to the relation $\Omega = \Omega^* + n\Omega_{ext}$ where $n = -3, -2, -1, 1, 2$; they become broadband. If $U_0 = 1$ then the spectral picture is again changed. Now, the pedestal level has been elevated significantly. The spectral peaks have no smooth form. The spectral component $F_{\langle\zeta\rangle}(\Omega_1)$ is practically disappeared. But, there is the collection of spectral lines on the frequencies $n\Omega_{ext}$, n is an integer number. The system loses the memory about transition frequencies. But, the pedestal is also composed from the spectral lines distributed between the neighbor components on Ω_{ext}.

The more complex regimes of oscillations are generated if the amplitudes $U_0 > 1$. Here, we discuss the situation at $U_0 = 5$, $\Omega_{ext} = \Omega_1$. For this case we use potential (4). Now, the energy spectrum and the transition frequencies are distincted from the regime investigated previously in section 2 for the stationary potential term (3) with coefficient 1. The Table 2 illustrates the several values ϵ_n and $\Delta\epsilon_n$ for the stationary potential term with coefficient 7/2 and the semi-width $\zeta_L = 3$.

The oscillatory process has been excited by the initial short pulse with $F_0 = -0.1$, the

frequency of the external force was equal to $2\Omega_1$ with $\Omega_! = 1.7256$. Without friction, the temporal realization $\langle\zeta(\tau)\rangle$ and the frequency response are shown in Fig. 9.

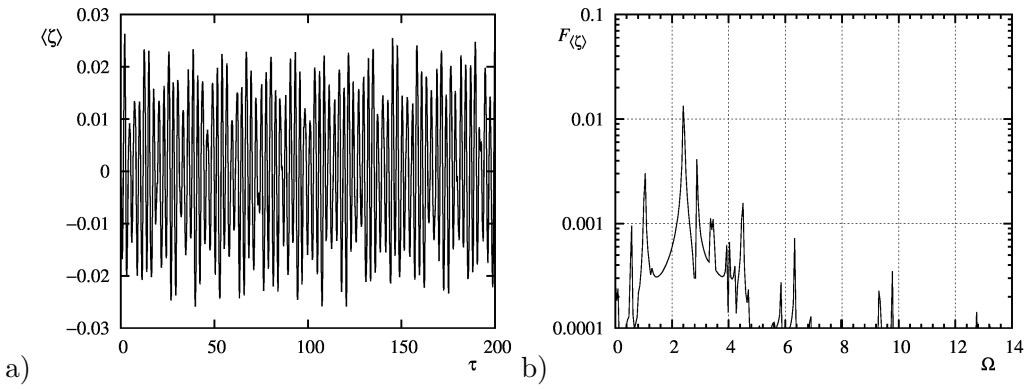


Figure 9: Temporal realization $\langle\zeta(\tau)\rangle$ (a) and Fourier spectrum (b) in regime with $U_0 > 1$

With the specified amplitude U_0 the frequency spectrum becomes more complicated. The spectral component $F_{\langle\zeta\rangle}(\Omega)$ on the frequency $\Omega = 2\Omega_1$ is detectable. In this case we studied about 30 spectral components which are sufficiently strong. The several of them noticed by numbers are governed by the determined relations between frequencies. For instance, the pairs $\Omega_3 - \Omega_2$, $\Omega_5 - \Omega_4$, $\Omega_6 - \Omega_5$, $\Omega_{11} - \Omega_9$ are equal to the same pair $\Omega_2 - \Omega_1$; $\Omega_7 - \Omega_6$, $\Omega_9 - \Omega_8$ are equal to Ω_1; $\Omega_{11} - \Omega_8 = \Omega_2$ and $\Omega_4 - \Omega_3 = 2(\Omega_1 + \Omega_2)$. This demonstrates the more complex properties of the signal than at $U_0 \leq 1$. The numerical values of Ω are closely distributed in interval $[1, 4.6]$. The transition frequencies are hidden. The influence of the external harmonic force on the frequency spectra is essential. The further calculations of spectra at the high amplitudes U_0 are necessary for the better understanding of dynamical laws.

4 Summary

The quantum Matheu oscillator with quartic potential and ohmic friction has been discussed as a generalization of the classical analogue. The temporal realizations of the mean dynamical variables and frequency spectra were numerically analyzed. At the weak external force being harmonic in time and cubic in space, the oscillatory regimes with combined frequencies are established, they are caused by the Floquet states of system. At strong external force the realizations and frequency spectra are more complicated; the frequencies of spectral components are governed by the determined relations. The friction in the investigated quantum system causes the damping as in classical Matheu oscillator with cubic force.

References

[1] M.V. Bartuccelli, A. Berretti, J.H.B. Deane, G. Gentile, S.A. Gourley, Selection rules for periodic orbits and scaling laws for a driven damped quartic oscillator, Nonlinear Analysis: Real World Applications, Vol.9(5), 2008, P.1966-1988

[2] I. Katz, R. Lifshitz, A. Retzker, R. Straub. Classical to quantum transition of a driven nonlinear nanomechanical resonator. New Journal of Physics, Vol. 10, 2008, P.125023.

[3] G.W. Ford,M. Kac, P. Mazur. Statistical mechanics of assemblies of coupled oscillators. J. Math. Phys., Vol.6, p.504, 1965.

[4] M.D. Kostin. On the Schrödinger-Langevin equation. J. Chem. Phys., Vol.57(9), p.3589, 1972.

[5] M.D. Kostin. Friction in dissipative phenomena in quantum mechanics. J. St. Phys. Vol.12. 1975.

[6] A.L. Sanin, A.A. Smirnovsky. Oscillatory motion in confined potential systems with dissipation in the context of the Schrödinger–Langevin–Kostin equation. Phys. Lett. A, Vol.372, p.21–27, 2007.

[7] A.L. Sanin, A.A. Smirnovsky, Quantum spatially confined oscillator in system with friction and feedback, Izvestiya VUZ. Nonlinear dynamics, Vol.16, p.18–34, 2008.

[8] A.L. Sanin, A.A. Smirnovsky, Physics. Quantum dynamics, St. Petersburg Polytechnical University Press, 280 p., 2012.

[9] Van P., Fulop T., Stability of stationary solutions of the Schrödinger–Langevin equation, Phys. Lett. A, Vol.323, p.374–381, 2004.

[10] H.–D. Doebner, G.A. Goldin, On a general nonlinear Schrödinger equation admitting diffusion currents, Phys. Lett. A, Vol.162, p.397–401, 1992.

[11] H.–D. Doebner, G.A. Goldin, P. Nattermann. Gauge transformation in quantum mechanics and the unification of nonlinear Schrödinger equations. J. Math. Phys. – Vol. 40. – N.1. – 1999. – P.49–63.

[12] D. de Falco, D. Tamascelli. Quantum annealing and the Schrödinger–Langevin–Kostin equation. J. Phys. Rev. A. – 2009. – Vol.79. – P. 012315.

Cox's particle in magnetic and electric fields on the background of hyperbolic Lobachevsky geometry

O.V. Veko
vekoolga@mail.ru; Gymnasium, Kalinkovichi, Belarus

Abstract

Generalized non-relativistic Schrödinger equation for Cox's not point-like scalar particle with intrinsic structure has been solved exactly in presence of external uniform magnetic and electric fields in the case of Minkowski space. Extension of these problems to the case of the hyperbolic Lobachevsky 3-space is examined. Complete separations of the variables in the system of special cylindric coordinates in this curved model has performed for cases of magnetic and electric fields. In presence of the magnetic field, the quantum problem in radial variable has been solved exactly, wave functions and corresponding energy level have been found; the quantum motion in z-direction is described by 1-dimensional Schrödinger-like equation in an effective potential which turns out to be too difficult for analytical treatment. In the presence of electric field on the background of curved model, the situation is similar: radial equation is solved exactly in hypergeometric functions, an equation in z-variable can be be treated only qualitatively because its complexity.

Keywords: Cox's particle, generalized Schrödinger equation, magnetic field, electric field, Minkowski space, Lobachevsky space

1 The Schrödinger equation in the magnetic field

Let the uniform magnetic field $\vec{A} = \frac{1}{2}\vec{x} \times \vec{B}$ be directed along the axis z. Recalculating the potential to cylindrical coordinates, we obtain

$$A_r = 0,\ A_\phi = -\frac{Br^2}{2},\ A_z = 0,\ F_{r\phi} = -Br\ . \tag{1}$$

The metric tensor in these coordinates and field variables are determined by

$$dS^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2,\ \sqrt{-g} = r\ ,\quad B_3 = -Br,\ B^3 = -Br^{-1},\ B_iB^i = B^2\ . \tag{2}$$

We start with the Schrödinger wave equation for Cox's [1] particle in the following form [2], [3] (the mass of the particle is denoted by the letter M)

$$D_t\Psi = \frac{1}{2M}(\overset{\circ}{D}_1\overset{*}{D}_1 + \overset{\circ}{D}_2\, r^{-2}\, \overset{*}{D}_2 + \overset{\circ}{D}_3\overset{*}{D}_3)\Psi, \tag{3}$$

where

$$D_1 = i\hbar\partial_r,\ D_2 = i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2},\ D_3 = i\hbar\partial_z,$$
$$\overset{\circ}{D}_1 = i\hbar(\partial_r + \frac{1}{r}),\quad \overset{\circ}{D}_2 = i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2},\quad \overset{\circ}{D}_3 = i\hbar\partial_z,$$

$$\overset{*}{D}_1 = \frac{1}{1+\Gamma^2 B^2}(D_1 - \Gamma B_3 D^2) = \frac{1}{1+\Gamma^2 B^2}\left(i\hbar\partial_r - \frac{\Gamma B}{r}(i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2})\right),$$

$$\overset{*}{D}_2 = \frac{1}{1+\Gamma^2 B^2}(\ D_2 + \Gamma B_3 D^1) = \frac{1}{1+\Gamma^2 B^2}\left((i\hbar\partial_\phi + \frac{e}{c}\frac{Br^2}{2}) + i\hbar\Gamma B r\partial_r\right),$$

$$\overset{*}{D}_3 = \frac{(D_3 + \Gamma^2 B^3\ B_3 D_3)}{1+\Gamma^2 B^2} = i\hbar\partial_z\ ; \tag{4}$$

below we will use the notation $eB/2\hbar c = b,\ \Gamma B = \gamma$. We compute

$$\frac{1}{2M}\ \overset{\circ}{D}_1\overset{*}{D}_1 = -\frac{\hbar^2}{2M(1+\gamma^2)}\left[\partial_r^2 + \frac{1}{r}\partial_r - \frac{\gamma}{r}\partial_r\partial_\phi + i\gamma b r\partial_r + 2i\gamma b\right],$$

$$\frac{1}{2M}\ \overset{\circ}{D}_2\overset{*}{D}_2 = -\frac{\hbar^2}{2M(1+\gamma^2)}\left[\frac{1}{r^2}(\partial_\phi - ibr^2)^2 + \gamma(\partial_\phi - ibr^2)\ \frac{1}{r}\partial_r\right],$$

$$\frac{1}{2M}\ \overset{\circ}{D}_3\overset{*}{D}_3 = -\frac{\hbar^2}{2M(1+\gamma^2)}(1+\gamma^2)\partial_z^2. \tag{5}$$

After using the substitution for the wave function

$$\Psi = e^{-iEt/\hbar}e^{im\phi}e^{ikz}R(r), \quad \epsilon = \frac{2mE}{\hbar^2}(1+\gamma^2) \tag{6}$$

we get the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 2i\gamma b + \epsilon - \frac{(m-br^2)^2}{r^2} - (1+\gamma^2)k^2\right]R = 0. \tag{7}$$

By physical reasons, parameter γ must be imaginary: $\gamma = -i\eta$; so the radial equation reads

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 2\eta b + \epsilon - \frac{(m-br^2)^2}{r^2} - (1-\eta^2)k^2\right]R = 0. \tag{8}$$

The term $2\eta b$ corresponds to the shift of the all energy levels of the nonrelativistic particle. With the use of notation $\epsilon - (1-\eta^2)k^2 + 2\eta b = \epsilon'$, equation (8) can be written as:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-br^2)^2}{r^2} + \epsilon'\right] = 0\ , \tag{9}$$

which coincides with the differential equation arising in the usual problem of the Schrödinger particle in the magnetic field. Its solutions are known. We present here only an expression for the energy spectrum (turning back from ϵ' to ϵ)

$$\epsilon = 4b(n + \frac{m+\ \mid m \mid\ +1}{2}) + (1-\eta^2)k^2 - 2\eta b\ ; \tag{10}$$

from this after translating to ordinary units we find

$$E = \frac{p^2}{2M} + \frac{1}{1-\eta^2}\frac{eB}{Mc}\hbar\left(n + \frac{m+\ \mid m \mid\ +1}{2}\right) - \frac{\eta}{2}\frac{1}{1-\eta^2}\frac{eB}{Mc}\hbar\,. \tag{11}$$

With the notation $\eta = \Gamma B,\ \Gamma^* = \Gamma,\ \omega = eB/Mc$, the formula for the energy levels can be written as

$$E = \frac{p^2}{2M} + \frac{\omega\hbar}{1-(\Gamma B)^2}\left(n + \frac{m+\ \mid m \mid\ +1}{2}\right) - \frac{\omega\hbar}{1-(\Gamma B)^2}\frac{\Gamma B}{2}\ . \tag{12}$$

Thus, the intrinsic structure of the Cox's particle modifies the frequency of the quantum oscillator

$$\omega \qquad \Longrightarrow \qquad \tilde{\omega} = \frac{\omega}{1-\Gamma^2 B^2},\ \omega = \frac{eB}{Mc}\ . \tag{13}$$

2 Cox's particle in the magnetic field in the Lobachevsky space

In a special (cylindrical) coordinate system in the Lobachevsky space, analogue of the uniform magnetic field is determined by the relations [4] (we use dimensionless coordinate r obtained by dividing on the curvature radius ρ):

$$dS^2 = c^2dt^2 - \mathrm{ch}^2 z(dr^2 + \mathrm{sh}^2 r d\phi^2) + dz^2,$$
$$\sqrt{-g} = \rho^3\ \mathrm{sh}\ r\ \mathrm{ch}^2 z, \quad A_\phi = -B\rho^2(\mathrm{ch}\ r - 1),\ F_{r\phi} = -B\rho\ \mathrm{sh}\ r,$$
$$B_3 = -B\rho\ \mathrm{sh}\ r, \quad B^3 = -\frac{B}{\rho\ \mathrm{sh}\ r\ \mathrm{ch}^4 z}, \quad B_iB^i = B^2\mathrm{ch}^{-4} z. \tag{14}$$

We start with the representation of the wave equation in the form [2]

$$D_t\Psi = \frac{1}{2M\rho^2}\left[\overset{\circ}{D}_1\overset{*}{D}_1 + \overset{\circ}{D}_2\ \frac{1}{\mathrm{sh}^2 r}\ \overset{*}{D}_2 + \overset{\circ}{D}_3\overset{*}{D}_3\right]\Psi,$$

where

$$D_1 = i\hbar\partial_r, \quad D_2 = i\hbar\partial_\phi + \frac{e}{c}B\rho^2(\mathrm{ch}\ r - 1), \quad D_3 = i\hbar\partial_z,$$
$$\overset{\circ}{D}_1 = i\hbar(\partial_r + \frac{\mathrm{ch}\ r}{\mathrm{sh}\ r}), \quad \overset{\circ}{D}_2 = i\hbar\partial_\phi + \frac{e}{c}B\rho^2(\mathrm{ch}\ r - 1), \quad \overset{\circ}{D}_3 = i\hbar(\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z}),$$
$$\overset{*}{D}_1 = \frac{1}{1 + \Gamma^2B^2\mathrm{ch}^{-4} z}\left[i\hbar\partial_r - \frac{\Gamma B\mathrm{ch}^{-2} z}{\mathrm{sh} r}(i\hbar\partial_\phi + \frac{e}{c}B\rho^2(\mathrm{ch}\ r - 1))\right],$$
$$\overset{*}{D}_2 = \frac{1}{1 + \Gamma^2B^2\mathrm{ch}^{-4} z}\left[(i\hbar\partial_\phi + \frac{e}{c}B\rho^2(\mathrm{ch}\ r - 1)) + i\hbar\Gamma B\mathrm{ch}^{-2} z\ \mathrm{sh}\ r\partial_r\right],$$
$$\overset{*}{D}_3 = \frac{(D_3 + \Gamma^2B^3\ B_3D_3)}{1 + \Gamma^2B^2\mathrm{ch}^{-4} z} = i\hbar\partial_z. \tag{15}$$

Below the notation is used: $(eB\rho^2/\hbar c) = b$, $\Gamma B\mathrm{ch}^{-2} z = \gamma(z)$. Then we compute

$$\frac{1}{2M\rho^2}\ \overset{\circ}{D}_1\ g^{11}\ \overset{*}{D}_1 = -\frac{\hbar^2\mathrm{ch}^{-2} z}{2M\rho^2(1 + \gamma^2(z))}\left(\partial_r^2 + (\frac{\mathrm{ch}\ r}{\mathrm{sh}\ r} + i\gamma(z)b\frac{\mathrm{ch}\ r - 1}{\mathrm{sh}\ r})\partial_r - \frac{\gamma(z)}{\mathrm{sh}\ r}\partial_r\partial_\phi + i\gamma(z)b\right),$$
$$\frac{1}{2M\rho^2}\ \overset{\circ}{D}_2\ g^{22}\ \overset{*}{D}_2 = -\frac{\hbar^2\mathrm{ch}^{-2} z}{2M\rho^2(1 + \gamma^2(z))}\left[\frac{1}{\mathrm{sh}^2 r}[\partial_\phi - ib(\mathrm{ch}\ r - 1)]^2 + \gamma(z)[\partial_\phi - ib(\mathrm{ch}\ r - 1)]\ \frac{1}{\mathrm{sh}\ r}\partial_r\right],$$
$$\frac{1}{2M\rho^2}\ \overset{\circ}{D}_3\ g^{33}\ \overset{*}{D}_3 = -\frac{\hbar^2}{2M\rho^2}(\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\partial_z.$$

We introduce the substitution for the wave function:

$$\Psi = e^{-iEt/\hbar}e^{im\phi}Z(z)R(r)\ , \quad \epsilon = \frac{E}{\hbar^2/2M\rho^2}\ , \tag{16}$$

then the Schrödinger equation (15) gives (by physical reasons function $\gamma(z)$ must be purely imaginary: $i\gamma(z \Longrightarrow \gamma(z))$

$$\left[\frac{\mathrm{ch}^{-2} z}{1 - \gamma^2(z)}\left(\partial_r^2 + \frac{\mathrm{ch}\ r}{\mathrm{sh}\ r}\partial_r - \frac{[m - b(\mathrm{ch}\ r - 1)]^2}{\mathrm{sh}^2 r} + b\gamma(z)\right) + \epsilon + (\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\partial_z\right]R(r)Z(z) = 0\ . \tag{17}$$

In this equation, the variables are separated:

$$\frac{1}{R}\left(\frac{d^2}{dr^2} + \frac{\mathrm{ch}\ r}{\mathrm{sh}\ r}\frac{d}{dr} - \frac{[m - b(\mathrm{ch}\ r - 1)]^2}{\mathrm{sh}^2 r}\right)R$$
$$+\frac{1}{Z}(1 - \gamma^2(z))\ \mathrm{ch}^2 z\left(\frac{b\ \gamma(z)\ \mathrm{ch}^{-2} z}{1 - \gamma^2(z)} + \epsilon + (\frac{d}{dz} + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\frac{d}{dz}\right)Z = 0. \tag{18}$$

The radial equation for the function $R(r)$ reads

$$\left(\frac{d^2}{dr^2}+\frac{\mathrm{ch}\ r}{\mathrm{sh}\ r}\frac{d}{dr}-\frac{[m-b(\mathrm{ch}\ r-1)]^2}{\mathrm{sh}^2 r}+\Lambda\right)R=0, \tag{19}$$

and the equation for $Z(z)$ is (remember that $\gamma = B\Gamma$)

$$\left(\frac{d^2}{dz^2}+2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z}\frac{d}{dz}+\epsilon+\frac{b\gamma-\Lambda\ \mathrm{ch}^2 z}{\mathrm{ch}^4 z-\gamma^2}\right)Z=0. \tag{20}$$

3 Analysis of the equation in the variable z

In equation (20), let us eliminate the first derivative term:

$$Z=\frac{1}{\mathrm{ch}\ z}f(z),\quad U(z)=-\frac{b\gamma-\Lambda\ \mathrm{ch}^2 z}{\mathrm{ch}^4 z-\gamma^2},\quad \left(\frac{d^2}{dz^2}+\epsilon-1-U(z)\right)f(z)=0. \tag{21}$$

Eq. (21) can be viewed as the Schrödinger equation in the effective potential field $U(z)$. The corresponding effective force is

$$F_z=-\frac{dU}{dz}=2\ \mathrm{ch}\ z\ \mathrm{sh}\ z\ \frac{\Lambda\ \mathrm{ch}^4 z-2b\gamma\ \mathrm{ch}^2 z+\gamma^2\Lambda}{(\mathrm{ch}^4 z-\gamma^2)^2}. \tag{22}$$

We find the points of local extremum: $z=0$ and the roots of a quadratic equation

$$\Lambda\ \mathrm{ch}^4 z-2b\gamma\ \mathrm{ch}^2 z+\gamma^2\Lambda=0\quad\Longrightarrow\quad\left(\mathrm{ch}^2 z\right)|_{1,2}=\frac{b}{\Lambda}\gamma\pm\sqrt{(\frac{b^2}{\Lambda^2}-1)\gamma^2}\ . \tag{23}$$

In the section **4** it will be shown that when considering the bound states (for motion in the variable r) we have $\Lambda^2 > b^2$. This means that the square root in (23) is an imaginary number. Consequently, the point of zero force (equilibrium points) except $z=0$ cannot exist. The situation is illustrated in the Fig. 1.

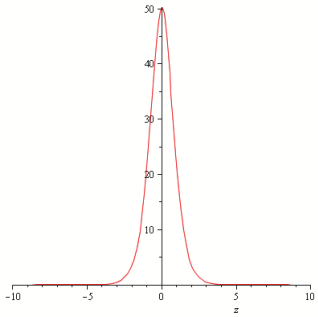


Figure 1: *Effective potential $U(z)$:*

After the change of variables $\mathrm{ch}^2 z = y$, the differential equation (21) reads

$$\left[\frac{d^2}{dy^2}+\left(\frac{3}{2}\frac{1}{y}+\frac{1}{2}\frac{1}{y-1}\right)\frac{d}{dy}+\frac{\epsilon}{4y(y-1)}+\frac{b\gamma-\Lambda\ y}{(y-\gamma)(y+\gamma)4y(y-1)}\right]Z(y)=0\ . \tag{24}$$

We consider this equation around the singular points (note that the points $y = 0, \pm\gamma(|\ \gamma\ | << 1)$ are located outside the physical range of the variable):

$$y \sim 1 \ \ (z \to 0), \quad \left[\frac{d^2}{dy^2} + \left(\frac{1}{2}\frac{1}{y-1}\right)\frac{d}{dy} + \frac{\epsilon}{4(y-1)} + \frac{b\gamma - \Lambda}{(1-\gamma^2)4(y-1)}\right] Z(y) = 0\ ,$$

$$Z(y) = \exp\left[\pm i\sqrt{A(y-1)}\right],\ A = \epsilon + \frac{b\gamma - \Lambda}{1-\gamma^2};$$

$$y \sim \infty \qquad (z \to \pm\infty), \quad \left(\frac{d^2}{dy^2} + \frac{2}{y}\frac{d}{dy} + \frac{\epsilon}{4y^2}\right) Z(z) = 0, \quad Z = y^D\ ,\ \ D = \frac{-1 \pm i\sqrt{\epsilon - 1}}{2}\ .$$

Further progress in analytical treatment of eq. (24) (with 5 singular points) is hardly possible.

4 Solution of the radial equation

Let us turn to the radial equation (19) for the function $R(r)$

$$\left[\frac{d^2}{dr^2} + \frac{\mathrm{ch}\ r}{\mathrm{sh}\ r}\frac{d}{dr} - \frac{[m - b(\mathrm{ch}\ r - 1)]^2}{\mathrm{sh}^2 r} + \Lambda\right] R(z) = 0\,. \tag{25}$$

After the change of the variable

$$\cosh r - 1 = -2y,\ y = -\sinh^2\frac{r}{2} \in (-\infty, 0], \tag{26}$$

eq. (25) reads (for definiteness, we assume that the field is oriented along the the negative direction of the axis $z:\ b = -B, B > 0$)

$$\left[y(1-y)\frac{d^2}{dy^2} + (1-2y)\frac{d}{dy} - \frac{1}{4}\left(\frac{m^2}{y} - 4B^2 + \frac{(m-2B)^2}{1-y}\right) - \Lambda\right] R = 0. \tag{27}$$

We find the behavior of R near singular points 0 and 1 (note that the point $y = 1$ does not belong to the physical range of the coordinate y):

$$y \longrightarrow 0\ ,\ \ R(y) \sim y^a,\ a = \pm\frac{m}{2}; \quad y \longrightarrow 1\ ,\ \ R(y) \sim (1-y)^b,\ b = \pm\frac{m-2B}{2}.$$

Making the substitution $R = y^a(1-y)^b F$, at $a = \pm m/2,\ b = \pm(m-2B)/2$, we obtain the equation

$$R = z^a(1-z)^b F, \quad y(1-y)F'' + [(2a+1) - 2(a+b+1)y]F'$$
$$-[a(a+1) + 2ab + b(b+1) - B^2 + \Lambda]F = 0,$$

which is the equation of hypergeometric type

$$y(1-y)F + [\gamma - (\alpha + \beta + 1)y]F' - \alpha\beta F = 0\ .$$

Thus, the solutions are constructed as follows: (bound states correspond to positive values of the parameter a and negative values of the parameter b):

$$z = -\sinh^2\frac{r}{2}\ , \quad z \in (-\infty, +0]\ , \quad R = (-\sinh\frac{r}{2})^{|m|}(\cosh\frac{r}{2})^{-|m-2B|}F(\alpha, \beta, \gamma, -\sinh^2\frac{r}{2})\ , \tag{28}$$

where α and β are defined by the following expressions

$$a = +\frac{\mid m \mid}{2}\ ,\ b = -\frac{\mid m - 2B \mid}{2}\ ,\quad \gamma = 2a + 1 = + \mid m \mid + 1\ ,$$
$$\alpha = a + b + \frac{1}{2} - \sqrt{B^2 + \frac{1}{4} - \Lambda}\ ,\quad \beta = a + b + \frac{1}{2} + \sqrt{B^2 + \frac{1}{4} - \Lambda}\ . \tag{29}$$

Detailed study shows that here we have finite series of bound states described by the relation

$$m < 2B, \qquad \frac{m+ \mid m \mid}{2} + n + 1/2 \leq B, \qquad n = 0, 1, \ldots, N_B,$$
$$\Lambda - 1/4 = 2B\left(\frac{m+ \mid m \mid}{2} + n + 1/2\right) - \left(\frac{m+ \mid m \mid}{2} + n + \frac{1}{2}\right)^2\ . \tag{30}$$

In usual units the last relation can be written as:

$$\Lambda - \frac{1}{4} = \rho^2\Lambda_0 - \frac{1}{4}\ ,\quad \lim_{\rho\to\infty}\Lambda_0 = \frac{2M}{\hbar^2}(E - \frac{P^2}{2M}),\quad m < 2B,\ m + n + 1/2 \leq \frac{eB}{\hbar c}\rho^2\ ,$$
$$\rho^2\Lambda_0 - \frac{1}{4} = 2\frac{eB}{\hbar c}\rho^2(\frac{m+ \mid m \mid}{2} + n + 1/2) - (\frac{m+ \mid m \mid}{2} + n + 1/2)^2, n = 0, 1, \ldots, N_B. \tag{31}$$

In the limit of extinction curvature (31), we obtain the known result in flat space

$$E - \frac{P^2}{2M} = \frac{eB\hbar}{Mc}\ (\frac{m+ \mid m \mid}{2} + n + 1/2)\ . \tag{32}$$

To get the results for the opposite orientation of the magnetic field $B < 0$, it is necessary to use the symmetry of the original differential equation (25):

$$m \to m' = -m\ ,\qquad B \to B' = -B\ . \tag{33}$$

From (30) it follows $\Lambda - 1/4 = 2BN - N^2,\ n = 0, 1, \ldots, N_B$, where

$$\frac{1}{2} \leq N = \frac{m+ \mid m \mid}{2} + n + 1/2 \leq B;$$

therefore Λ obeys the restriction

$$B \leq \Lambda \leq B^2 + \frac{1}{4}\ . \tag{34}$$

5 Cox's particle in the electric field, Minkowski space

Schródinger equation for Cox's particle in the electric field has the form [2]

$$\left(D_t - c\ \frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1 + \Gamma^2 E_i E^i)}\right)\Psi = -\frac{1}{2M}\ \overset{\circ}{D}_k\ g^{kj}\left[\ D_j + \frac{\Gamma^2 E_j(E^i D_i) + \mu\Gamma E_j}{1 + \Gamma^2 E_i E^i}\right]\Psi; \tag{35}$$

the notation is used:

$$A_0 = -eEz,\qquad E_i = (F_{01}, F_{02}, F_{03})\ ,\quad g^{11}E_1 = E^1,\ g^{22}E_2 = E^2,\ g^{33}E_3 = E^3,$$
$$i\hbar\partial_t - eA_0 = D_t,\qquad i\hbar\partial_k = D_k,\quad \frac{i\hbar}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} = \overset{\circ}{D}_k\ .$$

Let us use cylindric coordinates

$$dS^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2,\quad E_3 = E,\ E^3 = -E,\ E_3E^3 = -E^2. \tag{36}$$

First, we get (let it be $\Gamma E = \gamma$)

$$\left(D_t - c\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)} \right) = i\hbar\partial_t + eEz + c\,\frac{\gamma^2\mu + \gamma D_3}{2(1-\gamma^2)}. \tag{37}$$

Next, we consider the Hamiltonian

$$H = \frac{1}{2M}\left[\overset{\circ}{D}_1\, D_1 + \overset{\circ}{D}_2\, \frac{1}{r^2} D_2 + \overset{\circ}{D}_3 \left(D_3 + \frac{\mu\gamma}{1-\gamma^2} \right) \right]. \tag{38}$$

In explicit form, the extended Schrödinger equation looks as follows (to allow for the imaginary character of γ, we make formal change $i\gamma \longrightarrow \gamma$)

$$\left(i\hbar\partial_t + eEz - \frac{Mc^2\gamma^2}{2(1+\gamma^2)} + \frac{\gamma}{2(1+\gamma^2)}\hbar c\partial_z \right)\Psi = \frac{-\hbar^2}{2M}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{\partial_\phi^2}{r^2} + \partial_z^2 - \frac{(mc/\hbar)\gamma}{1+\gamma^2}\partial_z \right)\Psi. \tag{39}$$

With the substitution $\Psi = e^{-iWt/\hbar} e^{im\phi} Z(z) R(t)$ and the notation

$$\frac{M^2c^2}{\hbar^2} = \frac{1}{\lambda^2}\ , \quad \frac{2M}{\hbar^2}W = w\ , \quad \frac{2M}{\hbar^2}\,eEz = \nu\ ,$$

we get

$$\frac{1}{Z(z)}\left(\partial_z^2 + \nu\, z + w - \frac{1}{\lambda^2}\frac{\gamma^2}{1+\gamma^2} \right) Z(z) + \frac{1}{R(r)}\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2} \right) R(r). \tag{40}$$

After separation of the variables ($w_\perp > 0$ stands for the separation constant) we derive

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2} + w_\perp \right) R(r) = 0, \tag{41}$$

$$(\partial_z^2 + \nu\, z + w')Z(z) = 0, \qquad w' = w - w_\perp + \frac{1}{\lambda^2}\frac{\gamma^2}{1+\gamma^2}. \tag{42}$$

In fact, (41) and (42) coincide with the well known equations for an ordinary particle in the uniform electric field. Equation in the variable z looks as a one-dimensional Schrödinger equation in the potential of the form $U(z) = -\nu\, z\ ,\ \nu > 0$:

$$(\frac{d^2}{dz^2} + w' + \nu\, z)Z(z) = 0. \tag{43}$$

The form of the curve $U(z)$ says that any particle moving from the right must be reflected by this barrier in vicinity of the point $z_0 = -\frac{w'}{\nu}$ (we assume that electric force acts in positive direction of the axis z).

Solutions of the equation (43) can be expressed in Airy functions. Indeed, in (43) let us change the variable

$$\nu z + w' = ax,\ (\frac{d^2}{dx^2} + \frac{a^3}{\nu^2}x)Z(x) = 0\ ;$$

let it be (for definiteness $\nu > 0$)

$$\frac{a^3}{\nu^2} = -1, \qquad a = -\nu^{2/3}, \quad x = \frac{\nu z + w'}{-\nu^{2/3}} = -\nu^{1/3}z - \frac{w'}{\nu^{2/3}}; \tag{44}$$

then we arrive at the Airy equation

$$\left(\frac{d^2}{dx^2} - x\right) Z(x) = 0 \; ; \tag{45}$$

to the turning point $z_0 = -w'/\nu$ there corresponds the value $x_0 = 0$. Eq. (45) can be related to the Bessel equation. Indeed, let us introduce the variable

$$\xi = \frac{2}{3}x^{3/2} \; , \qquad x = \frac{3}{2}\xi^{2/3} \; , \tag{46}$$

then Airy equation gives

$$(\frac{1}{3\xi}\frac{d}{d\xi} + \frac{d^2}{d\xi^2} - 1)Z = 0 \; .$$

Applying the substitution $Z = \xi^{1/3} f(\xi)$, we arrive at the Bessel equation

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - 1 - \frac{1/9}{\xi^2}\right) f(\xi) = 0 \tag{47}$$

with two solutions

$$f_1(\xi) = J_{+1/3}(i\xi) \; , \; f_2(\xi) = J_{-1/3}(i\xi) \; . \tag{48}$$

Thus, general solutions of Airy equation can be constructed as linear combinations of

$$Z_1(x) = \xi^{1/3} J_{+1/3}(i\xi) \; , \;\; Z_2(x) = \xi^{1/3} J_{-1/3}(i\xi) \; , \quad i\xi = i\frac{2\sqrt{\nu}}{3} \; (z + \frac{w'}{\nu})^{3/2} \; . \tag{49}$$

Far on the right from the turning point we have imaginary infinity and far on the right from the turning point we have real infinity respectively:

$$i\xi = i\frac{2\sqrt{\nu}}{3}(z + \frac{w'}{\nu})^{3/2} \sim +i\infty \quad \text{and} \quad i\xi = i\frac{2\sqrt{\nu}}{3}(z + \frac{w'}{\nu})^{3/2} \sim \infty.$$

With the use of the known relation

$$J_\mu(y) = \frac{(y/2)^\mu}{\Gamma(\mu+1)} \; e^{-iy} \; {}_1F_1(\mu + \frac{1}{2}, 2\mu + 1, 2iy)$$

and with the notation $y = i\xi, \; \mu = +1/3, -1/3$, one expresses two independent solutions of the Schrödinger equation as follows

$$Z_1 = \xi^{1/3} J_{+1/3}(i\xi) = \xi^{+1/3} \frac{(i\xi/2)^\mu}{\Gamma(\mu+1)} e^{\xi}_1 F_1(+\mu + \frac{1}{2}, +2\mu + 1, -2\xi) \; ,$$
$$Z_2 = \xi^{1/3} J_{-1/3}(i\xi) = \xi^{1/3} \frac{(i\xi/2)^{-\mu}}{\Gamma(-\mu+1)} \; e^{\xi}_1 F_1(-\mu + \frac{1}{2}, -2\mu + 1, -2\xi) \; .$$

6 Cox's particle in the electric field in the Lobachevsky model

We determine the concept of generalized electric field in the special system of cylindric coordinates in the curved space as follows

$$dS^2 = dt^2 - \mathrm{ch}^2 z(dr^2 + \mathrm{sh}^2 r d\phi^2) - dz^2, \quad \sqrt{-g} = \mathrm{sh}\, r\, \mathrm{ch}^2 z,$$
$$A_0 = -E\rho \tanh z, \quad E_3 = \frac{E}{\cosh^2 z},\ E^3 = -\frac{E}{\cosh^2 z}, \quad E_3 E^3 = -E^2\ \cosh^{-4} z. \tag{50}$$

Below we use operators (the coordinate $tc/\rho \longrightarrow t$ is dimensionless)

$$i\frac{\hbar c}{\rho}\partial_t - eA_0 = D_t,\ i\frac{\hbar}{\rho}\partial_k = D_k,\ \frac{i\hbar/\rho}{\sqrt{-g}}\frac{\partial}{\partial x^k}\sqrt{-g} = \overset{\circ}{D}_k\ .$$

We start with the extended Schrödinger equation [2] (let it be $E^2\ \cosh^{-4} z = -\gamma^2(x)$)

$$\left(D_t - c\,\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)}\right)\ \Psi = \frac{1}{2M\rho^2} = \overset{\circ}{D}_k\ (-g^{kj})\left(D_j + \frac{\Gamma^2 E_j (E^i D_i) + \mu\Gamma E_j}{1+\Gamma^2 E_i E^i}\right)\Psi.$$

Allowing for relations

$$\left(D_t - c\,\frac{\Gamma^2 E_i E^i \mu + \Gamma E^j D_j}{2(1+\Gamma^2 E_i E^i)}\right) = \frac{\hbar c}{\rho}\left(i\partial_t + \frac{eE\rho}{\hbar c/\rho}\tanh z + \frac{1}{2}\frac{Mc\rho}{\hbar}\frac{\gamma^2(z)}{1-\gamma^2(z)} + \frac{1}{2}\frac{\gamma(z)}{1-\gamma^2(z)} i\partial_z\right),$$

and

$$H = -\frac{\hbar^2}{2M\rho^2}\left[\frac{1}{\cosh^2 z}\left(\partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r + \frac{\partial_\phi^2}{\sinh^2 r}\right) + (\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\left(\partial_z - \frac{Mc\rho}{\hbar}\frac{i\gamma(z)}{1-\gamma^2(z)}\right)\right],$$

we get an explicit form of the extended Schrödinger equation

$$\frac{\hbar c}{\rho}\left(i\partial_t + \frac{eE\rho}{\hbar c/\rho}\tanh z + \frac{1}{2}\frac{Mc\rho}{\hbar}\frac{\gamma^2(z)}{1-\gamma^2(z)} + \frac{1}{2}\frac{\gamma(z)}{1-\gamma^2(z)} i\partial_z\right)\Psi$$
$$= -\frac{\hbar^2}{2M\rho^2}\left[\frac{1}{\cosh^2 z}\left(\partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r + \frac{\partial_\phi^2}{\sinh^2 r}\right) + (\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\left(\partial_z - \frac{Mc\rho}{\hbar}\frac{i\gamma(z)}{1-\gamma^2(z)}\right)\right]\Psi;$$

Specially note that two terms proportional to $i\gamma(z)\partial_z$ compensate each other. Additionally, we should perform formal change $i\gamma \longrightarrow \gamma$:

$$\frac{\hbar c}{\rho}\left(i\partial_t + \frac{eE\rho}{\hbar c/\rho}\tanh z - \frac{1}{2}\frac{Mc\rho}{\hbar}\frac{\gamma^2(z)}{1+\gamma^2(z)}\right)\Psi$$
$$= -\frac{\hbar^2}{2M\rho^2}\left[\frac{1}{\cosh^2 z}\left(\partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r + \frac{\partial_\phi^2}{\sinh^2 r}\right)\right.$$
$$\left.+(\partial_z + 2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z})\partial_z - \frac{Mc\rho}{\hbar}\left(\frac{\partial}{\partial z}\frac{\gamma(z)}{1+\gamma^2(z)}\right) - \frac{Mc\rho}{\hbar}\frac{\gamma(z)}{1+\gamma^2(z)}2\frac{\mathrm{sh}\ z}{\mathrm{ch}\ z}\right]\Psi; \tag{51}$$

With the use of substitution

$$\Psi = \exp(-iwt)\ e^{im\phi}R(r)Z(z), \quad w = \frac{W\rho}{\hbar c},$$

and notation

$$W = w\frac{\hbar c}{\rho}\frac{1}{\hbar^2/2M\rho^2} = 2w\frac{M\rho c}{\hbar}, \quad \nu = \frac{eE\rho}{\hbar^2/2M\rho^2}\ , \quad \frac{1}{2}Mc^2\frac{1}{\hbar^2/2M\rho^2} = \frac{M^2\rho^2c^2}{\hbar^2} = \mu^2,$$

we get

$$\cosh^2 z \left(W + \nu \ \tanh z - \frac{\mu^2 \gamma^2(z)}{1+\gamma^2(z)} \right) R(r)Z(z) + \left(\partial_r^2 + \frac{\cosh r}{\sinh r}\partial_r - \frac{m^2}{\sinh^2 r} \right) R(r)Z(z)$$
$$+ \cosh^2 z \left[(\partial_z + 2\frac{\text{sh } z}{\text{ch } z})\partial_z - \mu \left(\frac{\partial}{\partial z} \frac{\gamma(z)}{1+\gamma^2(z)} \right) - \mu \frac{\gamma(z)}{1+\gamma^2(z)} 2\frac{\text{sh } z}{\text{ch } z} \right] R(r)Z(z) = 0 \, . \tag{52}$$

In this equation one can separate the variables

$$\left(\frac{d^2}{dr^2} + \frac{\cosh r}{\sinh r}\frac{d}{dr} - \frac{m^2}{\sinh^2 r} + \Lambda \right) R = 0, \tag{53}$$

$$\left[\frac{d^2}{dz^2} + 2\frac{\text{sh } z}{\text{ch } z}\frac{d}{dz} - \mu \left(\frac{d}{dz} \frac{\gamma(z)}{1+\gamma^2(z)} \right) - \mu \frac{\gamma(z)}{1+\gamma^2(z)} 2\frac{\text{sh } z}{\text{ch } z} \right.$$
$$\left. + W + \nu \ \tanh z - \mu^2 \frac{\gamma^2(z)}{1+\gamma^2(z)} - \frac{\Lambda}{\text{ch}^2 z} \right] Z = 0 \, ; \tag{54}$$

remember that $\gamma(z) = \gamma \ \text{ch}^{-2} z$. The most interesting is the equation in variable z. After elementary transformation it is reduced to the form

$$\left(\frac{d^2}{dz^2} + 2\frac{\text{sh } z}{\text{ch } z}\frac{d}{dz} - 2\mu\gamma \ \text{sh } z \ \text{ch } z \frac{-\text{ch}^4 z + \gamma^2}{(\text{ch}^4 z + \gamma^2)^2} - 2\mu\gamma \ \frac{\text{sh } z \ \text{ch } z}{\text{ch}^4 z + \gamma^2} \right.$$
$$\left. + W + \nu \ \tanh z - \frac{\mu^2 \gamma^2}{\text{ch}^4 z + \gamma^2} - \frac{\Lambda}{\text{ch}^2 z} \right) Z) = 0 \, . \tag{55}$$

This final equation turns out to be very complex and it hardly can be solved analytically.

7 Solving the radial equation in the Lobachewsky space

In eq. (53) let us introduce a new b=variable $x = (1 + \cosh r)/2, \ x \in [1, \ +\infty)$, so that

$$x\,(1-x)\,\frac{d^2R}{dx^2} + (1-2\,x)\,\frac{dR}{dx} - \left(w_\perp + \frac{1}{4}\,\frac{m^2}{x} + \frac{1}{4}\,\frac{m^2}{1-x} \right) R = 0\,; \tag{56}$$

with the substitution $R = x^a\,(1-x)^b\,F$ at $a = \pm \mid m \mid /2\,, \ b = \pm \mid m \mid /2$ we obtain the hypergeometric equation

$$x\,(1-x)\,\frac{d^2F}{dx^2} + [2\,a + 1 - (2\,a + 2\,b + 2)\,x]\ \frac{dF}{dx}\ - [(a+b)\,(a+b+1) + w_\perp]\ F = 0\,,$$

with parameters

$$F = F(\alpha, \beta, \gamma; x), \qquad \alpha = a + b + \frac{1}{2} - i\sqrt{w_\perp - 1/4},$$
$$\beta = a + b + \frac{1}{2} + i\sqrt{w_\perp - 1/4}, \quad w_\perp > \frac{1}{4}\,, \qquad \gamma = 2\,a + 1\,. \tag{57}$$

We will specify solutions tending to zero at $r = 0$:

$$F = u_2 = F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x)\ ; \tag{58}$$

when a and b take positive values $a = + \mid m \mid /2\,,\ b = + \mid m \mid /2\,$; the complete radial function is

$$R = x^a (1-x)^b F(\alpha, \beta, \alpha+\beta+1-\gamma, 1-x). \tag{59}$$

To find behavior at infinity, $r \to +\infty$, one should apply the following Kummer relationship

$$u_2 = \frac{\Gamma(\alpha+\beta+1-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\ u_3 + \frac{\Gamma(\alpha+\beta+1-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\ u_4\,, \tag{60}$$

$$u_2 = F(\alpha, \beta, \alpha+\beta+1-\gamma; 1-x)\ ,$$
$$u_3 = (-x)^{-\alpha} F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \frac{1}{x})\ ,$$
$$u_4 = (-x)^{-\beta} F(\beta, \beta+1-\gamma, \beta+1-\alpha, \frac{1}{x})\ .$$

Therefore, asymptotic behavior at $x \to 1$ ($r \to +\infty$) is given by

$$R \approx (-1)^{a+b} \Gamma(\alpha+\beta+1-\gamma)$$
$$\times \left(\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\ (-x)^{a+b-\alpha} + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\ (-x)^{a+b-\beta} \right). \tag{61}$$

From this it follows

$$x \approx \frac{e^r}{4}\,,\quad R \approx (-1)^{a+b} \Gamma(\alpha+\beta+1-\gamma)(-x)^{-1/2}$$
$$\times \left(\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+1-\gamma)\Gamma(\beta)} e^{-i\pi\alpha}\ (-x)^{+i\sqrt{\lambda-1/4}} + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1-\gamma)\Gamma(\alpha)} e^{-i\pi\beta}\ (-x)^{-i\sqrt{\lambda-1/4}} \right)\ . \tag{62}$$

Thus, constructed solutions represent standing radial waves. The factor $e^{-r/2}$ is not significant for probability interpretation $dW = \sqrt{-g}\ \psi^* \psi$, and the term $e^{-r/2}$ will be compensated by the factor $\sinh r \approx e^{+r}/2$ entering the volume element $dV = \sqrt{-g}\ dr\ dz\ d\phi$.

8 Conclusion

Assumed by Cox [1] not point-like structure for a relativistic spin zero particle provides us with a highly modified non-relativistic Schrödinger equation for such a particle This non-relativistic wave equation has been investigated in presence of the external uniform magnetic and electric field in the case of Minkowski space. Extension of these problems to the case of open hyperbolic Lobachevsky 3-space is given.

The author is grateful to V.V. Kisel, E.M. Ovsiyuk, and V.M. Red'kov for advices and help.

References

[1] W. Cox. Higher-rank representations for zero-spin field theories. J. Phys. Math. Gen. 1982. Vol. 15, no 2. P. 627–635.

[2] E.M. Ovsiyuk. Spin zero Cox's particle with an intrinsic structure: general analysis in external electromagnetic and gravitational fields. Ukr. J. Phys. 2015. Vol. 60. P. 485–496.

[3] K.V. Kazmerchuk, E.M. Ovsiyuk, Cox's particlele in m agnetic and electric field against the background of Euclidean and spherical geometries. Ukr. Phys. J. 2015. Vol. 60. P. 389–400.

[4] V.M. Redkov, E.M. Ovsiyuk. Quantum Mechanics in space of constant curvature. – Nova Science Publishers: New York, 2012.

Varying fundamental physical constants – dark energy, dark matter and strange stars

Hidezumi Terazawa

terazawa@mrj.biglobe.ne.jp

Center of Asia and Oceania for Science(CAOS)
3-11-26 Maesawa, Higashi-kurume, Tokyo 203-0032, Japan

Midlands Academy of Business & Technology(MABT),
Mansion House, 41 Guildhall Lane, Leicester LE1 5FR, United Kingdom

Abstract

A theory of special inconstancy, in which some fundamental physical constants such as the fine-structure and gravitational constants may vary, is proposed in pregeometry. In the special theory of inconstancy, the $\alpha - G$ relation of $\alpha = 3\pi/[16\ln(4\pi/5GM_W^2)]$ between the varying fine-structure and gravitational constants (where M_W is the charged weak boson mass) is derived from the hypothesis that both of these constants are related to the same fundamental length scale in nature. Furthermore, it leads to the prediction of $\dot{\alpha}/\alpha = (-0.8 \pm 2.5) \times 10^{-14} yr^{-1}$ from the most precise limit of $\dot{G}/G = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$ by Thorsett, which is not only consistent with the recent observation of $\dot{\alpha}/\alpha = (0.5 \pm 0.5) \times 10^{-14} yr^{-1}$ by Webb *et al.* but also feasible for future experimental tests. In special inconstancy, the past and present of the Universe are explained and the future of it is predicted, which is quite different from that in the Einstein theory of gravitation. Also, a theory of general inconstancy, in which any fundamental physical constants may vary, is proposed in "more general relativity", by assuming that the space-time is "environment-dependent". In the general theory of inconstancy, the $G - \Lambda$ relation between the varying gravitational and cosmological constants is derived from the hypothesis that the space-time metric is a function of τ, the "environment coordinate", in addition to x^μ, the ordinary space-time coordinates. Furthermore, it leads to the prediction of the varying cosmological constant, which is consistent with the present observations. In addition, the latest observation of spatial variation in the fine-structure constant from VLT/UVES of $(1.1 \pm 0.2) \times 10^{-6} GLyr^{-1}$ by King *et al.* is suggested to be taken as a clear evidence for environment-dependent fundamental physical constants. Finally, simple solutions to the most intriguing problems in current cosmology, the problems of the dark energy and dark matter, are proposed in my theory of general inconstancy and in the so-called Bodmer-Terazawa-Witten hypothesis of strange matter.

I. Introduction

Is a physical constant really constant? In 1937, Dirac [1] discussed possible time variation in the fundamental constants of nature. He made not only the large number hypothesis(LNH) but also, as a consequences of the LNH, the astonishing prediction that the gravitational constant G varies as a function of time. Since then, Jordan [2] and many others[3,4] have tried to construct new theories of gravitation or general relativity in order to accomodate such a time-varying G. Although the LNH has been inspiring many theoretical developements and has recently led

myself [5] to many new large number relations, the prediction of the varying G has not yet received any experimental evidence. Recently, Thorsett [6] has shown that measurements of the masses of young and old neutron stars in pulsar binaries lead to the most precise limit of

$$\dot{G}/G = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$$

at the 68% confidence level.

More recently, on the other hand, Webb *et al.* [7] have investigated possible time variation in the fine structure constant α by using quasar spectra over a wide rage of epoches, spanning redshifts $0.2 < z < 3.7$, in the history of our Universe, and derived the remarkable result of

$$\dot{\alpha}/\alpha = (6.40 \pm 1.35) \times 10^{-16} yr^{-1}$$

for $0.2 < z < 3.7$, which is consistent with a time-varying α. Note, however, that in 1976 Shylakhter [8] obtained the very restrictive limit of $|\Delta\alpha/\alpha| < 10^{-7}$ or, more precisely,

$$\Delta\alpha/\alpha\Delta t = (-0.2 \pm 0.8) \times 10^{-17} yr^{-1}$$

for $z \sim 0.16$ (but over a narrower and latest range of epochs between now and about 1.8 billion years ago) from the "Oklo natural reactor". Very lately, Srianand *et al.* [9] have made a detailed many-multiplet analysis performed on a new sample of Mg II systems observed in high quality quasar spectra obtained using the Very Large Telescope and found a null result of $\Delta\alpha/\alpha = (-0.06 \pm 0.06) \times 10^{-5}$ for the fractional change in α or 3σ constraint of

$$-2.5 \times 10^{-16} yr^{-1} \leq (\Delta\alpha/\alpha\Delta t) \leq +1.3 \times 10^{-16} yr^{-1}$$

for $0.4 \leq z \leq 2.3$, which seems to be inconsistent with the result of Webb *et al.* [7]. However, a careful comparison of these different results [7-8] indicates that they are all consistent with a time-varying α as

$$\dot{\alpha}/\alpha = (0.5 \pm 0.5) \times 10^{-14} yr^{-1}$$

for $2.2 < z < 3.7$. More lately, Kanekar *et al.* have constrained the fundamental constant evolution with HI and OH lines as $\Delta\alpha/\alpha = (-1.7 \pm 1.4) \times 10^{-6}$ over a lookback time of 6.7Gyrs or, equivalently,

$$\dot{\alpha}/\alpha = (-2.5 \pm 2.1) \times 10^{-16} yr^{-1}$$

for $0 \leq z \leq 0.765$ [8].

In this paper, I am going to propose a theory of special inconstancy, in which some fundamental physical constants such as the fine-structure and gravitational constants may vary. In the special theory of inconstancy, the $\alpha - G$ relation of

$$\alpha = 3\pi/[16 \ln(4\pi/5GM_W^2)]$$

(where M_W is the charged weak boson mass) is derived from the hypothesis that both of α and G are related to the same fundamental length scale in nature. Furthermore, from the limit on $\dot{G}$ by Thorsett, it leads to the prediction of

$$\dot{\alpha}/\alpha = (-0.8 \pm 2.5) \times 10^{-14} yr^{-1}$$

which is not only consistent with the above result on $\dot{\alpha}$ but also feasible for future experimental tests. In special inconstancy, the past and present of the Universe are explained and the future of it is predicted, which is quite different from that in the Einstein theory of gravitation. Also, a general theory of inconstancy, in which any fundamental physical constants may vary, is proposed

in "more general relativity", by assuming that the space-time is "environment-dependent". In the general theory of inconstancy, the "$G-\Lambda$ relation" between the varying gravitational and cosmological constants is derived from the hypothesis that the space-time metric is a function of τ, the "environment coordinate", in addition to x^μ, the ordinary space-time coodinates. Furthermore, it leads to the prediction of the varying cosmological constant, which is consistent with the present experimental observations.

I will organize this note as follows: in Section II, I will briefly review pregeometry in which a theory of special inconstancy is constructed. In Sections III and IV, I will present the theories of special and general inconstancy and their predictions, respectively. Finally in Section V, I will discuss future prospects by proposing simple solutions to the most intriguing problems in current cosmology, the problems of dark energy and dark matter, in my theory of general inconstancy and in the so-called Bodmer-Terazawa-Witten hypothesis of strange quark matter.

II. Pregeometry

Pregeometry is a theory in which Einstein geometrical theory of gravity in general relativity can be derived from a more fundamental principle as an effective and approximate theory at low energies (or at long distances). In 1967, Sakharov [10] suggested possible approximate derivation of the Einstein-Hilbert action from quantum fluctuations of matter. A decade later, we [11] demonstrated that not only Einstein theory of gravity in general relativity but also the standard model of strong and electroweak interactions in quantum chromodynamics and in the unified gauge theory can be derived as an effective and approximate theory at low energies from the more fundamental unified composite model of all fundamental particles and forces [12].

Let us explain what pregeometry means more explicitly in a simple model of

$$S_0 = \int d^4x\sqrt{-g}L_0(g_{\mu\nu}(x), A_\mu(x), \phi_i(x))$$

where $g_{\mu\nu}$ is the space-time metric, $g = det(g_{\mu\nu})$, A_μ is an Abelian gauge field, and ϕ_i $(i = 1 \sim n)$ are n complex scalar fields of matter with the charge e. The fundamental Lagrangian L_0 consists of the gauge-invariant kinetic terms of the matter fields only as

$$L_0 = g^{\mu\nu}[\partial_\mu + iA_\mu)\phi_i^\dagger](\partial_\nu - iA_\nu)\phi_i - F^{-1}$$

(where F is an arbitrary constant) but does not contain either the kinetic term of the space-time metric or that of the gauge field so that both of $g_{\mu\nu}$ and A_μ are auxiliary fields. The effective action for the space-time metric and gauge field can be defined by the path-integral over the matter fields as

$$exp(iS_{eff}) = \int \prod_i [d\phi_i^\dagger][d\phi_i] exp(iS_0)$$

and it can be expressed formally as

$$S_{eff} = -iTrln[(\partial_\nu - iA_\nu)\sqrt{-g}g^{\mu\nu}(\partial_\mu + iA_\mu)] - \int d^4x\sqrt{-g}F^{-1}$$

after the path-integration over ϕ_i. For small scalar curvature R and Ricci curvature tensor $R_{\mu\nu}$, the effective action can be calculated to be

$$S_{eff} = \int d^4x\sqrt{-g}[2\lambda + (1/16\pi G)R + c(R^2 + dR^{\mu\nu}R_{\mu\nu}) + (1/4e^2)F^{\mu\nu}F_{\mu\nu} + ...]$$

with

$$2\lambda = [n\Lambda^4/8(4\pi)^2] - F^{-1},$$

$$(1/16\pi G) = n\Lambda^2/24(4\pi)^2,$$
$$c = nln\Lambda^2/240(4\pi)^2,$$
$$d = 2,$$

and

$$(1/4e^2) = nln\Lambda^2/3(4\pi)^2,$$

where λ and Λ are the cosmological constant and the momentum cut-off of the Pauli-Villars type, respectively. Notice that the arbitrary constant F^{-1} plays a role of counter term so that the cosmological constant may become as small as or as large as it is observed. Notice also that the momentum cut-off Λ must be of order of the Planck mass $G^{-1/2}(\sim 10^{19}GeV)$. Furthermore, not only the R^2 and $R^{\mu\nu}R_{\mu\nu}$ terms but also the remaining terms in the expansion of S_{eff} are practically negligible. This complete a simple demonstration that not only the Einstein-Hilbert action of gravity but also the Maxwell action of electromagnetism in general relativity can be derived as an effective and approximate theory at low energies from the simple model in pregeometry, provided that there exists a natural momentum cut-off at around the Planck mass in nature [13].

One of the most remarkable consequences of pregeometry is the $\alpha - G$ relation, a simple relation between the fine-structure and gravitational constant, which can be easily derived from the results for α and for G by eliminating the momentum cut-off Λ. In our unified quark-lepton model of all fundamental forces [14,15], the $\alpha - G$ relation is given by [16]

$$\alpha = 3\pi/\sum_i Q_i^2 ln(12\pi/nGm_i^2),$$

where Q_i and m_i are the charge and mass of quarks and leptons, respectively. For three generations of quarks and leptons and their mirror- or super-partners, the $\alpha - G$ relation simply becomes

$$\alpha \cong 3\pi/16ln(4\pi/5GM_W^2)$$

where M_W is the charged weak boson mass. Notice that this $\alpha - G$ relation is very well satisfied by the experimental data of $\alpha \cong 1/137$, $G^{-1/2} \cong 1.22 \times 10^{19}GeV$, and $M_W \cong 80.4GeV$ [17].

III. Special Inconstancy

Special inconstancy is a principle in which some fundamental physical constants such as the fine-structure and gravitational constants may vary. Let us first make it clear that in this note we use the natural unit system of $h/2\pi = c = 1$ (where h is the Planck constant and c is the speed of light in vacuum). Notice, however, that it does not mean that in discussing the relevant possibility of the varying fine-structure and gravitational constants [18], we exclude another intriguing possibility of the varying light velocity recently discussed by some authors [19] since varying either h or c is inevitably related to varying the fine-structure constant α ($\equiv e^2/2hc$)(if the unit charge e stays constant). It simply means that we must set up a certain reference frame on which we can discuss whether physical quantities such as the fine-structure, gravitational, and cosmological [20] constants be really constant. Our basic hypothesis is that both of the fine-structure and gravitational constants are related to the more fundamental length scale of nature as in the unified (pregauge [21] and) pregeometric [10-12] theory (or "pregaugeometry" in short) of all fundamental forces [14,15] reviewed in the last Section.

To be more explicit, in the simple model of pregaugeometry discussed in the last Section, assert that

$$< (\partial_\mu + iA_\mu)\sqrt{-g}g^{\mu\nu}(\partial_\nu - iA_\nu)\phi_i >_\Lambda = 0,$$

$$g_{\mu\nu} = F < [(\partial_\mu + iA_\mu)\phi_i^\dagger](\partial_\nu - iA_\nu)\phi_i >_\Lambda,$$

and

$$A_\mu = (i/2) < [\phi_i^\dagger \partial_\mu \phi_i - (\partial_\mu \phi_i^\dagger)\phi_i]/(\phi_j^\dagger \phi_j) >_\Lambda,$$

where $<>_\Lambda$ denotes the expectation value in the space-time with the fundamental length scale parameter of Λ^{-1}. The first equation is the usual field equation for ϕ_i while the last two can be taken either as the "equations of motion" for $g_{\mu\nu}$ and A_μ, which can be derived from the fundamental action S_0, or as the "fundamental field equations", which can reproduce the effective Einstein-Hilbert-Maxwell action S_{eff} at low energies($\ll \Lambda$) or at long distances($\gg \Lambda^{-1}$).

The most important consequence of special inconstancy in pregaugeometry is the $\dot{\alpha} - \dot{G}$ relaton for the varying fine-structure and gravitational constant of

$$\dot{\alpha}/\alpha^2 = (16/3\pi)[(\dot{G}/G) + 2(\dot{M_W}/M_W)],$$

which can be derived from differentiating both hand sides of the $\alpha - G$ relation with respect to any parameter for varying fundamental physical constants. This immediately leads to the remarkable predictions of

$$\dot{\alpha}/\alpha = (-0.8 \pm 2.5) \times 10^{-14} yr^{-1}$$

for constant M_W and

$$\dot{M_W}/M_W = (0.5 \pm 1.2) \times 10^{-12} yr^{-1}$$

from the limit of $\dot{G}/G = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$ by Thorsett [6] and from the experimental data of $\dot{\alpha}/\alpha = (0.5 \pm 0.5) \times 10^{-14} yr^{-1}$ by Webb *et al.*[7]. The first prediction is not only consistent with the experimental data by Webb *et al.* [7] but also feasible for future experimental tests. The second prediction, however, seems too small to be feasible for experimental tests in the near future although such prediction for the possible varying particle masses seems extremely interesting at least theoretically. Notice that the varying M_W is perfectly possible through the varying electroweak gauge coupling constant g (which is related to the fine-structure constant in the standard unified electroweak gauge theory of Glashow-Salam-Weinberg [22]) and/or the varying vacuum expectation value of the Higgs scalar v (which is related to the momentum cut-off Λ in the unified composite model of the Nambu-Jona-Lasinio type for all fundamental forces [23]) since $M_W = gv/2$. We must mention, however, that neither one of these predictions may assert for the fine-structure constant α or the weak boson mass M_W to vary since both of these predicted values are consistent with zero within their errors.

Let us now first add that in some pregaugeometric model [24], the $\alpha - G$ relation is not of the type of $\alpha \sim 1/ln(1/GM^2)$ but of the type of $\alpha \sim GM^2$(where M is a parameter of mass dimension) so that the $\dot{\alpha}$-$\dot{G}$ relation becomes

$$\dot{\alpha}/\alpha = (\dot{G}/G) + 2(\dot{M}/M).$$

This type of relation predicts

$$\dot{\alpha}/\alpha = (-0.6 \pm 2.0) \times 10^{-12} yr^{-1}$$

for constant M and

$$\dot{M}/M = (0.3 \pm 1.0) \times 10^{-12} yr^{-1}$$

from the limit by Thorsett [6] and from the experimental data by Webb *et al.* [7]. We suspect, however, that the predicted value for $\dot{M}/M$ seems too small to be feasible for experimental tests in the near future although the one for $\dot{\alpha}/\alpha$ is consistent with the experimental data by Webb *et al.* [7]. We must also mention that both of these predicted values are consistent with zero

within their errors so that neither the fine-structure constant α nor the mass parameter M may be predicted to vary.

Next, remember that in the principle of special inconstancy we do not assert that physical constants may vary as a function of time but do that they may vary in general, depending on any parameters including the cosmological time, temperature, *etc.* What is the origin of varying the physical constants? The answer to this question may be related to the answer to another fundamental question: What is the origin of the fundamental length scale Λ^{-1} in nature? It can be spontaneous breakdown of scale-invariance in the Universe, which has been proposed by myself [5] for the last quater century. It can be the natural, dynamical, automatic, *a priori*, but somewhat "wishful-thinking" cut-off at around the Planck length $G^{1/2}$ where gravity would become as strong as electromagnetism, which was suggested by Landau [13] in 1955. It can also be due to the Kaluza-Klein extra dimension [25], which is supposed to be compactified at an extremely small length scale of the order of $G^{-1/2}$ or at a relatively large length scale of the order of $1/TeV$ recently emphasized by Arkani-Hamed *et al.* [26]. It seems, however, the most natural and likely that the origin of the fundamental length comes from the substructure of fundamental particles including quarks, leptons, gauge bosons, Higgs scalars, *etc.* [27,28]. In the unified composite model of all fundamental particles and forces [28], the fundamental energy scale Λ in pregaugeometry can be related to some even more fundamental parameters such as the masses of subquarks, the more fundamental constituents of quarks and leptons, and the energy scale in quantum subchromodynamics, the more fundamental dynamics confining subquarks into a quark or a lepton. In either way, the fundamental length scale Λ^{-1} can be identified with the size of quarks and leptons, the fundamental particles.

In pregeometric special inconstancy, let us now briefly explain the past and present of the Universe and try to predict the future of it, which may differ from that in the conventional Einstein theory of gravitation in general relativity [29]. The history of our Universe goes as follows: Long, long time ago there was no physical space-time, in which the space-time metric was finite and non-vanishing so that the distance was well defined, but the only matter "existed" in the mathematical space-time. Suddenly, there appeared the big bang of our Universe as a phase transition of the space-time from the pregeometric phase to the geometric one due to quantum fluctuations of matter, as suggested by us [30] in the early nineteen eighties, and our Universe had happened to be either flat or open. Then, not only all fundamental particles but also all fundamental forces between them were created and they started obeying the effective theory of all fundamental particles and forces including the Einstein theory of gravity with the non-vanishing and varying cosmological constant. In the earliest era during which the matter density had been extremely small, our Universe had been expanding almost exponentially. It had been the "almost inflationary Universe". In the next era of the radiation dominated Universe, our Universe was expanding less fast. Furthermore, in the last era of the matter dominated Universe, our Universe has still been expanding even faster. This history of our Universe is well simulated by a simple model of $(\Omega_m, \Omega_\lambda, -q) = (0, 1, 1)$, $(1/4, 3/4, 1/3)$, or $(1/4, 3/4, 1/2)$ for the early inflationary era, for the radiation dominated era, or for the matter dominated era, respectively, where Ω_m, Ω_λ, and q are the "pressureless-matter-density", "scaled cosmological constant", and deceleration parameter of the Universe, respectively. Notice that there must be another "phase transition" in which Ω_λ changed from 1 to 3/4 in between the early inflationary era and the radiation dominated era. Concerning the cosmological constant, I have been most impressed by the recent observation of the "farthest supernova ever seen" by Hubble Space Telescope [31]. "This supernova shows us the universe is behaving like a driver who slows down approaching a red stoplight and then hits the accelerator when the light turns green." Notice that this behavior of the Universe is what our model simulates. Note also that our model of the Universe is consistent with the recent measurement of the cosmological mass density

from clustering in the Two-Degree-Field Galaxy Redshift Survey [32] which strongly favors a low density Universe with $\Omega_m \cong 0.3$. Very recently, the CBI Collaboration [33,17] has found $\Omega_\lambda = 0.64 + 0.11/-0.14$ and $\Omega_m + \Omega_\lambda = 1.006 \pm 0.006$. More recently, the Wilkinson Microwave Anisotropy Probe (WMAP) team [34,17] has found 1) the first generation of stars to shine in the Universe first ignited only 200 million years after the big bang, 2) the age of the Universe(t_0) is 13.69 ± 0.13 billion years old, and 3) $\Omega_m = 0.26 \pm 0.02$ and $\Omega_\lambda = 0.74 \pm 0.03$.

The future of the Universe in our special inconstant picture can be quite different from that in the Einstein-Friedmann picture: 1) Since the cosmological constant may vary in special inconstancy, the space-time of our Universe which is almost flat and expanding faster and faster may not continue to be flat and accelerating forever. Our Universe may even encounter a "topological phae transition", which was first discussed by Wheeler [35] in 1959, from the open Universe to the closed one. 2) If the gravitational constant increases, the expansion of the space-time may not continue forever. The Universe may well stop expanding, start contracting, and even be bouncing forever. If G decreases, it will be more accelerated ever. 3) If the fine-structure constant (and/or other fundamental coupling constants such as the strong and weak coupling constants) varies, our Universe may encounter as "obsolete phase transition" from the matter-dominated Universe to the radiation-dominated one. In short, we can expect anything about the future of our Universe or, in other words, we can predict nothing definite on the destiny of our Universe.

IV. General Inconstancy

General inconstancy is a principle in which any fundamental physical constants may vary. In order to accomodate general inconstancy with general relativity, let us assume that the space-time is environment dependent. More explicitly, let us make the hypothesis that the space-time metric, $g_{\mu\nu}$, is a function of the "environment coordinate", τ, in addition to the ordinary space-time coodinates, x^μ, where $\mu, \nu = 0, 1, 2, 3$, so that the infinitesimal distance, ds, is given by

$$ds^2 = g_{\mu\nu}(x, \tau) dx^\mu dx^\nu.$$

Notice that the physical meaning of τ is arbitrary as it can be the temperature(T), the cosmological time(t), a parameter related to the Kaluza-Klein extra dimensions, or anything else. Notice also that τ may not be a continuous valuable but be a discrete number and that there may exist more than one "environment coordinates". I will consider such extensions of the environment-dependent space-time metric later.

In this "more general relativity", let us consider a simple model for general inconstancy given by the generalized Einstein-Hilbert action of

$$S_I = \int d\tau \int d^4x \sqrt{-g}[R(x, \tau) + ...] = \int d^4x \sqrt{-g}[2\lambda + (1/16\pi G)R(x) + ...].$$

In this equation we may find the equations of

$$2\lambda = \int d\tau \int d^4x \sqrt{-g}[R(x, \tau) + ...] / \int d^4x \sqrt{-g}$$

and

$$(1/16\pi G) = \int d\tau \int d^4x \sqrt{-g}[R(x, \tau) + ...] / \int d^4x \sqrt{-g} R(x)$$

so that we may obtain the $G - \Lambda$ relation of

$$32\pi G\lambda = \int d^4x \sqrt{-g} R(x) / \int d^4x \sqrt{-g}.$$

Futhermore, by differentiating both hand-sides of the relation with respect to τ, we can obtain the relation between the varying gravitaional and cosmological constants:

$$\dot{G}/G = -\dot{\lambda}/\lambda.$$

Since in the homogeneous and isotropic Einstein-Friedmann model of the Universe, the scalar curvature is given by $R = 6[(1-q)H^2 + (k/a^2)]$, where a is the scale parameter in the Robertson-Walker metric and $k = +1, 0, -1$ for the closed, flat, or open universe, the first relation explains the positivity of the cosmological constant, λ, for the present Universe with $-q > 0$ and, probably, $k = 0$. The second relation leads to the prediction of

$$\dot{\lambda}/\lambda = (0.6 \pm 2.0) \times 10^{-12} yr^{-1},$$

from the limit by Thorsett [6]. This prediction is not only consistent with the present experimental observation but also feasible for future experimental tests.

V. Dark Energy, Dark Matter, and Strange Stars – Future Prospects

In conclusion, let us first point out that not only continuous physical constants such as α and G but also discrete physical numbers such as the number of the space-time dimensions n, the number of quark colors N_c, the number of quark-lepton generations N_g, *etc.* may vary. In fact, an astonishing "dimensional phase transition", which was discussed by myself [36] about three decades ago, may be possible in the history of our Universe. If n is related to N_c as in the "space-color corespondence", which was proposed by myself about four decades ago [37], both of these fundamental physical natural numbers must vary simultaneously.

Before concluding this note, let us introduce two more recent reports on finding an evidence for environment-dependent fundamental physical constants in addition to many recent reports [38]. Very lately, Kanekar *et al.* have obtained a very strong constraint of $\Delta\alpha/\alpha = (-1.7 \pm 1.4) \times 10^{-6}$ over a lookback time of 6.7Gyrs with HI and OH lines[39]. Even more lately, Webb *et al.* have found spatial variation of the fine-structure constant with the amplitude of $(1.1 \pm 0.2) \times 10^{-6} GLyr^{-1}$ by using UVES on the VLT [40], which may be taken as a clear evidence for the environment-dependent fundamental physical constant.

Also, as an addendum to my latest note on cosmology in 2014[41,42], let me propose simple solutions to the most intriguing problems in current cosmology, the problems of the dark energy and the dark matter:

The problem of the dark energy consists of the two questions: 1) "Where does the cosmological constant λ come from?" and 2)"Why is it so large as or so small as $\Omega_\lambda = 1, 3/4$¿'. A simple answer to the first question is that it comes from the quantum fluctuations of matter as in pregeometry while another to the second is that it is arbitrary since it would not vanish without any miraculous cancellation between the quantum fluctuation and the constant F^{-1} in pregeometry as seen in Section II and since it is time-varying in general inconstancy as seen in Section IV. On the other hand, the problem of the dark matter can be simply solved by either one of the following possibilities: 1) It consists of exotic matter such as strange quark matter in the so-called Bodmer-Terazawa-Witten hypothesis that super-hypernuclear matter consisting of almost equal numbers of up, down, and strange quarks is stable [43, 44]. In fact, many obsevations of the possible candidates for strange stars consisting of strange quark matter have recently been reported by astronomical experiments [45,46], inducing many extensive theoretical investigations on strange stars [47]; 2) It consists of "color-balls"[48], the color-singlet complex objects consisting of an arbitrary number of gluons; 3) It consists of axions [49]; 4)It consists of WIMPs(Weakly Interacting Massive Particles) such as superpartners of the SM(Standard

Model) neutlinos, γ, and Z in supersymmetry [50]. Future astronomical experiments would tell us which one of these possibilities is a right answer to one of the most misterious questions in current cosmology.

Acknowledgement

The author would like to thank Professor Yu.A.Kurochkin and Professor V.M.Red'kov for inviting him to present a key note at the 9-th International Conference on Non-Euclidean Geometry in Physics and Mathematics, Minsk, Belarus, October 27-29, 2015, to which this key note has been contributed. He also wishes to thank Professor Hironari Miyazawa, the founder of many paradigms in particle physics including supersymmetry, to whom this paper has been dedicated for celebrating the eighty-eighth anniversary of birthday, for many useful helps which he had received since 1965 when he was a graduate student in the elementary-particle theory laboratory, Department of Physics, Faculty of Science, University of Tokyo.

References

[1] P.M.A. Dirac, Nature **139**, 323 (1937); Proc. R. Soc. London **A165**, 199 (1938); **A333**, 403 (1973); **A338**, 439 (1974).

[2] P. Jordan, Ann.der Phys. **6**, Folge **1**, 219 (1947). For the recent related article, see E.L. Schuking, Physic Today, October 1999, Vol. **52**, no. 10, p. 26 (1999).

[3] F. Hoyl and J.V. Narlikar, Nature **233**, 41 (1971).

[4] V. Canuto, P.J. Adams, S.-H. Hsieh, and E. Tsiang, Phys. Rev. D **16**, 1643 (1977).

[5] H. Terazawa, Phys. Lett. **101B**, 43(1981); in Proc. 3rd Alexander Friedmann International Seminar on Gravitation and Cosmology, St. Petersburg, 1995, edited by Yu.N. Gnedin, A.A. Grib, and V.M. Mostepanenko (Friedmann Laboratory Pub., St.Petersburg, 1995), p. 116; Mod. Phys. Lett. **A11**, 2971 (1996); **A12**, 2927 (1997); **A13**, 2801 (1998).

[6] S.E. Thorsett, Phys. Rev.Lett. **77**, 1432 (1996). See also V.M. Kaspi, J.H. Taylor, and M.F. Ryba, Astrophys. J. **42**, 713 (1996) and D.B. Guenther, L.M. Krauss, and P. Demarque, *ibid.* **498**, 871(1998). For a recent review, see J. Uzan, Rev. Mod.P hys. **75**, 403 (2003). More recently, Copi *et al.* have inferred $-4 \times 10^{-13} yr^{-1} < (\dot{G}/G)_{today} < 3 \times 10^{-13} yr^{-1}$ (by assuming a monotonic power law time dependence $G \propto t^{-a}$) from the big-bang nucleosynthesis constraint. See C.J. Copi, A.N. Davis, and L.M. Krauss, Phys. Rev. Lett. **92**, 171301 (2004). Very lately, Jofré *et al.* have estimated one of the most restrictive upper limits of $|\dot{G}/G| < 4 \times 10^{-12} yr^{-1}$ by using the surface temperature of the nearest millisecond pulsar, PSR J0437-4715, (inferred from ultraviolet observations) and by considering only modified Urca reactions. See P. Jofré, A. Reisenegger, and R. Fernández, Phys.Rev. Lett. **97**, 131102 (2006).

[7] J.K. Webb *et al.*, Phys. Rev. Lett. **82**, 884 (1999); *ibid.* **87**, 091301 (2001); M.T. Murphy, J.K. Webb, and V.V. Flambaum, Mon. Not. R. Astron. Soc. **345**, 609 (2003). Recently, de Souza has claimed that the velocity in which quasars evolve into normal galaxies make us to see the limit of quasars slightly Doppler shifted which axplains the "misleading" variation of the fine structure constant. See M.E. De Souza, astro-ph/0301085, 6 Jan 2003. However, it seems that his estimate of the shift is overestimated by ignoring the isotropy of quasar expansions. Also recently, J.N.Bahcall *et al.* have set a robust upper limit on $\dot{\alpha}$,

$\dot{\alpha}/\alpha < 10^{-13} yr^{-1}$, by using the strong nebular emission lines of O III. See J.N. Bahcall, C.L. Steinhardt, and D.Schlegel, Astrophys. J. **600**, 520 (2004). See also L.L. Cowie and A. Songaila, Astrophys. J. **453**, 596 (1995); R.L. White, R.H. Becker, X. Fan, and M.A.Strauss. Astron. J. **126**, 1 (2003); A. Songaila, astro-ph/0402347, to be published in Astron. J. For a recent review, see A. Songaila and L.L. Cowie, Nature **398**, 667 (1999). Very recently, Tzanavaris *et al.* have estimated the time variation of $x(= \alpha^2 g_p \mu$ where g_p is the proton g factor and μ is the electron/proton mass ratio, $m_e/m_p)$ from 21-cm and ultraviolet quasar absorption lines over a redshift range $0.24 < z < 2.04$ as $\dot{x}/x = (-1.43 \pm 1.27) \times 10^{-15} yr^{-1}$ for a linear fit. Using the $\Delta\alpha/\alpha$ results and assuming $\Delta g_p/g_p \sim 0$, they have obtained $\Delta\mu/\mu = (2.31 \pm 1.03) \times 10^{-5}$ and $\Delta\mu/\mu = (1.29 \pm 1.01) \times 10^{-5}$. See P.Tzanavaris *et al.*, Phys.Rev.Lett. **95**, 041301(2005). More recently, Reinhold *et al.* have obtained $\Delta\mu/\mu = (2.4 \pm 0.6) \times 10^{-5}$ for a weighted fit and $\Delta\mu/\mu = (2.0 \pm 0.6) \times 10^{-5}$ for an unweighted fit, which indicates, at a 3.5σ confidence level, that μ could have decreased in the past 12 Gyr. See E.Reinfold *et al.*, Phys.Rev.Lett. **96**, 151101(2006). Note, however, that the possible variation of μ may not necessarily mean that of the fundamental mass scale in nature but simply do that of the effective electron and/or proton masses due to different environmental parameters such as density, temperature, *etc.*

[8] A.I.Shylakhter, Nature **264**, 340(1976); ATOMKI Report no. A/1(1983); T. Damour and F. Dyson. Nucl. Phys. **B480**, 37 (1996); Y. Fujii *et al.*, *ibid.***B573**, 377 (2000), hep-ph/0205206. For the latest reexamination of the Oklo and other constrains, see Y. Fujii and A. Iwamoto, Phys. Rev. Lett. **91**, 241302 (2003). Very recently, Marion *et al.* have set a stringent upper bound on $\dot{\alpha}$, $\dot{\alpha}/\alpha = (-0.4 \pm 16) \times 10^{-16} yr^{-1}$, by comparing the hyperfine frequencies of Cs and Rb atoms. See H. Marion *et al.*, Phys. Rev. Lett. **90**, 150801 (2003). See also S. Bize *et al.*, *ibid.***90**, 150802 (2003); J.N. Chengalur and N. Kanekar, *ibid.***91**, 241302 (2003); T. Ashenfelter, G.J. Mathews, and K.A. Olive, *ibid.***92**, 041102 (2004). Even more recently, Fisher *et al.* have deduced a seperate limit of $\dot{\alpha}/\alpha = (-0.9 \pm 2.9) \times 10^{-15} yr^{-1}$ from combining the $1S - 2S$ transition frequency in H atom with the optical transition frequency in Hg^+ against the hyperfine splitting in Cs and the microwave Rb and Cs clock comparison. See M.Fisher, Phys.Rev.Lett. **92**, 230802(2004). Very lately, Peik *et al.* have measured an optical transition frequency at 688THz in $^{171}Yb^+$ with a cesium atomic clock at 2 times seperated by 28yr and find a value for the fractional variation of the freauency ratio f_{Yb}/f_{Cs} of $(-1.2 \pm 4.4) \times 10^{-15} yr^{-1}$, consistent with zero. Combined with recently published values for the constancy of other transition frequencies this measurement sets an upper limit on the present variability of α at the level of $2.0 \times 10^{-15} yr^{-1} (1\sigma)$ corresponding so far to the most stringent limit from laboratory experiments. See E. Peik *et al.*, Phys. Rev. Lett. **93**, 170801 (2004). For a recent brief review, see C. Seife, Science **306**, 793 (2004).

[9] H. Chand, R. Srianand, P. Petitjean, and B. Aracil, Astophs. **417**, 853 (2004); R. Srianand *ibid.*, Phys. Rev. Lett. **92**, 121302 (2004). For a latest comparison of this data with the Oklo data and the quasar data, see L.L. Cowie and A. Songaila, Nature **428**, 132 (2004).

[10] A.D. Sakharov, Dokl. Akad. Nauk SSSR **177**, 70 (1967) [Sov.Phys.JETP **12**, 1040 (1968)].

[11] K. Akama, Y. Chikashige, T. Matsuki, and H. Terazawa, Prog. Theor. Phys. **60**, 868 (1978); S.L. Adler, Phys. Rev. Lett. **44**, 1567(1980); A. Zee, Phys. Rev.D **23**, 858 (1981); D. Amati and G. Veneziano, Phys. Lett. **105B**, 358 (1981).

[12] For a review, see, for example, H. Terazawa, in Proc. 1st A.D.Sakharov Conference on Physics, Moscow, 1991, edited by L.V. Keldysh and V.Ya. Fainberg (Nova Science, New York, 1992), p. 1013.

[13] L. Landau, in **Niels Bohr and the Development of Physics**, edited by W.Pauli(Mc Graaw-Hill, New York, 1955), p.52. In 1977, we suggested abnormally strong gravity at short distances which breaks some presently reliable basic principles in quantum field theories such as special relativity and microscopic causality at a distance of the order of the Planck length. We also suggested that baryon number generation occurred in the very early Universe, breaking CPT invariance which is based on these basic principles. See H.Terazawa, M.Kasuya, and K.Akama, Report No. INS-Report-331 (INS, Univ. of Tokyo, 1979); H.Terazawa, in Proc. Workshop on Unified Theories and Baryon Number in the Universe, Tsukuba, 1979, edited by O.Sawada and A.Sugamoto (National Lab. for High Energy Physics, Tsukuba, 1979), p. 99; Gen.Relativ.Gravit. **12**, 93 (1980). For a recent idea similar to the latter suggestion, *i.e.*, dynamical breaking of CPT in an expanding Universe and, combined with baryon-number-violating interactions, driving the Universe towards an equilibrium baryon asymmetry that is observationally acceptable, see H.Davoudiasl *et al.*, Phys. Rev. Lett. **93**, 201301(2004). For a very recent search for signatures of Lorentz and CPT violations, see B. Feng *et al.*, Phys. Rev. Lett. **96**, 221302 (2006). Also, recently Calmet *et al.* have derived fundamental limits on measurements of position arising from quantum mechanics and classical general relativity and shown that any primitive probe or target used in an experiment must be larger than the Planck length. See X. Calmet, M. Graesser, and S.D.H. Hsu, Phys. Rev. Lett. **93**, 211101(2004). Very lately, Robinson and Wilczek have calculated the contribution of graviton exchange to the running of gauge couplings at lowest nontrivial order in pertubation theory and found that, when extrapolated formally, it renders all gauge couplings asymptotically free. See P.Robinson and F. Wilczek, Phys. Rev. Lett. **96**, 231601 (2006). Their results may lead to a modified version of the Landau conjecture that the effect of gravity suppresses all gauge couplings at extremely short distances.

[14] H. Terazawa, Y. Chikashige, and K. Akama, Phys.Rev. D **15**, 480 (1977).

[15] For reviews, see, for example, H.Terazawa, in Proc. 19th International Conf. on High Energy Physics, Tokyo, 1978, edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Phys. Soc. Japan, Tokyo, 1979), p. 617; in Proc. 22nd International conf. on High energy Physics, Leipzig, 1984, edited by A. Meyer and E. Wieczoek (Akademie der Wissenschaften der DDR, Zeuten, 1984), vol.I, p. 63.

[16] H. Terazawa, Y. Chikashige, K. Akama, and T. Matsuki, Phys. Rev.D **15**, 1181 (1977); H. Terazawa, *ibid.* **16**, 2373 (1977); **22**, 1037 (1980); **41**, 3541(E) (1990).

[17] J. Beringer *et al.* (Particle Data Group), Phys. Rev.D **86**, 010001 (2012), including the latest world averages of the cosmological data, $\Omega_m = 0.27 \pm 0.03$, $\Omega_\lambda = 0.73 \pm 0.03$, and $t_0 = 13.75 \pm 0.13 Gyr$.

[18] J.D. Bekenstein, Phys. Rev. D **25**, 1528 (1982); J.W. Moffat, Int. J. Mod. Phys. **D2**, 351 (1992); J.D. Barrow and J. Magueijo, Phys.Lett. **B443**, 104 (1998).

[19] M.A. Clayton and J.W. Moffat, Phys. Lett. **B460**, 263 (1999); A. Albrecht and J. Magueijo, Phys. Rev.D **59**, 043516(1999); J. D.Barrow, **ibid. 59**, 043515 (19999); P.P. Avelino and C.J.A.P. Martins, Phys. Lett. **B459**, 468 (1999); P.C.W. Davies *et al.*, Nature **418**, 603 (2002).

[20] A.Vilenkin, Phys. Rev. Lett. **81**, 5501 (1998); A.A. Starobinsky, JETP Lett. **68**, 757 (1998); L.O. Pimental and L.M. Diaz-rivera, Int. J. Mod. Phys. **A14**, 1523 (1999).

[21] J.D. Bjorken, Ann.Phys. (N.Y.) **24**, 174 (1963); H. Terazawa, Y. Chikashige, and K. Akama, Ref. [14].

[22] S.L. Glashow, Nucl. Phys. **22**, 579 (1961); A. Salam in **Elementary Particle Physics**, edited by N. Svartholm (Almqvist and Wilsell, Stockholm, 1968), p. 367; S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).

[23] Y. Nambu and G. Jona-Lasinio, Phys.Rev. **122**, 345 (1961); H. Terazawa, Y. Chikashige, and K. Akama, in Ref. [14].

[24] See, for example, H. Terazawa, Phys. Lett. **133B**, 57 (1983).

[25] T.Kaluza, Sitzker. Preuss. Akad. Wiss. **K1**, 966 (1921); O. Klein, Z. Phys. **37**, 896 (1926).

[26] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B429**, 263 (1998); Phys. Rev.D **59**, 086004 (1999); I. Antoniadis *et al.*, Phys. Lett. **B436**, 257 (1998); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[27] J.C. Pati and A. Salam, Phys. Rev.D **10**, 275 (1974); H. Terazawa, Y. Chikashige, and K. Akama, in Ref. [14]; H. Terazawa, Phys. Rev.D **22**, 184 (1980).

[28] For classical reviews, see, for example, H.Terazawa, in Ref. [15] and for more recent reviews, see for example, H.Terazawa, in Proc. International Conf. New Trends in High-Energy Physics, Alushta, Crimea, 2003, edited by P.N. Bogolyubov, L.L. Jenkovszky, and V.V. Magas (Bogolyubov Institute for Theoretical Physics, Kiev, 2003), Ukrainian J.Phys. **48**, 1292(2003); in Proc. XXII International Conf. on New Trends in High-Energy Physics, Alshuta, Crimea, 2011, edited by P.N. Bogolyubov and L.L. Jenkovszky (Bogolyubov Institute for Theoretical Physics, Kiev, 2011), p. 352.

[29] For recent more detailed reviews including many other current topics in cosmology, see, for example, H.Terazawa, in Proc. 11th Interational Conf. and School Foundations & Advances in Nonlinear Science, Minsk, Belarus, 2003, edited by V.I.Kuvshinov and G.G.Krylov (Belarusian State University, Minsk, Belarus, 2004), p. 99.

[30] K. Akama and H. Terazawa, Gen. Rel. Grav. **15**, 201 (1983).

[31] For a latest review, see C.Seife, Science **303**, 1271 (2004). Even more lately, an extremely small galaxy has been found with the image taken by the Very Large Telescope. For a brief review, see C.Seife, *ibid.* **303**, 1597 (2004).

[32] J.A. Peacock *et al.*, Nature **410**, 169 (2001). For a recent brief review, see, for example, A.Watson, Science **295**, 2341 (2002).

[33] J.L.S ievers *et al.* (CBI Collaboration), Astrophys.J. **591**, 599 (2003).

[34] C.L. Bennett *et al.*, Astrophys. J. Suppl. **148**, 1 (2003); D.N. Spergel *et al.*, *ibid.* **148**, 175 (2003). See also T. Reichhardt, Nature **421**, 777 (2003); S. Carroll, *ibid.* **422**, 26 (2003); G. Brumfiel *ibid.* **422**, 108 (2003). For a review, see B.Schwarzschild, Physics Today, Vol.**56**, no. 4, April 2003, p. 21.

[35] J.A. Wheeler, in Conference on the Role of Gravitation in Physics, Chapel Hill, North Carolina, 1959 (WADC Technical report 57-216, ASTIA Document No. AD118180; B.DeWitt, in Proc. Third Seminar on Quantum Gravity, Moscow, 1984, edited by M.A. Markov, V.A. Berezin, and V.P. Frolov (World Scientific, Singapore, 1985), p.103; V.G. Gurzadyan and

A.A. Kocharyan, Zh. Eksp. Teor. Fiz. **95**, 3 (1989) [Sov.Phys.JETP **8**, 1 (1989); Mod. Phys. Lett. **A4**, 50 (1989).

[36] H. Terazawa, in Quantum field Theory and Quantum Statistics, edited by I.A. Batalin and G.A. Vilkovisky (Adam Hilger, London, 1987), Vol.I, p.637; in Particles and Nuclei, edited by H. Terazawa (World Scientific, Singapore, 1986), p. 04; in Modern Problems of the Unified Field Theory, edited by D.V. Galt'sov, L.S. Kuzmenkov, and P.I. Pronin (Moscow State University, Moscow, 1991), p. 124 (in russian translation); in Proc. 5th Seminar on Quantum Gravity, Moscow, 1990, edited by M.A. Markov, V.A. Berezin, and V.P. Frolov (World Scientific, Singapore, 1991), p.643. For a recent similar idea of emergence of a four dimensional world from causal quantum gravity, see J. Ambjørn, J. Jurkierwicz, and R. Loll, Phys. Rev. Lett. **93**, 131301 (2004).

[37] H. Terazawa, in ref. [26]; in Wandering in the Fields, edited by K. Kawarabayashi and A. Ukawa (World Scientific, Singapore, 1987), p. 388.

[38] For a recent review, see J.C. erengut and V.V. Flambaum, Nuclear Physics News, Vol. **20**, no.3, p.19 (2010).

[39] N. Kanekar *et al.*, arXiv:1201.3372v1 [astro-ph.CO], Mon, 16 Jan 2012, accepted for publication in Astrophys.J. Letters (2012).

[40] J.K.Webb textitet al., Phys. Rev. Lett. **107**, 191101 (2011); J.A. King *et al.*, arXiv:1202.4758v1 [astro-ph.CO], Tue, 21 Feb 2012; Mon. Not. R. Astron. Soc. **422**, 3370 (2012). See also P. Avelino and L. Sousa, arXiv:1404.3419v3 [astro-ph.CO] 5 May 2015.

[41] H. Terazawa, in Proc. XXI International Seminar on Nonlinear Phenomena in Complex Systems, Minsk, Belarus, 2014, edited by L.F.Babichev, V.I.Kuvshinov, and V.A.Shaparau(National Academy of Sciences of Belarus, Minsk, Belarus, 2014), Nonlinear Dynamics and Applications. **20** (2014), p. 241, arXiv:1202.1859v13 [physics.gen-ph] 20 Apr 2015, and many references therein.

[42] For a recent review on experimental obsevations of varying fundamental physical constants, see Y.V. Stadnik and V.V. Flambaum, arXiv:1412.7801v4 [hep-ph] 21 Mar 2015; arXiv:1503.08540v3 [astro-ph.CO] 28 Apr 2015. For a recent review on observational probes of cosmic acceleration, see D.H. Weinberg *et al.*, arXiv:1201.2434v2 [astro-ph.CO] 6 Mar 2013.

[43] H. Terazawa, INS-Report-336(INS, University of Tokyo, Tokyo) May, 1979; J. Phys. Soc. Jan. **58**, 3555 (1989); **58**, 4388 (189); **59**, 1199 (1990). For a classic review on superhypernuclei, see H. Terazawa, in Proc. 2nd Conf. on Nucler and Particle Physics, Cairo, 1999, edited by N.M.H. Comsan and K.M. Hanna (Nuclear Research Center, Atomic Energy authority, Cairo, 2000), p.28. For a more recent review, see H.Terazawa, in Proc. International Conf. on New Trends in High -Energy Physics, Yalta, Crimea (Ukraine), 2005, edited by P.N. Bogolyubov, P.O. Fedosenko, L.L. Jenkovszky, and Yu.A. Karpenko (Bogolyubov Institute for Theoretical Physics, Kiev, 2005), p. 259, arXiv:1304.5655v2 [physics.gen-ph] 15 Apr 2015.

[44] S.A. Chin and A.K. Karman, Phys. Rev. Lett. **43**, 1292 (1979); E. Witten, Phys. Rev.D **30**, 272 (1986); E. Farhi and R.L. Jaffe, Phys. Rev.D **30**, 2379 (1984); **32**, 2452(1985).

[45] For a recent review on exotic nuclei and strange stars, see H. Terazawa, Nonlinear Phenomena in Complex Systems, **18**, 25 (2015), and many references therein.

[46] For early proposals of strange stars, see N.Itoh, Prog. Theor. Phys. **44**, 291 (1970); A.R.Bodmer, Phys.Rev.D **4**, 1601 (1971). For a classical review, see F. Weber, Ch. Schaab, M.K. Weigel, and N.K. Glendenning, Report No.LBL-37264, UC-413(LBL, Berkeley, 1995), in Proc.Ringer Workshop, Tegernsee, Germany, 1995, presented at Confernce: C95-03-06. See also N.K. Glendenning, Compact Stars, Nuclear Physics, Particle Physics, and General Relativity, 2nd ed. (Springer-Verlag, New York, 2000). For a more recent review, see F.Weber, arXiv:astro-ph/0407155v2, 27 Sep 2004, published in Prog. Par. Nucl. Phys. **56**, 193 (205). For a latest review, see F. Weber *et al.*, arXiv:1210.1910 [astro-ph.SR] 6 Oct 2012, published in Proc.the IAU Symposium 291, edited by J. van Leeuwen(International Astronomical Union, 2013), p. 61.

[47] For some recent theoretical investigations on strange stars, see L. Paulucci and J.E. Horvath, arxiv:1405.1777v1 [astro-ph.HE] 7 Mar 2014; P.-C. Chu, L.-W. Chen, and X.Wang, arXiv:1406.5610v1[nucl-th] 21 Jun 2014; S. Biswas *et al.*, arXiv:1409.8366v4 [nucl-th] 22 Apr 2015, and many refernces therein.

[48] H. Terazawa, J. Phys. Soc. Jan. **69**, 2825 (2000); Nonlin.Phenom.Complex Syst.**3**, 293 (2000), and many references therein. For the latest theoretical investigations on color-balls, see H.Terazawa, in Ref. [45] and H.Terazawa and M. Yasue, arXiv:1508.00172v2 [hep-ph] 9 Sep 2015, and many references therein.

[49] For some recent theoretical investigations on axions as the dark matter, see Y.V. Stadnik and V.V. Flambaum, arXiv:1506.08364v1 [hep-ph] 28 Jun 2015, and many references therein. For a latest review, see Q. Yang, arXiv:1509.00673v2 [hep-ph] 10 Sep 2015. Very lately, however, P.H.Frampton has disputed whether axions exist. See P.H. Frampton, "Searching for Dark Matter Constituents with Many Solar Masses", arXiv:1510.00400v2 [hep-ph] 5 Oct 2015.

[50] H. Miyazawa, Prog. Theor.Phys. **36**, 1266 (1966); Yu.A. Gol'fand and L.P. Likhtman, ZhETF Pis.Red. **13**, 452 (1971) [JETP Lett.**13**, 323 (1971)]; D.V. Volkov and V.P. Aklov, *ibid.* **16**, 621 (1972)[*ibid.* **16**, 438 (1972)]; Phys. Lett. **46B**, 10 9(1973); J. Wess and B. Zumino, Nucl. Phys. **B70**, 39 (1974). Very lately, P.H.Frampton has queried arguments for WIMPs which arise from electroweak supersymmetry and argued that dark matter constituents must uniquely be primordial black holes if they constitute all dark matter. See the details in Ref. [49].

On solving the Schrodinger equation with hypersingular kernel in momentum space

Viktor Andreev
vik.andreev@gsu.by; Francisk Skorina Gomel State University, Gomel, Belarus

Abstract

This paper shows that the Schrödinger equation in the momentum representation for a linear confining potential for states with zero orbital angular momentum can be solved with high accuracy (far superior to other methods) using the special quadrature formulas for hypersingular integral.

1 Methods of solution of integral equations

After partial decomposition Schrödinger equation in the momentum space for centrally symmetric potentials, takes the form:

$$\frac{k^2}{2\mu}\phi_\ell(k) + \int\limits_0^\infty V_\ell(k,k')\phi_\ell(k')k'^2\mathrm{d}k' = E\phi_\ell(k)\ , \tag{1}$$

where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass; m_1, m_2 are mass of the constituents of a bound system; $\mathbf{k}$ is the momentum of the relative motion ($|\mathbf{k}| = k$); $\phi_\ell(k)$ is the radial part of the Fourier transform of the wave function in the coordinate representation; $V_\ell(k,k')$ is the operator ℓ -th component of the partial decomposition of the interaction potential; E is binding energy.

However, the description of bound states in the momentum representation is complicated by necessity of solving the integral equation (1), containing singular terms. So for a linear confining potential $V(r) = \sigma r$ we have that

$$V_\ell(k,k') = \frac{\sigma}{\pi(kk')^2}Q'_\ell(\frac{k^2+k'^2}{2kk'})\ . \tag{2}$$

where function $Q_\ell(y)$ is Legendre polynomial of 2nd kind. Since the function Q'_ℓ hypersingular if $k = k'$, then the potential $V_\ell(k,k')$ is also hypersingular. Standard methods of numerical solution of the equation (1) with the potential (2) gives relatively low accuracy of [1, 2]. The numerical solution of the integral equation (1) can be reduced to a problem on the eigenvalues, which arises when using quadrature formulas for the integrals in the equation.

As a result, the integral equation of the form (1) can be reduced to the problem

$$\sum_{j=1}^{N} H\left(k_i, k_j\right)\phi(k_j) = \sum_{j=1}^{N} H_{ij}\phi(k_j) = E\phi(k_i)\ , \tag{3}$$

where to obtain the eigenvalues and vectors need to know the elements of H_{ij}. And if $i \neq j$, the problem of calculating the elements H_{ij} for a linear confining potential is not complex, then the $i = j$ $(k = k')$ directly to do this is not possible, due to the presence of singularities.

2 Quadrature formulas for singular integrals

Receive quadrature formula for the integral

$$I(z) = \int_{-1}^{1} F(t) w(t) g(t,z)\,\mathrm{d}t \tag{4}$$

where $g(t,z)$ is function is singular at $t = z$. The functions $F(t)$ and $w(t)$ is part of the kernel that does not contain the singularities for all $-1 < t, z < 1$.

For this the function $F(t)$ in (4) with the help of interpolation polynomial

$$G_i(t) = \frac{P_N^{(\alpha,\beta)}(t)}{(t-\xi_{i,N})\,P_N'^{(\alpha,\beta)}(\xi_{i,N})} \tag{5}$$

replaced the expansion

$$F(t) \approx \sum_{i,=1}^{N} G_i(t)\,F(\xi_{i,N})\ , \tag{6}$$

where $\xi_{i,N}$ are the roots of the Jacobi polynomial

$$P_N^{(\alpha,\beta)}(\xi_{i,N}) = 0 \quad (i = 1, 2, \ldots, N)\ . \tag{7}$$

Substituting the expansion (6) in a ratio of $I(z)$ we find that the quadrature formula for the integral takes the form

$$I(z) \approx \sum_{i=1}^{N} \omega_i(z)\,F(\xi_{i,N})\ , \tag{8}$$

where

$$\omega_i(z) = \frac{1}{P_N'^{(\alpha,\beta)}(\xi_{i,N})} \int_{-1}^{1} g(t,z)\,w(t)\,\frac{P_N^{(\alpha,\beta)}(t)}{t-\xi_{i,N}}\,\mathrm{d}t\ . \tag{9}$$

Thus the calculation of (9) will help you find the weight coefficients for the quadrature formula (4), the singular values.

3 The analytical form of weighting factors

Consider the possibility of analytical calculation of the weighting factors for different types of singularities that is, depending on the function $g(t,z)$.

3.1 The singular Cauchy integral

The most famous option (4) in the literature is the Cauchy integral

$$g(t,z) = \frac{1}{t-z}\ , \quad -1 < z < 1\ .$$

For this case, there are a large number of works (see for examples [3, 4, 5]), which offered various options for quadrature formulas. In this case, you can get a formula for the weighting factors (9) direct calculation of the integral

$$\omega_i^C(z) = \int_{-1}^{1} \frac{w(t)}{P_N'^{(\alpha,\beta)}(\xi_{i,N})}\,\frac{P_N^{(\alpha,\beta)}(t)}{(t-\xi_{i,N})(t-z)}\,\mathrm{d}t\ . \tag{10}$$

With the help of identity

$$\frac{1}{(t-\xi_{i,N})(t-z)} = \frac{1}{z-\xi_{i,N}}\left[\frac{1}{t-z} - \frac{1}{t-\xi_{i,N}}\right] \tag{11}$$

coefficients (10)reducible to the form

$$\omega_i^C(z) = \begin{cases} \dfrac{1}{P_N'^{(\alpha,\beta)}(\xi_{i,N})} \dfrac{\Pi_N^{(\alpha,\beta)}(z) - \Pi_N^{(\alpha,\beta)}(\xi_{i,N})}{(z-\xi_{i,N})}\ , \text{ if } z \neq \xi_{i,N}\ , \\ \dfrac{\Pi_N'^{(\alpha,\beta)}(\xi_{i,N})}{P_N'^{(\alpha,\beta)}(\xi_{i,N})}\ , \qquad \text{if } z = \xi_{i,N} \end{cases}\ , \tag{12}$$

where

$$\Pi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1} w(t)\frac{P_n^{(\alpha,\beta)}(t)}{(t-z)}\mathrm{d}t\ . \tag{13}$$

To calculate the coefficients of $\omega_i^C(z)$ with a high degree of accuracy to be calculated analytically integral (13) for a variety of functions $w(t)$.

The most famous variant is the version of the function $w(t)$ is weight function of the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ that is

$$w(t) = w^{(\alpha,\beta)}(t) \equiv (1-t)^{\alpha}(1+t)^{\beta}\ .$$

Then the integral (13) have the form

$$\Pi_n^{(\alpha,\beta)}(z) = \mathcal{Q}_n^{(\alpha,\beta)}(z)\ ,$$

where

$$\mathcal{Q}_n^{(\alpha,\beta)}(z) = \int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\frac{P_n^{(\alpha,\beta)}(t)}{(t-z)}\mathrm{d}t\ . \tag{14}$$

In the most general case for arbitrary α and β, the function $\mathcal{Q}_n^{(\alpha,\beta)}(z)$ connected with the Jacobi polynomials of the second kind $Q_n^{(\alpha,\beta)}(z)$ ratio

$$\mathcal{Q}_n^{(\alpha,\beta)}(z) = (-2)(z-1)^{\alpha}(z+1)^{\beta}Q_n^{(\alpha,\beta)}(z)\ , \tag{15}$$

where

$$Q_n^{(\alpha,\beta)}(z) = 2^{\alpha+\beta+n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \times$$
$$\times(z+1)^{-\beta}(z-1)^{-\alpha-n-1}{}_2F_1\left(n+1, n+\alpha+1; 2n+\alpha+\beta+2; \frac{2}{1-z}\right)\ .$$

3.2 Hypersingular variant

Consider hypersingular variant the integral (9), when the function is $g(t,z) = 1/(t-z)^2$. The concept of the final calculation of the integrals of this type was first introduced by Hadamard (J. Hadamard, Lectures he Cauchy's Problem in Linear Partial Differential Equations, Yale University Press (1923).) and developed in the papers [6, 7, 8].

The final part hypersingular integral can be written as

$$\oint\limits_{-1}^{1} \frac{f(t)}{(t-z)^2}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}z}\left[\int\limits_{-1}^{1} \frac{f(t)}{t-z}\,\mathrm{d}t\right] \ , \ \ -1 < z < 1 \ . \tag{16}$$

Therefore, the weighting coefficients of the quadrature formula

$$\int\limits_{-1}^{1} \frac{f(t)}{(t-z)^2}\,\mathrm{d}t = \sum_{i=1}^{N} \omega_i^H\,(z)\, f(\xi_{i,N}) \tag{17}$$

are related with coefficients (ref fz3) ratio

$$\omega_i^H\,(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left[\omega_i^C\,(z)\right] \ . \tag{18}$$

Then the weights for the integral (4) function $g(t,z) = 1/(t-z)^2$ can be calculated by formulas

$$\omega_i^H\,(z) = \begin{cases} \dfrac{1}{P_N^{\prime(\alpha,\beta)}\,(\xi_{i,N})}\left\{\dfrac{\Pi_N^{\prime(\alpha,\beta)}\,(z)}{(z-\xi_{i,N})} - \dfrac{\Pi_N^{(\alpha,\beta)}\,(z) - \Pi_N^{(\alpha,\beta)}\,(\xi_{i,N})}{(z-\xi_{i,N})^2}\right\} \ , \text{ if } z \neq \xi_{i,N} \ , \\ \dfrac{\Pi_N^{\prime\prime(\alpha,\beta)}\,(\xi_{i,N})}{2P_N^{\prime(\alpha,\beta)}\,(\xi_{i,N})} \ , \qquad\qquad \text{if } z = \xi_{i,N} \ . \end{cases} \tag{19}$$

For the Cauchy integral ($g(t,z) = 1/(t-z)$) with $\alpha = -\beta = -1/2$, we have

$$\Pi_n^{(-1/2,1/2)}\,(z) = \int\limits_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\,\frac{V_n(t)}{(t-z)}\mathrm{d}t = \pi W_n(z) \ , \tag{20}$$

where $V_n(z)$ and $W_n(z)$ are Chebyshev polynomials 3 and 4 of kind, respectively (see [9]).

Then the quadrature formula for the Cauchy integral is of the form:

$$\int\limits_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\,\frac{f(t)}{(t-z)}\mathrm{d}t \approx \sum_{i=1}^{N} \omega_i^C\,(z)\, f\,(\xi_{i,N}) \ , \tag{21}$$

where

$$\omega_i^C\,(z) = \begin{cases} \dfrac{\pi}{V_N^{\prime}(\xi_{i,N})}\,\dfrac{W_N\,(z) - W_N\,(\xi_{i,N})}{(z-\xi_{i,N})} \ , \text{ if } z \neq \xi_{i,N} \ , \\ \pi\,\dfrac{W_N^{\prime}\,(\xi_{i,N})}{V_N^{\prime}\,(\xi_{i,N})} \ , \qquad\qquad \text{if } z = \xi_{i,N} \end{cases} \ . \tag{22}$$

Quadrature formula for hypersingular integral has the form:

$$\int\limits_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\,\frac{f(t)}{(t-z)^2}\mathrm{d}t \approx \sum_{i=1}^{N} \omega_i^H\,(z)\, f\,(\xi_{i,N}) \ , \tag{23}$$

where

$$\omega_i^H\,(z) = \begin{cases} \dfrac{\pi}{V_N^{\prime}(\xi_{i,N})}\left\{\dfrac{W_N^{\prime}\,(z)}{(z-\xi_{i,N})} - \dfrac{W_N\,(z) - W_N\,(\xi_{i,N})}{(z-\xi_{i,N})^2}\right\} \ , \text{ if } z \neq \xi_{i,N} \ , \\ \dfrac{\pi}{2}\,\dfrac{W_N^{\prime\prime}(\xi_{i,N})}{V_N^{\prime}(\xi_{i,N})} \ , \qquad\qquad \text{if } z = \xi_{i,N} \ . \end{cases} \tag{24}$$

Formula (24) to calculate weight coefficients allows to them with high accuracy and hence can be used to solve the Schrödinger equation with a linear confining potential in momentum space.

4 The calculation of the energy spectrum for a linear confining potential with $\ell = 0$

The Schrödinger equation with a linear confining potential

$$\frac{k^2}{2\mu}\phi_\ell(k) + \frac{\sigma}{\pi k^2}\int\limits_0^\infty Q'_\ell(y)\,\phi_\ell(k')\mathrm{d}k' = E\phi_\ell(k)\ ,\ \ y = \frac{k^2+k'^2}{2kk'}\ , \tag{25}$$

reducible to the form

$$\tilde{k}^2\,\phi_\ell(\tilde{k}) + \frac{1}{\pi\,\tilde{k}^2}\int\limits_0^\infty Q'_\ell(y)\,\tilde{k}'\phi_\ell(\tilde{k}')\mathrm{d}\tilde{k}' = \varepsilon\,\phi_\ell(\tilde{k}) \tag{26}$$

with the help of replacements

$$k = \beta\tilde{k}\ ,\ \ E = \frac{\beta^2}{2\mu}\varepsilon\ ,\ \ \beta = (2\mu\sigma)^{1/3}\ . \tag{27}$$

Using the mapping

$$\tilde{k} = \beta_0\sqrt{\frac{1+z}{1-z}}\,,\tilde{k}' = \beta_0\sqrt{\frac{1+t}{1-t}}\,, \tag{28}$$

we find that the equation (26) is transformed into

$$\frac{1}{\pi\,\beta_0}\left(\frac{1-z}{1+z}\right)\int\limits_{-1}^{1} Q'_\ell(y(t,z))\,\frac{\phi_\ell(t)\mathrm{d}t}{(1-t)\sqrt{1-t^2}} = \left(\varepsilon - \beta_0^2\frac{1+z}{1-z}\right)\phi_\ell(z)\ . \tag{29}$$

For the case of $\ell = 0$ the equation (29) after simplifications can be written as follows:

$$-\frac{1}{\pi\,\beta_0}\,(1-z)^2\int\limits_{-1}^{1}\phi_{\ell=0}(t)\sqrt{\frac{1+t}{1-t}}\,\frac{\mathrm{d}t}{(t-z)^2} = \left(\varepsilon - \beta_0^2\frac{1+z}{1-z}\right)\phi_{\ell=0}(z)\ . \tag{30}$$

Thus, for a linear confining potential we have hypersingular kernel $\sim 1/(t-z)^2$ and therefore for the numerical solution is necessary to use weighting factors(24).

Function $w(t)$ naturally chosen in the form

$$w(t) = \sqrt{\frac{1+t}{1-t}}\ .$$

As a result, the matrix for eigenvalue problems It takes the form:

$$H_{ij} = \left[\beta_0^2\,\delta_{ij}\,\left(\frac{1+\xi_{j,N}}{1-\xi_{j,N}}\right) - \frac{\omega_j^H\,(\xi_{i,N})}{\pi\,\beta_0}\,(1-\xi_{i,N})^2\right]\ , \tag{31}$$

where $z \to \xi_{i,N}$ and $t \to \xi_{j,N}$, $\xi_{i,N}$ are zeros of the polynomial $V_N(t)$ and matrix $\omega_j^H\,(\xi_{i,N})$ is calculated using the (24).

For a linear confining potential in the $\ell = 0$ is known that

$$\varepsilon = -z_n\ ,\ \ n = 1,2,3\ldots \tag{32}$$

where z_n are the zeros of the Airy function $Ai(z)$. Therefore, it is possible to compare the results of numerical calculations of the matrix (31) and accurate values (see, table 1)

Thus, the choice of weighting coefficients in which the singularity treated analytically and functions $w(t)$ associated with interpolating polynomials $P_N^{(\alpha,\beta)}(t)$ allows us to solve the equation (25) for $\ell = 0$ in momentum space with high accuracy.

The accuracy of calculations many orders of magnitude higher than similar calculations in momentum space [10, 11, 12, 13, 1]

Table 1: Relative error of δ of the solution of equation (31) ($\beta_0 = 0{,}9999$)

N	$n=1$	$=2$	$n=3$	$n=4$	$n=5$
50	3×10^{-22}	4×10^{-20}	3×10^{-17}	3×10^{-15}	8×10^{-14}
80	5×10^{-33}	2×10^{-29}	1×10^{-26}	3×10^{-24}	4×10^{-22}
100	2×10^{-39}	1×10^{-35}	1×10^{-32}	4×10^{-31}	5×10^{-28}
150	4×10^{-54}	8×10^{-50}	5×10^{-47}	1×10^{-43}	6×10^{-42}

5 Conclusion

I am grateful to the organizers for warm and kind hospitality throughout the Conference. The work was supported by the Belarusian Republican Foundation for Basic Research.

References

[1] A. Tang, J. W. Norbury The Nyström plus correction method for solving bound state equations in momentum space // Phys. Rev. — 2001. Vol. E63. P. 066703.

[2] J. W. Norbury, D. E. Kahana, K. Maung Maung Confining potential in momentum space // Can. J. Phys. 1992. Vol. 70. P. 86–89.

[3] M. A. Golberg Numerical Solution of Integral Equations Mathematical concepts and methods in science and engineering. — New York and London: Plenum Press , 1990. 436 P.

[4] A. A. Kornejchuk Quadrature formulae for singular integrals. (Russian)//Zh. Vychisl. Mat. Mat. Fiz. 1964. N 4, Suppl. P. 64–74.

[5] M. A. Sheshko Convergence of quadrature processes for a singular integral. (Russian) //Soviet Mathematics (Izvestiya VUZ. Matematika), 1976. N 20. P. 86-–94

[6] C.-Y. Hui, D. Shia Evaluations of hypersingular integrals using Gaussian quadrature // International Journal for Numerical Methods in Engineering. 1999. Vol. 44, N 2. P. 205–214.

[7] A. C. Kaya, F. Erdogan On the solution of integral equations with strongly singular kernels // Quart. Appl. Math. — 1987. Vol. XLV. P. 105–122.

[8] H. R. Kutt On the numerical evaluation of finite part integrals involving an algebraic singularity .Preprint. — National Research Institute for Mathematical Sciences: Pretoria, 1975.

[9] J. C. Mason, D. C. Handscomb Chebyshev polynomials — Chapman& Hall/Crc , 2002. 335 P.

[10] J.-K. Chen Spectral method for the Cornell and screened Cornell potentials in momentum space // Phys. Rev. D. 2013. Vol. 88. P. 076006. — Erratum Phys. Rev. D 89, 099904 (2014).

[11] A. Deloff Quarkonium bound-state problem in momentum space revisited // Annals Phys. 2007. Vol. 322. P. 2315–2326.

[12] H. Hersbach Relativistic linear potential in momentum space // Phys. Rev. D. Apr 1993. Vol. 47. P. 3027–3033 .

[13] S. Leitão, A. Stadler, M. T. Peña, E. P. Biernat Linear confinement in momentum space: singularity-free bound-state equations // Phys.Rev. 2014. Vol. D90 , N 9. P. 096003.

Spin dependent dipole polarizabilities and characteristics of the nucleon, related with parity violation

V.V. Andreev
vik.andreev@gsu.by; Francisk Skorina Gomel State University, Belarus
O.M. Deryuzhkova
dom@gsu.by;Francisk Skorina Gomel State University, Belarus
N.V. Maksimenko
maksimenko@gsu.by;Francisk Skorina Gomel State University, Belarus

Abstract

It is shown that in the offered model taking into account cross symmetry, gauge-invariant and properties of a Lagrangian to inversion of space spin dipole polarizabilities and a gyration make a contribution to decomposition of amplitude of Compton scattering since the third order on energy of photons according to low-energy theorems of Compton scattering on a nucleon.

1 Introduction

An important role in the understanding of the interaction of electromagnetic fields with hadrons play low-energy theorems as they are based on general principles of quantum theory and decomposition of Compton scattering amplitudes for the photon energy [1]. Currently, one of the most effective methods of investigation of electrodynamic processes is to use the effective Lagrangian obtained in the framework of field-theoretic approaches and consistent with the low-energy theorems [2]. With the development of the Standard Model of electroweak interactions in recent years introduced a new electroweak characteristics of hadrons, related to violation of P-parity [3, 4, 5] .

Effective relativistic-invariant Lagrangians possible to obtain not only the physical interpretation of electromagnetic and electroweak characteristics of hadrons, but also information on the mechanisms of electromagnetic and electroweak photon-hadron interactions. For a more reliable determination of polarizabilities and the characteristics of hadrons associated with parity violation, use a wide class of electrodynamic processes in which the dispersion is realized real and virtual photons, as well as two-photon production in hadron-hadron interactions. The solution of such problems is possible to perform in the framework of the relativistic field-theoretical approach, describe the interaction of electromagnetic fields with hadrons with regard to their electromagnetic and electroweak characteristics [6, 7].

In [8] for the construction of an effective relativistic invariant Lagrangian of the interaction of electromagnetic fields with the particles with constant electric and magnetic dipole moments introduced antisymmetric tensor of the dipole moments, which is independent of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

In this paper, a relativistic quantum field-invariant Lagrangian to [8, 9], which defines the tensor induced dipole moments. Also provided is a variant of relativistic-invariant definition of spin dipole polarizabilities of the nucleon, which is based on the construction of the covariant induced

dipole moments and phenomenological effective interaction Lagrangians of the electromagnetic field with these moments. On the basis of the relativistic properties of P-transformation, as well as cross-symmetry set covariant spin structure of the amplitude of the Compton scattering, consistent with the low-energy theorems. It is shown that the proposed model and the characteristics of the spin polarizability of the nucleon connected with parity nonconservation, contribute to the expansion of the amplitude of Compton scattering from the third order with respect to the photon energy.

2 The scattering amplitude of the electromagnetic field of the spin $1/2$ particle in the dipole approximation

To get the low-energy scattering amplitude of the electromagnetic field on the spin of particles with polarizabilities will follow [10]. However, the determination of the induced electric $\vec{d}$ and magnetic $\vec{m}$ dipole moments of the vectors of electric $\overrightarrow{E}$ and magnetic $\overrightarrow{H}$ electromagnetic field strengths using the relations [11, 12]:

$$\overrightarrow{d} = 4\pi\hat{\alpha}\overrightarrow{E}, \quad \overrightarrow{m} = 4\pi\hat{\beta}\overrightarrow{H}, \tag{1}$$

where $\hat{\alpha}$ and $\hat{\beta}$- the matrices, the matrix elements which are tensors of electric and magnetic polarizabilities. The diagonal elements of these matrices are expressed through the scalar electric and magnetic polarizability:

$$\alpha_{ij} = \alpha_1\delta_{ij}, \qquad \beta_{ij} = \beta_1\delta_{ij}.$$

Using (1) low-energy scattering amplitude of the electromagnetic field can be expressed through matrixes $\hat{\alpha}$ and $\hat{\beta}$ as follows [13]:

$$\begin{aligned} M\left(\overrightarrow{n_2}\right) = 4\pi\omega^2\Bigg\{ & \left(\overrightarrow{e}^{(\lambda_2)*}\hat{\alpha}\overrightarrow{e}^{(\lambda_1)}\right) + \left(\overrightarrow{n_2}\overrightarrow{e}^{(\lambda_1)}\right)\left(\overrightarrow{n_1}\hat{\beta}\overrightarrow{e}^{(\lambda_2)*}\right) + \\ & + \left(\overrightarrow{n_1}\overrightarrow{e}^{(\lambda_2)*}\right)\left(\overrightarrow{e}^{(\lambda_1)}\hat{\beta}\overrightarrow{n_2}\right) - \left(\overrightarrow{e}^{(\lambda_2)*}\overrightarrow{e}^{(\lambda_1)}\right)\left(\overrightarrow{n_1}\hat{\beta}\overrightarrow{n_2}\right) - \left(\overrightarrow{n_1}\overrightarrow{n_2}\right) \times \\ & \times \left(\overrightarrow{e}^{(\lambda_1)}\hat{\beta}\overrightarrow{e}^{(\lambda_2)*}\right) + \left[\left(\overrightarrow{n_2}\overrightarrow{n_1}\right)\left(\overrightarrow{e}^{(\lambda_2)*}\overrightarrow{e}^{(\lambda_1)}\right) - \left(\overrightarrow{n_2}\overrightarrow{e}^{(\lambda_1)}\right)\left(\overrightarrow{n_1}\overrightarrow{e}^{(\lambda_2)*}\right)\right] Sp\left(\hat{\beta}\right)\Bigg\}. \end{aligned} \tag{2}$$

In expression (2) we have introduced the following notation: $\overrightarrow{e}^{(\lambda_1)}$and $\overrightarrow{e}^{(\lambda_2)}$ – polarization vectors, $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ – single vectors of the falling and scattered radiation, ω – radiation frequency.

From definition $\overrightarrow{d}$ and $\overrightarrow{m}$ it agrees (1) follows, $\hat{\alpha}$ and $\hat{\beta}$ satisfy to the hermiticities condition.

In this case, as shown in work [14], tensors can α_{ij} and β_{ij} be presented as follows:

$$\alpha_{ij} = \alpha_1\delta_{ij} + i\alpha_2\varepsilon_{ijk}C_k, \quad \beta_{ij} = \beta_1\delta_{ij} + i\beta_2\varepsilon_{ijk}C_k,$$

where α_1, α_2, β_1 and β_2 – the real values, ε_{ijk} – a tensor Levi-Civita, C_k – pseudo-vector components. In case of a spin particle as such pseudo-vector it is possible to choose a pseudo-vector – operator of the spin particle $\overrightarrow{\hat{S}}$. If to consider that matrixes $\hat{\alpha}$ and $\hat{\beta}$ depend from$\overrightarrow{\hat{S}}$, using algebra of operators 1/2-spin:

$$\left[\hat{S}_i, \hat{S}_j\right] = i\varepsilon_{ijk}\hat{S}_k, \qquad \hat{S}_i\hat{S}_j = \frac{1}{4}\delta_{ij} + \frac{i}{2}\varepsilon_{ijk}\hat{S}_k,$$

these tensors can be presented as follows

$$\alpha_{ij} = \alpha_1 \delta_{ij} + i\alpha_2 \varepsilon_{ijk} \hat{S}_k, \qquad \beta_{ij} = \beta_1 \delta_{ij} + i\beta_2 \varepsilon_{ijk} \hat{S}_k. \tag{3}$$

Substituting (3) in the equation (2), we will obtain:

$$\begin{aligned} M\left(\overrightarrow{n_2}\right) = 4\pi\omega^2 \chi_f^+ \Bigg\{ \alpha_1 \left(\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{e}^{(\lambda_1)} \right) + \beta_1 \left(\left[\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{n_2} \right] \cdot \left[\overrightarrow{e}^{(\lambda_1)} \overrightarrow{n_1} \right] \right) + \\ + i\alpha_2 \left(\overset{\wedge}{\overrightarrow{S}} \cdot \left[\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{e}^{(\lambda_1)} \right] \right) + i\beta_2 \left(\overset{\wedge}{\overrightarrow{S}} \cdot \left[\left[\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{n_2} \right] \cdot \left[\overrightarrow{e}^{(\lambda_1)} \overrightarrow{n_1} \right] \right] \right) \Bigg\} \chi_i \ , \end{aligned} \tag{4}$$

where χ_i and χ_f – spinor of an initial and final particle.

If the amplitude (4) require the condition of crossing symmetry, the equation (4) will be only the first two terms

$$M\left(\overrightarrow{n_2}\right) = 4\pi\omega^2 \left\{ \alpha_1 \left(\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{e}^{(\lambda_1)} \right) + \beta_1 \left(\left[\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{n_2} \right] \cdot \left[\overrightarrow{e}^{(\lambda_1)} \overrightarrow{n_1} \right] \right) \right\} , \tag{5}$$

which is consistent with the spin structure of the amplitude of the low-energy Compton scattering with the electric and magnetic polarizabilities [15]. In the case of Compton forward scattering amplitude has a total spin structure of the form [16]

$$M = g\left(\omega\right) \left(\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{e}^{(\lambda_1)} \right) + ih\left(\omega\right) \left(\overrightarrow{S} \cdot \left[\overrightarrow{e}^{(\lambda_2)*} \overrightarrow{e}^{(\lambda_1)} \right] \right) \ . \tag{6}$$

In this definition, the amplitude of the scalar function $g\left(\omega\right)$ is even, and $h\left(\omega\right)$ - with respect to cross-odd symmetry. Consequently, since the polarizability contribute to the amplitude (6) starting from the second-order and higher, the spin structure of the second term in (6) is determined by the contributions polarizabilities from the third-order.

We now define the Lagrangian and the Compton scattering amplitude in the covariant representation of the dipole.

3 Amplitude of low-energy Compton scattering in covariant dipole representation

In [8] for the construction of an effective relativistic invariant Lagrangian of the interaction of electromagnetic fields with the particles with constant electric and magnetic dipole moments introduced antisymmetric tensor of the dipole moments, which is independent of the electromagnetic field tensor $F_{\mu\nu}$:

$$G^{\mu\nu} = \left(d^\mu u^\nu - u^\mu d^\nu\right) + \varepsilon^{\mu\nu\rho\sigma} m_\rho u_\sigma, \tag{7}$$

where d^μ and m^μ – the components of the electric and magnetic moments presented in a covariant form; u^μ – particle 4-speed components, $\varepsilon^{\mu\nu\rho\sigma}$ – 4-dimensional tensor Levi-Civita.

An effective Lagrangians interaction of an electromagnetic field with particles with the constant dipole moments is represented as follows:

$$L = -\frac{1}{2}\left(e_\mu d^\mu + h_\mu m^\mu\right), \tag{8}$$

where $e_\mu = F_{\mu\nu} u^\nu$, $h_\mu = \tilde{F}_{\mu\nu} u^\nu$, $\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

We are assuming that the form of a tensor (7) can be given and for the induced dipole moments. We will write down in a covariant form taking into account conservation law of parity and definition of a vector of Paulie-Lyubansky W^{μ} components of vectors of the electric and magnetic moments

$$d^{\mu} = 4\pi\alpha^{\mu\nu}e_{\nu} + 4\pi\kappa^{\mu\nu\delta}\left(\partial_{\delta}\right)e_{\nu}\ , \tag{9}$$

$$m^{\mu} = 4\pi\beta^{\mu\nu}h_{\nu} + 4\pi\tilde{\kappa}^{\mu\nu\delta}\left(\partial_{\delta}\right)h_{\nu}\ . \tag{10}$$

In equations (9) and (10) introduced the notation:

$$\alpha^{\mu\nu} = \alpha_1 g^{\mu\nu}, \kappa^{\mu\nu\delta} = k\varepsilon^{\mu\nu\rho\varepsilon}W_{\rho}\ , \qquad \beta^{\mu\nu} = \beta_1 g^{\mu\nu}, \tilde{\kappa}^{\mu\nu\delta} = \tilde{k}\varepsilon^{\mu\nu\rho\delta}W_{\rho}\ .$$

In case of a particle the spin 1/2 vector $\hat{W}{}^{\mu}$ has the form:

$$\hat{W}{}^{\mu} = -\frac{1}{2m}\gamma^5\left(\gamma^{\mu}\hat{p} - p^{\mu}\right)\ ,$$

where $\hat{p} = \gamma_{\mu}p^{\mu}$, p^{μ}– a particle 4-momentum, γ^{μ} – the matrixes satisfying to permutable ratios $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$. Equations (9) and (10) follows from expressions, d^{μ} as m^{μ} consist of symmetric and antisymmetric parts of the permutation of indexes μ and ν. As will be shown below, this presentation is consistent with the condition of crossing symmetry amplitude Compton scattering.

The Lagrangian (8), with which you can get the Compton scattering amplitude and align it with the low-energy theorems, within the field-theoretical covariant approach has the form [17]:

$$L\left(x\right) = \frac{i\pi}{4m}\left[\bar{\Psi}\gamma^{\nu}\hat{L}_{\nu\sigma}\overset{\leftrightarrow}{\partial^{\sigma}}\Psi + \bar{\Psi}\hat{L}_{\nu\sigma}\gamma^{\nu}\overset{\leftrightarrow}{\partial^{\sigma}}\Psi + \bar{\Psi}\gamma^{\sigma}\hat{L}_{\nu\sigma}\overset{\leftrightarrow}{\partial^{\nu}}\Psi + \bar{\Psi}\hat{L}_{\nu\sigma}\gamma^{\sigma}\overset{\leftrightarrow}{\partial^{\nu}}\Psi\right], \tag{11}$$

where $\Psi\left(x\right)$ is the bispinor of Dirac field, $\overset{\leftrightarrow}{\partial^{\nu}} = \overset{\rightarrow}{\partial^{\nu}} - \overset{\leftarrow}{\partial^{\nu}}$, shooters specify the directions of action of derivatives.

As was it is shown in work [13] tensor $\hat{L}_{\nu\sigma}$ in expression (11) has to be is presented definitely that Lagrangian $L\left(x\right)$ satisfied to parity conservation law, and spin structures of amplitude of Compton scattering – cross symmetry:

$$\hat{L}_{\nu\sigma} = \hat{L}{}^{(\alpha_1)}_{\nu\sigma} + \hat{L}{}^{(\beta_1)}_{\nu\sigma} + \hat{L}{}^{(\kappa)}_{\nu\sigma} + \hat{L}{}^{(\tilde{\kappa})}_{\nu\sigma}\ . \tag{12}$$

In turn, the tensor of (12) are consistent with the definitions (9) and (10) are as follows:

$$\hat{L}{}^{(\alpha_1)}_{\nu\sigma} = F_{\nu\mu}\overset{\wedge}{\alpha}{}^{\mu\rho}\left(\alpha_1\right)F_{\rho\sigma}, \tag{13}$$

$$\hat{L}{}^{(\kappa)}_{\nu\sigma} = F_{\nu\mu}\overset{\leftrightarrow}{\partial_{\delta}}F_{\rho\sigma}\overset{\wedge}{\kappa}{}^{\mu\rho\delta}\left(\kappa\right)\ , \tag{14}$$

where have introduced the following notations $\alpha^{\mu\nu} = \alpha_1 g^{\mu\nu}, \kappa^{\mu\nu\delta}\left(\kappa\right) = k\varepsilon^{\mu\nu\rho\varepsilon}\hat{W}_{\rho}$. The derivative $\overset{\leftrightarrow}{\partial_{\delta}}$ operate only to tensors of an electromagnetic field $F_{\mu\nu}$, and the operator $\hat{W}_{\rho}$ operate to wave functions Ψ and $\bar{\Psi}$.

If in tensors (13) and (14) to make replacement $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, we will receive expressions for $\hat{L}{}^{(\beta_1)}_{\nu\sigma}$ and $\hat{L}{}^{(\tilde{\kappa})}_{\nu\sigma}$. Thus, effective relativistic-invariant Lagrangian, allowing to consider scalar electric and magnetic dipole polarizabilities of a nucleon, it is possible to present in the form:

$$L^{(\alpha_1)} + L^{(\beta_1)} = \frac{2\pi}{m}\left(\alpha_1 F_{\nu\mu}F^{\mu}_{\sigma} + \beta_1\tilde{F}_{\nu\mu}\tilde{F}^{\mu}_{\sigma}\right)\theta^{\nu\sigma}, \tag{15}$$

where $\theta^{\nu\sigma} = \frac{i}{2}\bar{\Psi}\gamma^{\nu}\overleftrightarrow{\partial}^{\sigma}\Psi$.

Amplitude of Compton scattering taking into account a Lagrangian (15) has the form [18]

$$\begin{aligned} M^{(\alpha_1)} + M^{(\beta_1)} = \left(\tfrac{2\pi}{m}\right)\Big[\alpha_1\left(F^{(2)}_{\nu\mu}F^{(1)\mu}_{\sigma} + F^{(1)}_{\nu\mu}F^{(2)\mu}_{\sigma}\right) + \\ +\beta_1\left(\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)\mu}_{\sigma} + \tilde{F}^{(1)}_{\nu\mu}\tilde{F}^{(2)\mu}_{\sigma}\right)\Big]\bar{U}^{(r_2)}\left(\vec{p}_{\,2}\right)\gamma^{\nu}P^{\sigma}U^{(r_1)}\left(\vec{p}_{\,1}\right). \end{aligned} \tag{16}$$

In the equation (16) have introduced the notations:

$$F^{(n)}_{\mu\nu} = \left(k_{(n)\mu}e^{(\lambda_n)}_{\nu} - k_{(n)\nu}e^{(\lambda_n)}_{\mu}\right),$$

$\tilde{F}^{(n)}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{(n)\rho\sigma}$, parameter n has the values 1 and 2, $e^{(\lambda_1)}_{\mu}$ and $e^{(\lambda_2)*}_{\mu}$ are vectors of polarization of initial and final photons, $P = \frac{1}{2}(p_1 + p_2)$, k_1, p_1 and k_2, p_2 are four-momenta of initial and final photons and nucleons, $U^{(r_1)}\left(\vec{p}_{\,1}\right)$ and $\bar{U}^{(r_2)}\left(\vec{p}_{\,2}\right)$ are bispinors of initial and final nucleons.

Follows from (16) a ratio that the part of amplitude of Compton scattering caused by electric α_1 and magnetic β_1 scalar polarizabilities meets a condition of cross symmetry and makes a contribution, since the second order on energy of photons. In system of rest of a target and in the second order on energy of photons from (16) the ratio follows:

$$M^{(\alpha_1)} + M^{(\beta_1)} = 4\pi\omega_1\omega_2\chi^{+}_{f}\left[\alpha_1\left(\vec{e}^{\,(\lambda_2)*}\vec{e}^{\,(\lambda_1)}\right) + \beta_1\left(\left[\vec{e}^{\,(\lambda_2)*}\vec{n}_2\right]\cdot\left[\vec{e}^{\,(\lambda_1)}\vec{n}_1\right]\right)\right]\chi_i,$$

which will be coordinated with (5).

4 Dipole spin polarizabilities and the characteristics of a nucleon, connected with violation parity

The electromagnetic characteristics of hadrons connected with not preservation of parity [3, 4] possess properties of the giration used in optics [11]. In this section we will consider relativistic-invariant determination of dipole spin polarizabilities and the giration of a nucleon connected with parity not preservation, and also we will pay attention to distinction of their deposits to amplitude of Compton scattering. Follows from (12) a ratio that effective Lagrangian, corresponding to deposits of spin dipole polarizabilities κ and $\tilde{\kappa}$, has an appearance:

$$\begin{aligned} L^{(\kappa)} + L^{(\tilde{\kappa})} = \frac{i\pi}{4m}\left(\varepsilon^{\mu\rho\kappa\delta}\right)\left[\kappa F_{\nu\mu}\overleftrightarrow{\partial}_{\delta}F_{\rho\sigma} + \tilde{\kappa}\tilde{F}_{\nu\mu}\overleftrightarrow{\partial}_{\delta}\tilde{F}_{\rho\sigma}\right]\times \\ \times\bar{\Psi}\left[\left(\gamma^{\nu}\hat{W}_{\kappa} + \hat{W}_{\kappa}\gamma^{\nu}\right)\overleftrightarrow{\partial}^{\sigma} + \left(\gamma^{\sigma}\hat{W}_{\kappa} + \hat{W}_{\kappa}\gamma^{\sigma}\right)\overleftrightarrow{\partial}^{\nu}\right]\Psi. \end{aligned} \tag{17}$$

The part of amplitude of Compton scattering calculated on the basis of this Lagrangian is defined as follows:

$$\begin{aligned} M^{(\kappa)} + M^{(\tilde{\kappa})} = \frac{i\pi}{4m^2}\left(\varepsilon^{\mu\rho\kappa\delta}\right)(k_1 + k_2)_{\delta}\left[\kappa\left(F^{(2)}_{\nu\mu}F^{(1)}_{\rho\sigma} - F^{(2)}_{\sigma\rho}F^{(1)}_{\mu\nu}\right) + \right. \\ \left. +\tilde{\kappa}\left(\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)}_{\rho\sigma} - \tilde{F}^{(2)}_{\sigma\rho}\tilde{F}^{(1)}_{\mu\nu}\right)\right]\bar{U}^{(r_2)}\left(\vec{p}_{\,2}\right)\gamma^{5}[(\delta^{\nu}_{\tau}\gamma_{\kappa} - \delta^{\nu}_{\kappa}\gamma_{\tau})P^{\sigma} + \\ +(\delta^{\sigma}_{\tau}\gamma_{\kappa} - \delta^{\sigma}_{\kappa}\gamma_{\tau})P^{\nu}]P_{\tau}U^{(r_1)}\left(\vec{p}_{\,1}\right). \end{aligned} \tag{18}$$

Expression (18) is testified of an invariant of cross symmetry. The contribution of spin polarizabilities κ also $\tilde{\kappa}$ begins with the third order on energy of photons. If determine amplitude

(18) in the rest frame and to neglect an impulse of return of a nucleon, we will obtain

$$M^{(\kappa)}+M^{(\tilde{\kappa})}=4\pi i\left(\omega_1+\omega_2\right)\left(\omega_1\omega_2\right)\left\{\kappa\left(\vec{S}\left[\vec{e}^{(\lambda_2)*}\vec{e}^{(\lambda_1)}\right]\right)+\right.$$
$$\left.+\tilde{\kappa}\left(\vec{S}\left[\left[\vec{e}^{(\lambda_2)*}\vec{n}_2\right]\cdot\left[\vec{e}^{(\lambda_1)}\vec{n}_1\right]\right]\right)\right\}. \tag{19}$$

According to from the equations (17) and (19) Lagrangian by means of which the contribution of spin dipole polarizabilities κ and $\tilde{\kappa}$ to amplitude of Compton scattering is considered is even concerning inversion of space.

By analogy with Lagrangian (17) we will construct new Lagrangian by which we will define contributions of girations (the characteristics connected with parity not preservation) to amplitude of Compton scattering. For this purpose it is enough in (17) to make replacement $\hat{W}_\kappa \to 1/m\,\overleftrightarrow{\partial}_\kappa$. As a result we will obtain:

$$L=\frac{i\pi}{2m^2}\left(\varepsilon^{\mu\rho\kappa\delta}\right)\left[\delta_E F_{\nu\mu}\overleftrightarrow{\partial}_\delta F_{\rho\sigma}+\delta_M\tilde{F}_{\nu\mu}\overleftrightarrow{\partial}_\delta\tilde{F}_{\rho\sigma}\right]\bar{\Psi}\left[\left(\gamma^\nu\overleftrightarrow{\partial}_\kappa\overleftrightarrow{\partial^\sigma}+\gamma^\sigma\overleftrightarrow{\partial}_\kappa\overleftrightarrow{\partial^\nu}\right)\right]\Psi\,, \tag{20}$$

where δ_E and δ_M are electric and magnetic girations.

Amplitude of Compton scattering which is obtained on the basis of a Lagrangian (20), in system of rest of a target and in neglect an impulse of return of a target, is defined so

$$M=4\pi\omega_1\omega_2\chi_f^+\left\{\delta_E\left(\left(\vec{k}_1+\vec{k}_2\right)\cdot\left[\vec{e}^{(\lambda_2)*}\vec{e}^{(\lambda_1)}\right]\right)+\delta_M\left(\left(\vec{k}_1+\vec{k}_2\right)\cdot\left[\vec{\Sigma}_2\vec{\Sigma}_1\right]\right)\right\}\chi_i\,, \tag{21}$$

where $\vec{\Sigma}_2=\left[\vec{e}^{(\lambda_2)*}\vec{n}_2\right]$, $\vec{\Sigma}_1=\left[\vec{e}^{(\lambda_1)}\vec{n}_1\right]$. The ratio (21) will be coordinated with low-energy determination of amplitude (2) if to present tensors of polarizabilities through δ_E and δ_M [11]

$$\alpha_{ij}=\alpha_1\delta_{ij}+i\delta_E\varepsilon_{ijk}\partial_k\,,\quad \beta_{ij}=\beta_1\delta_{ij}+i\delta_M\varepsilon_{ijk}\partial_k\,,$$

where the derivative ∂_κ action to vectors of an electromagnetic field. Thus, from the equations (19) and (21) follows:

1) in both amplitudes the condition of cross symmetry is satisfied;

2) if in the ratio (19) the invariance condition concerning inversion of space is satisfied, in the ratio (21) this condition is violated;

3) deposits of a giration and spin dipole polarizabilities to amplitude of Compton scattering on a nucleon begins with the third order on energy of photons.

5 Conclusion

In this work the proposal relativistic-invariant definition of spin dipole polarizabilities and girations which foundation on covariant build of the induced dipole moments and phenomenological effective Lagrangians of interaction of an electromagnetic field with these moments of a structural particle a spin 1/2 is offered. It is shown that in the offered model taking into account cross symmetry, gauge-invariant and properties of a Lagrangian to inversion of space spin dipole polarizabilities and a giration make a contribution to decomposition of amplitude of Compton scattering since the third order on energy of photons according to low-energy theorems of Compton scattering on a nucleon.

This work was supported by the Belarusian Republican Foundation for Basic Research (grants N F14-035 and N F15D-009).

References

[1] N.V. Maksimenko, S.G. Shulga Low-energy decomposition of amplitude of Compton scattering on a hadron and simultaneous switchboards of currents // Nuclear physics. 1990. Vol. 52. N 2(8). P. 524–534.

[2] R. J. Hill, G. Lee, G. Paz, M. P. Solon The NRQED lagrangian at order $\frac{1}{M^4}$ // Phys. Rev. D. 2013. Vol. 87. N 5. P. 053017-1–13.

[3] P.F. Bedaque, M.J. SavageP.F. Parity violation in $\gamma\vec{p}$ Compton Scattering // Phys. Rev. C. 2000. Vol. 62. P.018501-1–6.

[4] M. Gorchtein, X. Zhang Forward Compton Scattering with weak neutral current: constraints from sum rules // [Electronic resource]. – 2015. – Mode of access: http:// nucl-th/1501.0535. – Date of access: 22.01.2015.

[5] M. Gorchtein CP-violation in Compton Scattering// Phys. Rev. C. 2008. Vol.77. P.065501-1–6.

[6] C.E. Carlson, M. Vanderhaeghen Constraining off-shell effects using low-energy Compton scattering // [Electronic resource]. – 2011. – Mode of access: http://physics.atom-ph/1109.3779.– Date of access: 04.10.2011.

[7] N. Krupina, V. Pascalutsa Separation of proton polarizabilities with the beam asymmetry of Compton scattering // Phys. Rev. Lett.2013. Vol.110. N 26. P. 262001-1–4.

[8] J. S. Anandan Classical and quantum interaction of the dipole // Phys. Rev. Lett. 2000. Vol. 85. P. 1354–1357.

[9] A .J. Silenko Spin precession of a particle with an electric dipole moment: contributions from classical electrodynamics and from the Thomas effect // Phys. Scr. 2015. Vol. 90. P. 065303.

[10] L.D. Landau, E.M. Lifshits. Theory of field – M.: Science, 1967. 460 p.

[11] F.I. Fedorov Theory of gyrotropy – Minsk: Science and equipment, 1976. 456 p.

[12] V.G. Baryshevsky Nuclear optics of the polarized environments – Moscow: Energoatomizdat, 1995. 316 p.

[13] V.V. Andreev, O.M. Deryuzhkova, N.V. Maksimenko Covariant representation of spin polarizabilities of a nucleon//Problems of physics, mathematics and equipment. 2014. N 3(20). P. 7–12.

[14] M. V. Galynsky, F.I. FedorovO transformation of a tensor of a bunch at interaction of light with Wednesday // ZhPS. 1986. Vol. 44. N 2. P. 288–292.

[15] V. A. Petrunkin The electric and magnetic polarizabilities of hadrons // Elem. Chast. Atom. Yad. 1981. Vol. 12. N 3. P. 692–753.

[16] Damashek, M. Forward Compton scattering/ M. Damashek, F.J.Gilman// Phys. Rev. – 1970. – Vol. D1. – N 6. – P. 1319-1332.

[17] V.V. Andreev, N.V. Maksimenko Polarizabilities of elementary particles in theoretical-field approach // Problems of physics, mathematics and equipment. 2011. N 4(9). P. 7–11.

[18] V.V. Andreev, O.M. Deryuzhkova, N.V. Maksimenko Covariant equations of motion of a spin 1/2 particle in an electromagnetic field with allowance for polarizabilities // Russ. Phys. Journ. 2014. Vol. 56.N 9. P. 1069–1075.

Radiative decays of vector meson in Poincare-covariant quark model

V.V. Andreev,
vik.andreev@gsu.by; GSU, Gomel, Belarus
V.Yu. Haurysh,
mez0n@inbox.ru; GSU, Gomel, Belarus

Abstract

We research pseudoscalar and vector mesons in a relativistic constituent-quark model in the u-, d- and s-quark sector. Using experimental data for masses of mesons we determinate basic parameters of our model and analyzing radiative decay of light vector mesons using this paramets.

Keywords: Poincare-invariant quantum mechanics, quark, bounded systems, meson, electroweak decays, radiative decays, magnetic moment.

1 Introduction

Electroweak decays of pseudoscalar and vector mesons have always been usefull tool for understanding and researching mechanism of interaction of quarks inside the hadrons. We present one of the possible ways for describing electroweak properties of hadrons.

Generally, there are a lot of different models and approaches for describing basic parameters of pseudoscalar and vector mesons: lattice QCD [1], mock-meson models [2, 3], various approaches, based on parton assumptions [4], models, uses Bethe-Salpeter equation [5, 6] and different phenomenological models [7, 8]. In any approach the question of behaviour strong coupling constant is opened: in [9] was shown all different way for parametrization of strong coupling constant. However, "freezing" procedure is widely used in all approaches; we follow [3], where were used the convenient way for parametrization of strong coupling behavior, which based on data for first moments of spin functions of nucleons [10].

In our work we consider mesons as compounded two-body quark-antiquark system in a Relativistic Hamiltonian dynamics (further **RHD**) with potential, offered in [3]. For describing of electroweak decays we use point form of **RHD**: this form of dynamics with certain modifications is widely used for these purposes (see [11, 12]) . In sections 2-4 we briefly describe basic features of our model, in section 5 we will show process of calculation basic parameters of our model and, finally, we will calculate the quark magnetic moments using experimental data for $V \to P\gamma$ transition.

2 Modelling the strong coupling constant α_{QCD}

In complete 4-loop approximation the running coupling, obtained it the $\overline{MS}$-scheme, is given by [13]

$$\alpha_{\rm QCD}(Q^2) = \pi \left[\frac{1}{\beta_0 L_Q} - \frac{b_1 \ln L_Q}{(\beta_0 L_Q)^2} + \frac{1}{(\beta_0 L_Q)^2} \times \right.$$

$$\times \left[b_1^2(\ln^2 L_Q - \ln L_Q - 1) + b_2\right] + \frac{1}{(\beta_0 L_Q)^4}\left[b_1^3 \times \right. \tag{1}$$

$$\left. \times \left(-\ln^3 L_Q + 5/2 \ln^2 L_Q + 2\ln L_Q - \frac{1}{2}\right) - 3b_1 b_2 \ln L_Q + \frac{b_3}{2}\right],$$

where

$$L_Q = \ln z_Q = \ln(Q^2/\Lambda^2), \;\; b_i = \frac{\beta_i}{\beta_0},$$

and parameters of β - function given by equations:

$$\beta_0 = \frac{1}{4}\left(11 - \frac{2}{3}n_f\right), \;\; \beta_1 = \frac{1}{16}\left(102 - \frac{38}{3}n_f\right),$$

$$\beta_2 = \frac{1}{64}\left(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right),$$

$$\beta_3 = \frac{1}{256}\left(\frac{149753}{6} + 3564\zeta_3 + \left[-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right]n_f + \right.$$

$$\left. + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right]n_f^2 + \frac{1093}{729}n_f^3\right) \tag{2}$$

(details see in[13]).

As was said before, the behaviour of strong coupling constant has different ways of parametrization.

Since the main requirement for QCD running strong coupling constant dictated by the condition of matching the theoretical predictions with experimental data we, follow [3], chose the QCD coupling in form

$$\alpha_{QCD}(Q) = \sum_{i=1}^{7} \alpha_i \mathrm{e}^{-Q^2/4\gamma_i^2}, \tag{3}$$

where α_i and γ_i - some parameters, which determine the behavior of strong coupling constant.

Basically, sets of the parameters α_i and γ_i could be divided into two categories:

1. parameters, which forming a peak in non-perturbative region ($Q \leq 1$ GeV) with following critical value $\alpha_{QCD}^{crit.} = \alpha_{QCD}(Q = 0) = 0$.

2. parameters, which imply "freezing" of strong coupling constant with some value $\alpha_{QCD}^{crit.} = 0.6 \div 1.0$.

In our work we use "freezing" mode with $\alpha_{QCD}^{crit.}$ fixed by the requirement of the correspondence the experimental data for the difference in the first moments $\Gamma_1^{p-n}(x, Q^2)$ proton and neutron spin structure functions $g_1^{p,n}(x, Q^2)$ (so-called QCD-modified Bjorken sum rule [10]):

$$\Gamma_1^{p-n}(x, Q^2) = \Gamma_1^p(Q^2) - \Gamma_1^n(Q^2) = \frac{g_A}{6} C_{NS}(Q^2) + \Delta_{HT}^{p-n}(Q^2). \tag{4}$$

In (4) $g_a = 1.2701 \pm 0.0025$ is isovector axial charge, and the coefficients C_{NS} calculated within the pQCD (in $\overline{MS}$-scheme) up to the fourth order of α_{QCD} (see).

As a result, we obtained

$$\alpha_{QCD}^{crit.} = \sum_{i=1}^{7} \alpha_i = 0.660 \pm 0.007. \tag{5}$$

3 Poincare-covariant quark model of mesons

For describing bound states of quarks we use Poincare-invariant quantum mechanics or relativistic Hamiltonian dynamics (**RHD**), which is simple generalization of the quantum mechanics. Let us discuss the basic features of this model.

For the construction of the state vectors of the free particles in Poincare-invariant quantum mechanics use a set of ten generators of Poincare group $\hat{P}^{\mu}$ and $\hat{M}^{\mu\nu}$, which have a simple physical meaning : operators $\hat{\vec{P}}$ are three-dimensional momentum, while a mass operator $\hat{M}$ (or Hamiltonian $\sqrt{\hat{\vec{P}}^2 + M^2}$) represented by $\hat{P}_0$. Components of tensor $\hat{M}^{23}, \hat{M}^{31}, \hat{M}^{12}$ are three-dimensional momentum of a system and $\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03}$ are boosts components: $\hat{\vec{N}} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$.

In [14] was shown, that there is no definite division of operators on the kinematic and dynamic set. Since there are three ways of introduction of interaction in **RHD**: instant, point and front forms of **RHD** (for details, see [15]). However, in any form of dynamic, interaction contains mass operator $\hat{M} \equiv M_0 + \hat{V}$, where M_0 is an effective mass of a system of noninteracting particles with masses m_q and $m_{\bar{Q}}$:

$$M_0 = \sqrt{\mathbf{k}^2 + m_q^2} + \sqrt{\mathbf{k}^2 + m_{\bar{Q}}^2}. \tag{6}$$

In (6) $\mathbf{k}$ - the relative momentum

$$\mathbf{k} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) + \frac{P}{\widetilde{M}_0\left(\omega_{\widetilde{M}_0}(\mathrm{P}) + \widetilde{M}_0\right)}\left(m_{\bar{Q}}^2 - m_q^2 - \widetilde{M}_0\left[\omega_{m_{\bar{Q}}}(\mathrm{p}_2) - \omega_{m_q}(\mathrm{p}_1)\right]\right) \tag{7}$$

and $\mathbf{P}$ is the total momentum of free system:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \tag{8}$$

$$\widetilde{M}_0 = \sqrt{\left(\omega_{m_q}(\mathrm{p}_1) + \omega_{m_{\bar{Q}}}(\mathrm{p}_2)\right)^2 - \mathbf{P}^2}.$$

In all paper we assume, that $\omega_m(\mathrm{k}) = \sqrt{\mathbf{k}^2 + m^2}$ and $\mathrm{k} = \sqrt{\mathbf{k}^2}$.

Using state vectors of free particles one can easily build a state vector of bound system [15]:

$$|\mathbf{P}, J\mu, M\rangle = \sum_{\lambda_1,\lambda_2}\sum_{\nu_1,\nu_2}\int \mathbf{dk}\sqrt{\frac{\omega_{m_q}(\mathrm{p}_1)\omega_{m_{\bar{Q}}}(\mathrm{p}_2)M_0}{\omega_{m_q}(\mathrm{k})\omega_{m_{\bar{Q}}}(\mathrm{k})\omega_{M_0}(\mathrm{P})}}\Phi_{LS}^{J\mu}(\mathrm{k})\times$$

$$\times Y_{Lm}(\theta_k, \phi_k)\ C_{\nu_1,\nu_1,\lambda}^{s_1,s_2,S}\ C_{m,\lambda,\mu}^{L,S,J}\ D_{\lambda_1,\nu_1}^{1/2}(\mathbf{n}_{W_1}) D_{\lambda_2,\nu_2}^{1/2}(\mathbf{n}_{W_2})|\mathbf{p}_1, \lambda_1; \mathbf{p}_2, \lambda_2\rangle, \tag{9}$$

where $\Phi_{LS}^{J\mu}(\mathrm{k})$ - wave function of the bound system, which satisfies the equation [16] :

$$\sum_{\lambda_1,\lambda_2}\int \langle \mathbf{k}, \sigma_1, \sigma_2||\hat{V}||\mathbf{k}', \lambda_1, \lambda_2\rangle \Phi_{\mathbf{Q};\lambda_1,\lambda_2}^{J\mu}(\mathbf{k}')\mathbf{dk}' = (M - M_0)\,\Phi_{\mathbf{Q};\sigma_1,\sigma_2}^{J\mu}(\mathbf{k}). \tag{10}$$

At the present time one of the three forms of relativistic dynamics for describing composite systems most widely used the light front dynamic (see [17, 18]). This is explained by the fact that in this form of the dynamic missing diagram of the pairs creation from a vacuum. However,as well as in any form of dynamics, there are certain difficulties.

We believe, that point form of **RHD** the most convenient for describing compounded systems of two quarks (mesons), since in this form of dynamics 4-velocities with and without interaction are the same; this caused the Lorentz-invariance of the wave functions of coupled systems. We use this advantage in all paper for calculation leptonic and hadronic decay constants.

4 The quark-antiquark potential, based on QCD

For describing two-body bound system, it's necessary to determinate interaction between particles. It should be noticed, that different potentials can be used for describing bound system of the same composition. Such freedom gives the opportunity to construct new models, based on the Poincare-covariant model.

We use the quark-antiquark potential in coordinate representation, which was offered in [3]. That potential considered a sum of Coulomb, linear confinement and spin-spin parts of pseudoscalar and vector mesons:

$$\hat{V}(\mathrm{r}) = \hat{V}_{\mathrm{Coul}}(\mathrm{r}) + \hat{V}_{\mathrm{conf}}(\mathrm{r}) + \hat{V}_{\mathrm{SS}}(\mathrm{r}), \tag{11}$$

where

$$\hat{V}_{\mathrm{Coul}}(\mathrm{r}) = -\frac{4}{3}\frac{\alpha_{QCD}(\mathrm{r})}{\mathrm{r}} = -\frac{4}{3\mathrm{r}}\sum_{k=1}^{7}\alpha_k \mathrm{erf}(\tau_k \mathrm{r})\ , \tag{12}$$

$$\hat{V}_{\mathrm{conf}}(\mathrm{r}) = w_0 + \sigma \mathrm{r}\left(\frac{\exp(-b^2\mathrm{r}^2)}{\sqrt{\pi}b\mathrm{r}} + \left(1+\frac{1}{2b^2\mathrm{r}^2}\right)\mathrm{erf}(b\mathrm{r})\right) \tag{13}$$

and

$$\hat{V}_{\mathrm{SS}}(\mathrm{r}) = -\frac{32\left(\mathbf{S}_q\mathbf{S}_{\bar{Q}}\right)}{9\sqrt{\pi}m_q m_{\bar{Q}}}\sum_{k=1}^{7}\alpha_k\tau_k^4\exp(-\tau_k^2\mathrm{r}^2). \tag{14}$$

In (12) $\alpha_{QCD}(\mathrm{r})$ - strong coupling constant in coordinate representation, which we discussed in section 2.

Parameter of linear confinement part in large number of models lies in the range $\sigma = 0.18 \div 0.20\ \mathrm{GeV}^2$, that's why we assume, that

$$\sigma = (\sigma \pm \Delta\sigma) = (0.19 \pm 0.01)\ \ \mathrm{GeV}^2. \tag{15}$$

Parameter of spin-spin part τ_k is deduced from the relation $\tau_k^2 = \gamma_k^2 b^2/\left(\gamma^2 + b^2\right)$, where b - "smearing" parameter, which was used in [3] by the following rule:

$$\widetilde{f}(\mathbf{r}) = \int \mathbf{dr}'\rho(\mathbf{r}-\mathbf{r}')f(\mathbf{r}'). \tag{16}$$

Using (16) "smearing" function with parameter b is chosen in the form

$$\rho(\mathbf{r}-\mathbf{r}') = \frac{\mathrm{b}^3}{\pi^{3/2}}\exp\left[-\mathrm{b}\left(\mathbf{r}-\mathbf{r}'\right)^2\right]. \tag{17}$$

The others parameters of of model (masses of quarks etc.) must be calculated. In following section we will show process of calculation.

5 Detection of model parameters

For model parameters calculation we use variational method [19] with oscillator wave functions

$$\Phi(\mathrm{k},\beta) = \frac{2}{\pi^{1/4}\beta^{3/2}}\exp\left(-\frac{\mathrm{k}^2}{2\beta^2}\right), \tag{18}$$

which is reduced to the calculation of the functional minimum:

$$M(m_q, m_{\bar{Q}}, w_0, \beta) = \langle\Phi(\mathrm{k},\beta)|\hat{M}|\Phi(\mathrm{k},\beta)\rangle = \langle\Phi(\mathrm{k},\beta)|\hat{M}_0|\Phi(\mathrm{k},\beta)\rangle + \langle\Phi(\mathrm{k},\beta)|\hat{V}|\Phi(\mathrm{k},\beta)\rangle. \tag{19}$$

Minimum of functional requirements leads to the equation

$$\left.\frac{\partial M(m_q, m_{\bar{Q}}, \beta, w_0, \mathrm{b})}{\partial \beta}\right|_{\beta_{min}} = 0. \tag{20}$$

Another equations can be obtained from the leptonic decays of pseudoscalar $P(q\bar{Q})$ and vector $V(q\bar{Q})$ mesons. For decay $P(q\bar{Q}) \to \ell + \tilde{\nu}_\ell$ decay constant is defined by the relation:

$$_{\text{out}}\left\langle 0 \left| \hat{J}^\mu_A \right| \mathbf{P}, M_P \right\rangle_{\text{in}} = \mathrm{i}\frac{1}{(2\pi)^{3/2}}\frac{P^\mu}{\sqrt{2\omega_{M_P}}(\mathrm{P})} f_P, \tag{21}$$

where $\hat{J}^\mu_A$ - the electroweak axial current,which is taken in the Heisenberg representation. For the case of leptonic decays of vector mesons $V(q\bar{Q}) \to \ell + \bar{\ell}$ decay constant is defined by

$$_{\text{out}}\left\langle 0 \left| \hat{J}^\mu_V \right| \mathbf{P}, M_V, \lambda \right\rangle_{\text{in}} = \mathrm{i}\frac{1}{(2\pi)^{3/2}}\frac{\varepsilon^\mu_\lambda M_V}{\sqrt{2\omega_{M_V}}(\mathrm{P})} f_V, \tag{22}$$

where ε^μ_λ - polarization vector of a vector meson with mass M_V. In [9, 20] was calculated integral representations for the constants of leptonic pseudoscalar f_P and vector f_V mesons within the point the framework of Poincare-covariant models:

$$f_P(m_q, m_{\bar{Q}}, \beta) = \sqrt{\frac{3}{2}}\frac{1}{\pi}\int \mathrm{dk}\ \Phi(\mathrm{k}, \beta)\ \mathrm{k}^2\ \times$$
$$\times \frac{\sqrt{(\omega_{m_q}(\mathrm{k}) + \mathrm{k})(\omega_{m_{\bar{Q}}}(\mathrm{k}) - \mathrm{k})} + \sqrt{(\omega_{m_{\bar{Q}}}(\mathrm{k}) + \mathrm{k})(\omega_{m_q}(\mathrm{k}) - \mathrm{k})}}{\sqrt{\omega_{m_q}(\mathrm{k}) + \omega_{m_{\bar{Q}}}(\mathrm{k})}\sqrt{\omega_{m_q}(\mathrm{k})\omega_{m_{\bar{Q}}}(\mathrm{k})}} \tag{23}$$

and

$$f_V(m_q, m_{\bar{Q}}, \beta) = \sqrt{\frac{3}{2}}\frac{1}{\pi}\int \mathrm{dk}\ \Phi(\mathrm{k}, \beta)\ \mathrm{k}^2\ \times$$
$$\times \frac{\sqrt{(\omega_{m_q}(\mathrm{k}) + \mathrm{k})(\omega_{m_{\bar{Q}}}(\mathrm{k}) - \mathrm{k})} + \sqrt{(\omega_{m_{\bar{Q}}}(\mathrm{k}) + \mathrm{k})(\omega_{m_q}(\mathrm{k}) - \mathrm{k})}}{3\sqrt{\omega_{m_q}(\mathrm{k}) + \omega_{m_{\bar{Q}}}(\mathrm{k})}\sqrt{\omega_{m_q}(\mathrm{k})\omega_{m_{\bar{Q}}}(\mathrm{k})}}. \tag{24}$$

Thus, we have following system of equations

$$M_P(m_q, m_{\bar{Q}}, w_0, \beta) = M_P^{\text{exp}} \pm \Delta M_P, \tag{25}$$

$$\left.\frac{\partial M_P(m_q, m_{\bar{Q}}, w_0, \beta)}{\partial \beta}\right|_{\beta_{min}} = 0. \tag{26}$$

$$M_V(m_q, m_{\bar{Q}}, w_0, \beta) = M_V^{\text{exp}} \pm \Delta M_V, \tag{27}$$

$$M_V(m_q, m_{\bar{Q}}, w_0, \beta) - M_P(m_q, m_{\bar{Q}}, w_0, \beta) = M_V^{\text{exp}} - M_P^{\text{exp}} \pm \delta M_{V,P}^{\text{exp}}, \tag{28}$$

$$f_P(m_q, m_{\bar{Q}}, \beta) = f_P^{\text{exp}} \pm \Delta f_P^{\text{exp}}, \tag{29}$$

$$f_V(m_q, m_{\bar{Q}}, \beta) = f_V^{\text{exp}} \pm \Delta f_V^{\text{exp}}, \tag{30}$$

where $\delta M_{V,P}^{\text{exp}} = \sqrt{\Delta M_V^2 + \Delta M_P^2}$.

Assuming, that [3]

$$m_d - m_u = (4 \pm 1)\,\mathrm{MeV} \tag{31}$$

we get following parameters of Poincare-covariant quark model [9]:

$$m_u = (239.9 \pm 2.3)\,\mathrm{MeV},\ \ m_d = (243.8 \pm 2.3)\,\mathrm{MeV},\ \ m_s = (466.6 \pm 28)\,\mathrm{MeV},$$
$$\beta_{uu} \simeq \beta_{dd} \simeq \beta_{ud} = (328.78 \pm 2.1)\,\mathrm{MeV},\ \ \beta_{us} \simeq (360.3 \pm 12.1)\,\mathrm{MeV}. \tag{32}$$

6 Radiative decay of vector mesons in Poincare-covariant quark model

Matrix element of the radiative decay process $V \to P\gamma^*$ could be parameterized using 4-velocity of the vector and pseudoscalar mesons by expression [17]

$$\langle Q', M'|\hat{J}^\mu|Q, M\rangle = \mathrm{e}\ g_{VP\gamma}\frac{1}{(2\pi)^3}\frac{\epsilon^{\mu\nu\rho\sigma}\varepsilon_\nu(\lambda)Q_\rho Q'_\sigma}{\sqrt{2\omega_M(\mathrm{Q})}\sqrt{2\omega_{M'}(\mathrm{Q}')}}, \tag{33}$$

or, using 4-velocities of mesons

$$V_Q = \frac{Q}{M}, \quad V_{Q'} = \frac{Q'}{M'} \tag{34}$$

we obtain

$$\langle Q', M'|\hat{J}^\mu|Q, M\rangle = \mathrm{e}\ g_{VP\gamma}\frac{1}{(2\pi)^3}\frac{K^\mu}{\sqrt{2V_0}\sqrt{2V'_0}}\sqrt{MM'}, \tag{35}$$

where $K^\mu = \epsilon^{\mu\nu\rho\sigma}\varepsilon_\nu(\lambda)V_\rho V'_\sigma$.

In framework of Poincare invariant quantum mechanics we consider mesons P and V as relativistic constituent quark-antiquark system: using (9) one can easily construct state vectors of pseudoscalar and vector mesons

$$|Q, M\rangle = \frac{1}{2\sqrt{\pi}}\int \mathrm{d}^3\mathrm{k}\ \ \Phi_{\ell,S}(\mathrm{k})\sqrt{\frac{\omega_{m_q}(\mathrm{p}_1)\omega_{m_{\bar{Q}}}(\mathrm{p}_2)}{\omega_{m_q}(\mathrm{k})\omega_{m_{\bar{Q}}}(\mathrm{k})V_0}}\sum_{\lambda_1,\lambda_2}\sum_{\nu_1,\nu_2}\sqrt{\frac{3+4\nu_1\nu_2}{4}}\delta_{\nu_2,\lambda-\nu_1}$$

$$D_{\lambda_1,\nu_1}(\mathbf{n}_{W_1})D_{\lambda_2,\nu_2}(\mathbf{n}_{W_2})|\mathbf{p}_1,\lambda_1;\mathbf{p}_2,\lambda_2\rangle, \tag{36}$$

$$|Q', M'\rangle = \frac{1}{2\sqrt{\pi}}\int \mathrm{d}^3\mathrm{k}'\ \ \Phi_{\ell,S}(\mathrm{k}')\sqrt{\frac{\omega_{m_q}(\mathrm{p}'_1)\omega_{m_{\bar{Q}}}(\mathrm{p}'_2)}{\omega_{m_q}(\mathrm{k}')\omega_{m_{\bar{Q}}}(\mathrm{k}')V'_0}}\sum_{\lambda'_1,\lambda'_2}\sum_{\nu'_1,\nu'_2}\sqrt{2}\nu'_1\delta_{\nu'_1,-\nu'_2}$$

$$D_{\lambda'_1,\nu'_1}(\mathbf{n}'_{W_1})D_{\lambda'_2,\nu'_2}(\mathbf{n}'_{W_2})|\mathbf{p}'_1,\lambda'_1;\mathbf{p}'_2,\lambda'_2\rangle. \tag{37}$$

Using the current operator $\hat{J}^\mu$ in the quark basis

$$\hat{J}^\mu = \bar{\psi}_{q',\bar{Q}'}\Gamma^\mu\psi_{q,\bar{Q}} \tag{38}$$

after some calculation in generalized Breit system, we get

$$g_{VP\gamma^*} = \frac{1}{4\pi\sqrt{MM'}}\int \mathrm{d}^3\mathrm{k}\ \ \Phi_{\ell,S}(\mathrm{k})\Phi_{\ell,S}(\mathrm{k}_2)\sqrt{\frac{1}{\omega_{m_q}(\mathrm{k})}}\sqrt{\frac{1}{\omega_{m_q}(\mathrm{k}_2)}}\sqrt{\frac{\omega_{m_{\bar{Q}}}(\mathrm{k}_2)}{\omega_{m_{\bar{Q}}}(\mathrm{k})}}\times$$

$$\times\sum_{\nu_1}\sum_{\nu'_1}\sqrt{\frac{3+4\nu_1(\lambda-\nu_1)}{2}}\nu'_1\frac{e_q}{\mathrm{e}}\bar{u}_{\nu'_1}(\mathbf{k}_2,m_q)\frac{(\Gamma\cdot K^*)}{(K\cdot K^*)}B(\boldsymbol{v}_\mathbf{Q})u_{\nu_1}(\mathbf{k},m_q)D^{1/2}_{-\nu'_1,\lambda-\nu_1}+ \tag{39}$$

$$\frac{1}{4\pi\sqrt{MM'}}\int \mathrm{d}^3\mathrm{k}\Phi_{\ell,S}(\mathrm{k})\Phi_{\ell,S}(\mathrm{k}_2)\sqrt{\frac{1}{\omega_{m_{\bar{Q}}}(\mathrm{k})}}\sqrt{\frac{1}{\omega_{m_{\bar{Q}}}(\mathrm{k}_2)}}\sqrt{\frac{\omega_{m_q}(\mathrm{k}_2)}{\omega_{m_q}(\mathrm{k})}}\times$$

$$\times\sum_{\nu_1}\sum_{\nu'_1}\sqrt{\frac{3+4\nu_1(\lambda-\nu_1)}{2}}\nu'_1\frac{e_{\bar{Q}}}{\mathrm{e}}\bar{\upsilon}_{\lambda-\nu_1}(\mathbf{k},m_{\bar{Q}})\frac{(\Gamma\cdot K^*)}{(K\cdot K^*)}B(\upsilon_\mathbf{Q})\upsilon_{-\nu'_1}(\mathbf{k}_2,m_{\bar{Q}})D^{1/2}_{\nu'_1,\nu_1},$$

where

$$\Gamma^\mu = F_1(t)\gamma^\mu + \mathrm{i}F_2(t)\frac{\sigma^{\mu\nu}(k_2-k)}{2m_{q,\bar{Q}}}. \tag{40}$$

In relation (40) form-factors $F_1(t)$ and $F_2(t)$ normalized in the natural units magnetic μ_q and anomalous magnetic moments κ_q of quarks :

$$F_1(t=0)+F_2(t=0)=\mu_q,\;\; F_2(t=0)=\kappa_q, \tag{41}$$

and in (39) $\mathbf{k}_2$ and $\omega_{m_{q,\bar{Q}}}(\mathrm{k}_2)$ given by:

$$\mathbf{k}_2=\mathbf{k}+\boldsymbol{\upsilon}_{\mathbf{Q}}\left((\varpi+1)\omega_{m_{q,\bar{Q}}}+\sqrt{\varpi^2-1}\;\mathrm{k}\cos\theta_k\right), \tag{42}$$

$$\omega_{m_{q,\bar{Q}}}(\mathrm{k}_2)=\omega_{m_{q,\bar{Q}}}(\mathrm{k})\varpi-\sqrt{\varpi^2-1}\;\mathrm{k}\cos\theta_k, \tag{43}$$

where

$$\varpi=\frac{M_0^2+M_0'^2-t}{2M_0M_0'}\,. \tag{44}$$

7 Numerical calculation and discussion

Radiative constant decay $g_{VP\gamma}$ obtained limiting $g_{VP\gamma}=g_{VP\gamma^*}(t\to 0)$. Using the experimental data for the light $\rho, K^{*\pm}$ and K^{*0}–mesons (Table 1)) after numerical calculation we obtain the

Table 1: *Experimental values of light mesons decay width*

Meson	Decay width
ρ	68 ± 7 KeV
$K^{*\pm}$	50 ± 5 KeV
K^{*0}	116 ± 10 KeV

values of the magnetic moments of the quarks μ, in natural units, in relativistic quark model based on point form Poincare-invariant quantum mechanics:

Table 2: *The magnetic moments of quarks*

Quark	Quark magnetic moment ,μ_N	[25]	[26]
u	2.080 ± 0.082	2.066	2.08 ± 0.07
d	-1.261 ± 0.015	-1.110	-1.31 ± 0.06
s	-0.621 ± 0.011	-0.633	-0.77 ± 0.06

The comparative analysis shows the values obtained in the framework of relativistic quark model based on point form Poincare-invariant quantum mechanics correspond to the results, obtained from the analysis of the magnetic moments of the baryon.

8 Conclusion

This work dedicated calculation of integral representation for the form-factor transition $V\to P\gamma$ within the framework of the relativistic quark model based on point form of the Poincare-invariant quantum mechanics is obtained. From the condition of matching the width of the model calculations with the experimental values found values of the magnetic moments of quarks, which are correlated with the data, obtained from using the experimental values of the magnetic moments of baryons.

References

[1] K. Jansen *et al.*. Meson masses and decay constants from unquenched lattice QCD. Phys. Rev. D. 2008. Vol.80. P. 1–36.

[2] D.S.Hwang and G.H.Kim. Decay constants of B, B* and D, D* mesons in relativistic mock meson model. Phys. Rev. D. 1997. Vol.55. P. 1–15.

[3] S.Godfrey and N.Isgur. Mesons in a Relativized Quark Model with Chromodynamics. Phys. Rev. D. 1985. Vol.32. P.1–34.

[4] O.W.Greenberg. The Parton model. Compendium of Quantum Physics. 2009. Springer-Verlag. P.255–258.

[5] L.Zhi-Feng, W.Lucha and F.F.Schoberl. Exact-propagator instantaneous Bethe-Salpeter equation for quark-antiquark bound states. Mod. Phys. Lett. A.2006. Vol.21. P.1–16

[6] W.Lucha and F.F.Schoberl. Bound states by the spinless Salpeter equation. Fizika. B.1999. Vol.8. P.1–11

[7] F.U.Bernlochner and H.Lacker. A phenomenological model for radiative corrections in exclusive semileptonic B-meson decays to (pseudo)scalar final state mesons. arXiv:1003.1620.

[8] W.I.Eshraim. Phenomenology of a pseudoscalar glueball and charmed mesons. arXiv:1509.09117.

[9] V.V.Andreev. Nonperturbative region of effective strong coupling. arXiv:1305.4266.

[10] J.D.Bjorken. Inelastic Scattering of Polarized Leptons from Polarized Nucleons. Phys. Rev. D.1970.Vol.1. P.1–13.

[11] A.Amghar, B.Desplanques and L.Theussl. The Form-factor of the pion in 'point form' of relativistic dynamics revisited. Phys. Lett. B.2003 Vol. 574. P.1–8.

[12] B.Desplanques. Relativistic quantum mechanics: A Dirac's point-form inspired approach. Nucl. Phys. A. 2005 Vol.748. P.1–26.

[13] K.G.Chetyrkin, B.A.Kniehl and M.Steinhauser. Strong coupling constant with flavor thresholds at four loops in the $\overline{MS}$-scheme. Phys. Rev. Lett.1997. Vol. 79.P.1–10.

[14] P.A.M. Dirac. Forms of relativistic dynamics. Rev.Mod. Phys.1949. Vol. 21.P.1–8.

[15] B.D. Keister, W.N. Polyzou. Relativistic Hamiltonian dynamics in nuclear and particle physics. Adv.Nucl.Phys.1991. Vol. 20. P.225–479.

[16] V.V.Andreev. Poincare covariant model of two-particle systems with quantum field potentials. Gomel: GSU. 2008. 294 P.

[17] W.Jaus. Relativistic constituent quark model of electroweak properties of light mesons. Phys. Rev. D.1991.Vol.44. P.2851-2859.

[18] Ho-Meoyng, Choi. Light front quark model analysis of electroweak decays of pseudoscalar and vector mesons. arXiv:9911.2711.

[19] A.Messiah. Quantum Mechanics. Amsterdam: Dover Publications. 1961. 1152 P.

[20] V.V.Andreev, A.F.Krutov. Electromagnetic form factors of mesons. PFMT. 2011. Vol. 1(6). P. 7–19.

[21] W. Cox. Higher-rank representations for zero-spin field theories. PFMT. 2011. Vol. 1(6). P. 7–19.

[22] E.M. Ovsiyuk. Spin zero Cox's particle with an intrinsic structure: general analysis in external electromagnetic and gravitational fields. Ukr. J. Phys. 2015. Vol. 60. P. 485–496.

[23] K.V. Kazmerchuk, E.M. Ovsiyuk, Cox's particlele in m agnetic and electric field against the background of Euclidean and spherical geometries. Ukr. Phys. J. 2015. Vol. 60. P. 389–400.

[24] V.M. Redkov, E.M. Ovsiyuk. Quantum Mechanics in space of constant curvature. – Nova Science Publishers: New York, 2012.

[25] R. Bijker,E. Santopinto and E.Santopinto. Unquenched quark model for baryons: Magnetic moments, spins and orbital angular momenta. Phys. Rev. C. 2009. Vol.80. 10 P.

[26] Y. Dothan. Quark Magnetic Moments. Physica.1982. Vol. 114A. P.216–220

A new high precision method for solution of integral equations for two particles bound systems with the Cornell potential in momentum space

K.S. Babich, V.V. Andreev
Gomel State University, Belarus
kbabich@gsu.by

Abstract

A new high precision method for solution of integral equations in the momentum space with Cornell potential is suggested. The method can be used effectively for the bound state equations.

1 Relativistic equation in momentum space

In the general case, the wave function (WF) of a bound system of spinor quarks with masses m_q, m_Q and respectively with 4-momentums p_1, p_2 and helicities $\lambda_{1,2}$ in Relativistic Hamiltonian Dynamics (RHD) satisfies the three-dimensional integral equation [1]:

$$\sum_{\lambda_1,\lambda_2}\int < \mathbf{k}',\sigma_1,\sigma_2 \parallel \hat{V} \parallel \mathbf{k},\lambda_1,\lambda_2 > \Phi^{J\mu}_{\lambda_1\lambda_2}(\mathbf{k}')\,\mathrm{d}\mathbf{k}' =$$
$$= \left(M - \sqrt{\mathbf{k}^2+m_q^2} - \sqrt{\mathbf{k}^2+m_Q^2}\right)\Phi^{J\mu}_{\sigma_1\sigma_2}(\mathbf{k})\,. \qquad (1)$$

The knowledge of the kernel of the RHD integral equation makes it possible to switch to the procedure of numerical solution and calculation of the spectrum of masses of the quarkonium.

In the description of the meson, as a system of a quark and an antiquark is widely used the Cornell potential. Effective centrally symmetric potential interaction between quarks with constituent masses m_q and m_Q contains Coulomb and linear part:

$$\hat{V}(\mathrm{r}) = -\frac{4\alpha_s}{3\mathrm{r}} + \sigma\mathrm{r} + w\ , \quad \mathrm{r} = |\mathbf{r}|\ , \qquad (2)$$

where σ , w – model parameters, α_s – QCD strong coupling constant.
This potential satisfies the requirement of quark confinement and has been widely used in calculations of the spectra of heavy mesons [2, 3].

In a common way the potential of interaction in momentum space can be constructed using the amplitude of the elastic scattering of particles that construct the system [4]

$$q_i(k_1,\lambda_{k_1}) + \bar{q}_j(k_2,\lambda_{k_2}) \to q_k(p_1,\lambda_{p_1}) + \bar{q}_l(p_2,\lambda_{p_2})\ , \qquad (3)$$

where particle momenta and spin numbers are given between parentheses and $i,j,k,l = 1,2,3$ – color degrees of freedom.

For non-zero order of the perturbation theory the main contribution to the amplitude of elastic scattering T_{fi} of quark-antiquark determined by the diagram of one-gluon exchange (see Figure 1).

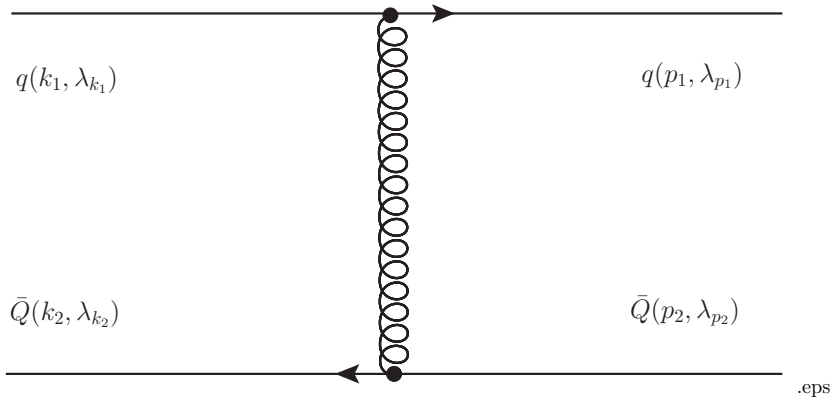


Figure 1: One-gluon exchange contribution V_{pert}

As a result the potential generated by diagram of gluon exchange including conservation of the currents and the running coupling constant of QCD will take the form:

$$V^{(\mathrm{pert})}_{\lambda_{p_1},\lambda_{p_2}\ \lambda_{k_1},\lambda_{k_2}}\left(\mathbf{k}',\mathbf{k}\right) = (-1)\frac{N_{k,k'}}{8\pi^2}\frac{4\alpha_s\left(q^2\right)}{3q^2}J^{\mu}\left(1\right)\widetilde{D}_{\mu\nu}\left(q\right)J^{\nu}\left(2\right) =$$
$$= (-1)\frac{N_{k,k'}}{8\pi^2}\frac{4\alpha_s\left(q^2\right)}{3q^2}\left[j^{\mu}\left(1\right)j_{\mu}\left(2\right) - \frac{q_{\mu}j^{\mu}\left(1\right)q_{\nu}j^{\nu}\left(2\right)}{q^2}\right] . \tag{4}$$

In non-relativistic limit $\left(\mathrm{k}^2, \mathrm{k}'^{\,2} \ll m_q^2, m_Q^2\right)$ potential (4) in coordinate space takes the form of the Coulomb potential [5]

$$\hat{V}_C\left(\mathrm{r}\right) = -\frac{4\alpha_s}{3\mathrm{r}} . \tag{5}$$

Therefore we can assume (4) as relativistic generalization of the Coulomb potential (5).

The confining component of interquark potential can be derived by analyzing the Lorentz structure of the potential and experimental data on the meson mass spectrum. The analysis results that the nonperturbative part of the interquark potential is determined as the sum of the vector ($\sim K_V(q^2)$) and scalar ($\sim K_S(q^2)$) confining parts [6]:

$$< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V}_{conf} \parallel \ \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} >=$$
$$= \frac{N_{k,k'}}{(2\pi)^3}\left[K_V(q^2)\bar{u}_{\lambda_{p_1}}(p_1)\left(\gamma^{\mu} + \frac{\mathrm{i}\kappa_q}{2m_q}\left(k_1 - p_1\right)_{\nu}\sigma^{\mu\nu}\right)u_{\lambda_{k_1}}(k_1) \times\right.$$
$$\times\bar{v}_{\lambda_{k_2}}(k_2)\left(\gamma^{\mu} + \frac{\mathrm{i}\kappa_Q}{2m_Q}\left(p_2 - k_2\right)_{\nu}\sigma^{\mu\nu}\right)v_{\lambda_{p_2}}(p_2) +$$
$$\left.+K_S(q^2)\bar{u}_{\lambda_{p_1}}(p_1)u_{\lambda_{k_1}}(k_1)\bar{v}_{\lambda_{k_2}}(k_2)v_{\lambda_{p_2}}(p_2)\right] , \tag{6}$$

where functions

$$K_V\left(q^2\right) = \frac{8\pi A_V}{\mathbf{q}^4} + \delta\left(\mathbf{k}' - \mathbf{k}\right)B_V\left(\mathrm{k}\right) \ , \ \ K_S\left(q^2\right) = \frac{8\pi A_S}{\mathbf{q}^4} + \delta\left(\mathbf{k}' - \mathbf{k}\right)B_S\left(\mathrm{k}\right) \ . \tag{7}$$

should provide confinement of the quarks in the meson.
Expression (6) for potential give in non-relativistic limit the linear confining potential $V\left(\mathrm{r}\right) = \sigma\mathrm{r} + w$ with parameters

$$\sigma = \left(A_V - A_S\right), \quad w = \left(B_V - B_S\right) . \tag{8}$$

Thus we can say that sum

$$< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V}_{pert} + \hat{V}_{conf} \parallel \ \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} > \tag{9}$$

defined by relations (4) and (6) is relativistic generalization of the Cornell potential (2).

But in structure of (9) we meet terms which give singularities in initial equation. At this place we have strong reasons for good numerical method to stay making calculations in momentum space unlike to do different expansion by velocities and so on.

Absolute the same type of hypersingular integrals we meet in bound state problems with the Cornell potential in momentum space. Further, for simplicity, we demonstrate how to calculate numerically integrals of this type with high precision on the example of non-relativistic Schrödinger equation with the Cornell potential.

After transformation of integral $[0, \infty) \to [-1, 1]$ by making substitution $k = c\frac{1+t}{1-t}$ in

$$\frac{k^2}{2\mu}\phi_l(k) + \int_0^\infty V_l(k,k')\phi_l(k')k'^2 dk' = E\phi_l(k)\ , \tag{10}$$

singular integrals appear [7]:
for Coulomb part

$$V^C(r) = -\frac{\alpha}{r} \quad \Rightarrow \quad V_l^C(k,k') = -\frac{\alpha}{\pi}\frac{Q_l(y)}{kk'} \quad \Rightarrow \int\limits_{-1}^{1} f(t) ln|t-z| dt \quad |z| \le 1 \tag{11}$$

for linear part

$$V^L(r) = \frac{r}{a^2} \quad \Rightarrow \quad V_l^L(k,k') = \frac{Q_l'(y)}{\pi(akk')^2} \quad \Rightarrow \quad \int\limits_{-1}^{1} \frac{f(t)}{(t-z)^2} dt \quad |z| \le 1, \tag{12}$$

where Q_l – Legendre polynomial of the second kind,

$$y = \frac{k^2 + k'^2}{2kk'}. \tag{13}$$

In paper [7] it was shown that "power of singularity" in integral (12) can be "reduced"

$$\int\limits_{-1}^{1} \frac{f(t)}{(t-z)^2} dt \quad |z| \le 1, \qquad \Rightarrow \qquad \int\limits_{-1}^{1} \frac{f(t)}{(t-z)} dt \quad |z| \le 1 \tag{14}$$

by making integration by parts. But instead of integral equation we receive integral-differential equation. This method has relatively low precision ($\sim 10^{-6}$). In the same paper [7] was suggested Semispectral Chebyshev method for integrals (11) and (14).

2 New quadrature formula

In paper [8] was shown that using property $\int\limits_0^\infty Q_0'(y)\, dk' = 0$ we can introduce contour-term and rewrite terms $\sim Q_0'(y)$ in a form

$$\int\limits_0^\infty Q_0'(y)\, \phi_0(k') dk' \quad \Rightarrow \quad \int\limits_0^\infty Q_0'(y)\, (\phi_0(k') - \phi_0(k))\, dk' \tag{15}$$

and substitution of pure expression

$$Q_0'(y) = \frac{1}{1-y^2} = -\left(\frac{2kk'}{k'+k}\right)^2 \frac{1}{(k'-k)^2} \ . \tag{16}$$

leads (in the case of Non-relativistic Schödinger equation with Cornell potential) to integral like

$$\frac{4\sigma}{\pi}\int_0^\infty \frac{P_l(y)}{(k'+k)^2}\frac{[\phi_\ell(k')-\phi_\ell(k)]}{(k-k')^2}dk'; \tag{17}$$

After some calculations we have received new quadrature formula for numerical calculation of such type singular integrals which combine advantages of paper [7] and Lande subtraction method

$$\int_{-1}^{1} \frac{\phi_\ell(t)-\phi_\ell(z)}{(t-z)^2}dt = \sum_{j=1}^{N}\omega_j(z)\phi_\ell(t_j) \ . \tag{18}$$

$$\omega_j(z) = \frac{2}{N}\sum_{i=1}^{N}{}'T_{i-1}(t_j)\ X_{i-1}(z) \ , \tag{19}$$

$$X_n(z) = 2\sum_{k=0}^{n-1}{}'U_{n-1-k}(z)\left\{T_k(z)\ln\left|\frac{1-z}{1+z}\right| + R_k(z)\right\} \ , \tag{20}$$

$$R_n(z) = 2\sum_{k=0}^{n-1}{}'T_k(z)\left[\frac{(-1)^{(n-k)+1}+1}{(n-k)}\right] \ . \tag{21}$$

At the same time the type of initial equation remains to be integral as before in contrast to paper [7] and accuracy of numerical calculations increases.

3 Numerical tests

3.1 Tests for integrals with exact solution

Like a pure numerical test of formula (18) lets check quadrature formula in the case of integrals that have exact solution. For the first test we took function $\phi\,(t) = t^4$. In this case exact solution will have the next form

$$I_1(\phi) = \int_{-1}^{1}\frac{t^4-z^4}{(t-z)^2}dt = \frac{2}{3}+6z^2+4z^3 ln\left|\frac{1-z}{1+z}\right|, |z|<1. \tag{22}$$

The results of calculations is presented in Table 1.

For the second test lets select function $\phi\,(t) = e^{-t}$. Than for the integral

$$I_2(\phi) = \int_{-1}^{1}\frac{e^{-t}-e^{-z}}{(t-z)^2}dt \tag{23}$$

we will have exact solution in a form

$$I_2(f_2) = \frac{e^{-1-z}}{z^2-1}\{e(z^2-1)[Ei(z+1)-Ei(z-1)] - e^{z+2}(z-1)+e^z(z+1)-2e\} \ , |z|<1 \tag{24}$$

where $Ei(z) = -\int_{-z}^{\infty}\frac{e^{-t}}{t}dt$ exponential integral. The results of calculations is presented in Table 2.

Table 1: Numerical test of quadrature formula for the integral (22).

z	Exact value, by (22)	Quadrature formula (18)	δ, %
-0.99	-13.997086845834733	-13.997086845834787	3.807×10^{-15}
-0.7	1.2267940186741848	1.2267940186741804	3.61992×10^{-15}
-0.3	1.1398104321587945	1.1398104321587932	1.16885×10^{-15}
-0.1	0.7258639838848181	0.7258639838848198	2.29428×10^{-15}
0.0	0.6666666666666666	0.6666666666666636	4.4964×10^{-15}
0.1	0.7258639838848181	0.7258639838848202	2.90609×10^{-15}
0.5	1.6173605223326117	1.6173605223326137	1.23559×10^{-15}
0.7	1.2267940186741848	1.2267940186741875	2.17195×10^{-15}
0.99	-13.997086845834733	-13.997086845834758	1.77673×10^{-15}

Table 2: Numerical test of quadrature formula for the integral (23).

z	Exact value, by (24)	Quadrature formula (18)	δ, %
-0.99	-18.597791752712833	-18.597791752712908	4.0116×10^{-15}
-0.6	-0.9833492587254643	-0.9833492587253971	6.83058×10^{-14}
-0.2	0.6793649549585903	0.6793649549585903	5.5563×10^{-15}
0.0	1.0283404811209695	1.0283404811209635	5.82998×10^{-15}
0.1	1.1445832768980895	1.1445832768980928	2.90994×10^{-15}
0.3	1.3046004784874934	1.3046004784874938	3.40402×10^{-16}
0.7	1.518977651147536	1.518977651147532	2.63125×10^{-15}
0.99	3.3470563271679046	3.3470563271677065	5.91755×10^{-14}

3.2 Non-relativistic Schrödinger equation with linear potential for the case $l = 0$

As a second numerical test we have done calculations of non-relativistic Schrödinger equation (10) with linear potential for the case $l = 0$ for which the exact solutions are well known [9].

Energy can be found by formula $\epsilon(0, s, 0) = -s^{2/3} z_\nu$, where $s \equiv 1/2\mu a$, and z_ν $(\nu = 1, 2, 3, \ldots)$ – zeros of Airy function $Ai(z)$. The results of this calculation is presented in Table 3. There is very good agreement between two results with excellent precision.

There is no results with accuracy better than $10^{-5} \div 10^{-6}$ in momentum space.

3.3 Comparison results in momentum space with results of in coordinate space

Like a crosscheck we also made comparison our results for solution of (10) with Cornell potential in momentum space for the first eigenvalue of energy E_1 for different values of coefficient α, with results of calculation in coordinate space by authors in [10] (see Table 4).

We should noted that our method let us repeat numbers from [10] with very good precisions already with $N = 100$ points, while that last was received on mesh $N = 300000$.

Table 3: Comparison of results of calculations for linear potential in momentum space by using our quadrature formula with exact value received by Airy function zeros. Number of points $N = 100$.

n	E_n (18)	E_n (exact value)	δ, %
1	2.3381074104597843	2.338107410459767	7.33×10^{-13}
2	4.0879494441309765	4.087949444130971	1.33×10^{-13}
3	5.520559828095326	5.520559828095551	4.06×10^{-12}
4	6.786708090071581	6.78670809007176	2.62×10^{-12}
5	7.944133587120411	7.944133587120854	5.58×10^{-12}
6	9.022650853340487	9.022650853340982	5.50×10^{-12}
7	10.040174341556877	10.040174341558087	1.21×10^{-11}

Table 4: Comparison of the results of $1S$-state energy calculations for Cornell potential in momentum space by using our quadrature formula with results for coordinate space (ζ_1) in [10].

α	$E_1(1S), N = 100$	$\zeta_1(1S)$ [10], $N = 300000$	$\Delta\zeta$
0.0	2.338 107 410 459 784 3	2.338 107 410 458 750	1.0×10^{-12}
0.2	2.167 316 208 772 692 5	2.167 316 208 771 731	1.0×10^{-12}
0.4	1.988 503 899 750 148 7	1.988 503 899 749 943	9.6×10^{-13}
0.6	1.801 073 805 647 306	1.801 073 805 646 145	8.5×10^{-13}
0.8	1.604 408 543 236 034 9	1.604 408 543 235 973	6.6×10^{-13}
1.0	1.397 875 641 659 084	1.397 875 641 659 578	3.8×10^{-13}
1.2	1.180 833 939 742 701	1.180 833 939 744 863	2.1×10^{-14}
1.4	0.952 640 495 217 967 7	0.952 640 495 219 193	5.8×10^{-13}

4 Conclusions

The new high precision quadrature formula for singular integrals like in bound-state equations with the Cornell potential was suggested.

Numerical tests of quadrature formula for the cases of exact solved mathematical integrals and problems with the Cornell potential were performed and was shown the good accuracy of method.

5 Acknowledgement

Authors thank to the Belarusian Republican Foundation for Fundamental Research for support of this work.

References

[1] Keister, B. Relativistic Hamiltonian dynamics in nuclear and particle physics / B. Keister, W. Polyzou // "Advances in Nuclear Physics" Plenum. New York. — 1991. — Vol. 20. — P. 225–479.

[2] Eichten, E. Spectrum of Charmed Quark-Antiquark Bound States / E. Eichten, T. Kinoshita, K. Gottfried, K.D. Lane, T.M. Yan // Phys.Rev.Lett. – 1975. – **34**. – P. 369.

[3] Eichten, E. Charmonium: comparison with experiment / E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane, Tung-Mow Yan // Phys.Rev. – 1980. – **D21**. – P. 203.

[4] Lucha, W. Relativistic treatment of fermion anti-fermion bound states / W. Lucha, H. Rupprecht, F. F. Schoberl // Phys. Rev. — 1991. — Vol. D44. — P. 242–249.

[5] Lucha, W. Effective potential models for hadrons / W. Lucha, F. F. Schoberl // e-Print arXiv:hep-ph/9601263v1 (Dated: January 13, 1996)

[6] Galkin, V. O. / V. O. Galkin, A. Y. Mishurov, R. F. Faustov // Sov. J. Nucl. Phys. — 1992. — Vol. 55. — P. 1207–1213.

[7] Deloff, A. Semi-spectral Chebyshev method in Quantum Mechanics / A. Deloff // Annals Phys. — 2007. — Vol. 322. — P. 1373–1419.

[8] Tang, A. MKN theory of bound states / A.Tang // e-Print arXiv: hep-ph/0103035v4.

[9] Abramowitz, M. Handbook of Mathematical Functions / M.Abramowitz, I.Stegun // Dover, New York – 1972.

[10] Kang, D. Precise numerical solutions of potential problems using Crank-Nicholson method / D.Kang, E.Won // e-Print arXiv: physics/060917v1 [physics.comp-ph] (Dated: February 2, 2008)

The Higgsstrahlung process in the Standard model extensions

O.M.Boyarkin
boyarkin@front.ru
International Sakharov Environmental Institute of Belarusian State University
G.G.Boyarkina
Belarusian State Pedagogical University
V.V. Makhnach
Institute of information technologies
of Belorussian State University of informanics and radioelectronics

Abstract

Within the left-right symmetric model (LRM) and the effective rank 5 models (ER5M's) the Higgsstrahlung process

$$e^+e^- \to ZH$$

is considered. It was shown that the deviations from the SM predicted by the LRM are larger than that predicted by the ER5M's. Investigation of the Higgsstrahlung disclosed that with its help one could make a choice between the SM and the SM extensions under consideration.

1 Introduction

On July 4, 2012 the discovery of a narrow resonance, with a mass near 125.7 GeV, in the search for the Standard Model (SM) Higgs boson at the Large Hadron Collider (LHC) was announced at CERN by both the ATLAS [1] and CMS [2]. This discovery shows once again that with the advent of the LHC, particle physics has entered a new exciting era. The Higgs boson observation not only confirmed the so-called Higgs mechanism for the electroweak symmetry breaking, but also opened a new possibility to perform a precise test of the SM. Within the current experimental and theoretical uncertainties the properties of the newly discovered particle are thus far in very good agreement with the predictions for a SM Higgs boson.

Due to its gauge charges the Higgs boson may interact with the beyond SM particles. This new interaction can also modify the couplings between the Higgs and SM particles at tree level or loop level. Some of the simplest extensions of the SM are those that add new neutral gauge boson Z' which could arise from a variety of contexts, ranging from simple extra $U(1)$ gauge symmetries [3], left-right models (LRM's)[3], 3-3-1 models [4], little Higgs models [5], technicolour models [6] and so on.

It is obvious that deviations from the SM predictions could be interpreted in terms of Z' parameters. Properties of Z' boson are mainly being investigated in collider experiments (see for review [7]). At hadron colliders the Z' boson could be produced via Drell-Yan production and would then be observed in the invariant mass distribution of the pair produced final state particles. In the case of e^+e^- colliders the basic processes for studying Z' boson parameters are fermion pair production, Bhabha and Möller scattering, along with W pair production.

When the LHC reaches its design energy and luminosity it should be able to see evidence for Z' up to ~ 5 TeV for a large variety of the SM extensions [8], and the HL-LHC will extend this

reach up to ~ 6 TeV. The high energy LHC (HE-LHC) would substantially extend this reach to ~ 11 TeV. In comparison, a 500 GeV e^+e^- collider with $\mathcal{L} = 500$ fb^{-1} will be sensitive to $m_{Z'}$ ~ 6 TeV that is comparable to potentiality of the HL-LHC while a 1 TeV e^+e^- collider with $\mathcal{L} = 1$ ab^{-1} will be able to detect Z' with the mass ~ 12 TeV which has the same order as the lower bound of $m_{Z'}$ at HE-LHC.

In this paper we shall centre on the two most popular GUT scenarios, namely, the LRM's and models coming from E_6 grand unification (effective rank-5 models — ER5M's). Within these SM extensions we shall consider the associated Higgs boson production with Z boson at the electron-positron annihilation

$$e^+ + e^- \to Z + H, \tag{1}$$

where H is an analog of the SM Higgs boson. The cross section of (1) are function of $m_{Z'}$ and contain the coupling constants g_{HZZ} and $g_{HZZ'}$ which describe interaction of the Higgs boson H with the neutral gauge bosons Z and Z'. The aim of the work is to investigate the influence of these parameters on the reaction in question. To avoid confusion, for every SM extensions we shall use the notations which have been universally accepted. For example, in the LRM the analog of the SM neutral gauge boson and the extra neutral gauge boson are symbolized by Z_1 and Z_2, respectively, while in the ER5M's they are indicated by the symbols Z and Z'.

2 Models

The LRM, based on the low-energy gauge group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, can arise from an $SO(10)$ or E_6 GUT. In this model, because of a symmetry between the two gauge $SU(2)$ groups quarks and leptons enter both into the left- and right-handed doublets. The LRM has three gauge coupling constants: g_L, g_R, and g' for the $SU(2)_L$, $SU(2)_R$, and $U(1)_{B-L}$ groups, respectively. In the more simplest case (canonical LRM) the Higgs sector includes one bi-doublet $\Phi\left(\frac{1}{2}, \frac{1^*}{2}, 0\right)$

$$\Phi = \begin{pmatrix} \Phi_1^0 & \Phi_2^+ \\ \Phi_1^- & \Phi_2^0 \end{pmatrix}, \tag{2}$$

and two triplets $\Delta_L(1,0,2)$, $\Delta_R(0,1,2)$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}_L) = \begin{pmatrix} \delta_L^+/\sqrt{2} & \delta_L^{++} \\ \delta_L^0 & -\delta_L^+/\sqrt{2} \end{pmatrix}, \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{\Delta}_R) = \begin{pmatrix} \delta_R^+/\sqrt{2} & \delta_R^{++} \\ \delta_R^0 & -\delta_R^+/\sqrt{2} \end{pmatrix}. \tag{3}$$

The symmetry breaking is triggered by the following VEVs

$$< \Phi >= \begin{pmatrix} k_1/\sqrt{2} & 0 \\ 0 & k_2/\sqrt{2} \end{pmatrix}, \qquad < \Delta_{L,R} >= \begin{pmatrix} 0 & 0 \\ v_{L,R}/\sqrt{2} & 0 \end{pmatrix}.$$

The bidoublet VEVs k_1 and k_2 break the SM symmetry and are of the order of the electroweak scale (EWS). Therefore, the VEVs obey to the following hierarchy

$$v_L << \max(k_1, k_2) << v_R. \tag{4}$$

After the spontaneous symmetry breaking we have 14 physical Higgs bosons: four doubly-charged scalars $\Delta_{1,2}^{(\pm\pm)}$, four singly-charged scalars $\tilde{\delta}^{(\pm)}$ and $h^{(\pm)}$, four neutral scalars $S_{1,2,3,4}$, and two neutral pseudoscalars $P_{1,2}$. The S_1 boson is an analog of the SM Higgs boson.

In the case $g_L = g_R$ the Lagrangians which will be needed in our calculation are as follows (see the book [9])

$$-L_{NC}^{LRM} = \sum_f \{\overline{\psi}_f(x)\gamma_\mu(g_{V_f}^{(1)} - g_{A_f}^{(1)}\gamma_5)Z_1^\mu(x)\psi_f(x) + \overline{\psi}_f(x)\gamma_\mu(g_{V_f}^{(2)} - g_{Af}^{(2)}\gamma_5)Z_2^\mu(x)\psi_f(x), \tag{5}$$

$$L_{S_1Z_nZ_k} = g_{S_1Z_nZ_k} S_1(x) Z_n^{\mu}(x) Z_{k\mu}(x), \qquad (n,k=1,2), \tag{6}$$

where

$$g_{Vf}^{(1)} = \frac{1}{2}\Bigg\{ e c_\phi c_W^{-1} s_W^{-1}\Big[S_3^W(f_L) - 2Q(f)s_W^2\Big] + \\ + \frac{e s_\phi c_W^{-1}}{\sqrt{e^{-2}g_R^2 c_W^2 - 1}}\Big[e^{-2}g_R^2 c_W^2 S_3^W(f_R) + S_3^W(f_L) - 2Q(f)\Big]\Bigg\},$$

$$g_{Af}^{(1)} = \frac{1}{2}\Bigg\{ e c_\phi c_W^{-1} s_W^{-1} S_3^W(f_L) - \frac{e s_\phi c_W^{-1}}{\sqrt{e^{-2}g_R^2 c_W^2 - 1}}\Big[e^{-2}g_R^2 c_W^2 S_3^W(f_R) - S_3^W(f_L)\Big]\Bigg\},$$

$$g_{Vf}^{(2)} = g_{Vf}^{(1)}\Big(\phi \to \phi + \frac{\pi}{2}\Big), \qquad g_{Af}^{(2)} = g_{Af}^{(1)}\Big(\phi \to \phi + \frac{\pi}{2}\Big), \qquad c_\phi = \cos\phi, \qquad s_\phi = \sin\phi,$$

$$g_{S_1Z_1Z_1} = -\frac{g_L^2(k_-^2 c_\theta + 2k_1k_2 s_\theta)}{2\sqrt{2}c_W^2 k_+}(c_\Phi - \sqrt{c_W^2 - s_W^2}s_\Phi)^2, \qquad k_- = \sqrt{k_1^2 - k_2^2},$$

$$g_{S_1Z_1Z_2} = \frac{g_L^2(k_-^2 c_\theta + 2k_1k_2 s_\theta)}{\sqrt{2}c_W^2 k_+}[2c_W^2 c_\Phi s_\Phi + \sqrt{c_W^2 - s_W^2}(c_\Phi^2 - s_\Phi^2)],$$

$$g_{S_1Z_2Z_2} = -\frac{g_L^2(k_-^2 c_\theta + 2k_1k_2 s_\theta)}{2\sqrt{2}c_W^2 k_+}(\sqrt{c_W^2 - s_W^2}c_\Phi + s_\Phi)^2,$$

$s_W = \sin\theta_W$, $c_W = \cos\theta_W$, θ_W is the Weinberg angle, and θ is the mixing angle in the sector of the neutral Higgs bosons S_1 and S_2 ($\tan 2\theta \sim k_-^2/v_R^2$). Since the mixing angle in the neutral gauge boson sector ϕ is very small (it was found $-0.0023 < \phi < 0.0027$) then in what follows we shall ignore by the mixing in the neutral gauge boson sector. In this case the couplings constants $g_{V_f}^{(1)}$ and $g_{A_f}^{(1)}$ are the same as in the SM ($g_{V_f}^{(1)} = g_{V_f}, g_{A_f}^{(1)} = g_{A_f}$).

There are two possibilities of defining the left-right (LR) symmetry as a generalized parity P and as a generalized charge conjugation C. However, instead of using parity or charge conjugation, one could choose D-parity, which is broken spontaneously [10]. In this case the discrete parity symmetry gets broken (by the VEV of a parity odd singlet scalar field) much before the $SU(2)_R$ gauge symmetry breaks. The gauge group is effectively $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times P$, where P is the discrete left-right symmetry which we call D-parity. This D-parity symmetry is different from the Lorentz parity in the sense that Lorentz parity interchanges left handed fermions with the right handed ones but the bosonic fields remain the same. Whereas, the D-parity also interchanges the $SU(2)_L$ Higgs fields with the $SU(2)_R$ Higgs fields. The parity odd singlet field breaks this gauge symmetry at high scale $\sim (10^{16} - 10^{19})$ GeV to $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ which further breaks down to the SM gauge group at a lower scale. In such a version of the LRM's the relation between gauge boson masses does not hold and the charged gauge boson $W_R^{\pm}$ appears to be more massive than the neutral gauge boson Z_R. Then a collider analysis using the 8 TeV ATLAS 20.3 fb^{-1} luminosity dilepton data [11] derives $m_{Z_R} \geq 2.5$ TeV for $g_L = g_R$.

Further we consider the simplest E_6-based low-energy group, $SU(2)_L \times U(1)_Y \times U(1)_{Y'}$ resulting from compactification of 10-dimensional $E_8 \times E_8'$ superstring theory to four dimensions on a manifold with a $SU(3)$ holonomy [12]. The compactification scheme of the model predicts that the matter fields occur in three supersymmetric chiral multiplets, each transforming according to the quantum numbers of the fundamental **27** of E_6. Among the fields associated with each such multiplet are five colorless neutral superfields: one is usually assigned nonzero lepton number, while two, $H_1^{(a)}$ and $H_2^{(a)}$ ($a = 1,2,3$), belong to doublets of the residual $SU(2)$, and

the remaining two, $N_1^{(a)}$ and $N_2^{(a)}$, are singlets under $SU(2)$. Usually one works in the basis when the breaking of the $SU(2)_L \times U(1)_Y \times U(1)_{Y'}$ symmetry down to $U(1)_{em}$ is realized when only the neutral components of the third-family Higgs fields $N_1^{(3)}$, $N_2^{(3)}$ and $H_1^{(3)}$ acquire nonzero VEV's — v_1, v_2 and n ($v_1, v_2 \ll n$), respectively. As usual, several of the degrees of freedom of these third-family Higgs bosons are eaten by the W, Z, and Z' in acquiring mass, and we are left with the six physical Higgs bosons: one neutral pseudoscalar H_3^0, three neutral scalars H_α^0 ($\alpha = 2, deg, Z'$), and two singly-charged scalars $H^\pm$ (the neutral scalar degenerate with H_3^0 and $H^\pm$ is denoted by H_{deg}^0 and that degenerate with Z' by $H_{Z'}^0$).

The third-family Higgs bosons will have trilinear tree-level couplings to vector-boson pairs and their Yukawa couplings to fermions of all generations will be tied to the fermion masses. In contrast, the Higgs fields of the first and second families by definition do not have VEV's and they possess only quartic couplings to vector-boson pairs, and their Yukawa couplings to fermions of their own and other generations cannot be very large. Therefore, only the Higgs bosons associated with the third-generation **27** multiplet of E_6 participate in the electroweak symmetry breaking.

Below we give the Lagrangians of the neutral Higgs boson H_2^0 being an analog of the SM Higgs with the fermions and neutral gauge bosons:

$$-L_{NC}^{E_6} = \sum_f \{\overline{\psi}_f(x)\gamma_\mu(g_{V_f} - g_{A_f}\gamma_5)Z^\mu(x)\psi_f(x) + \overline{\psi}_f(x)\gamma_\mu(g'_{V_f} - g'_{A_f}\gamma_5)Z'^\mu(x)\psi_f(x), \qquad (7)$$

$$L_{NGB}^{E_6} = g_{H_2^0ZZ}H_2^0(x)Z_\mu(x)Z^\mu(x) + g_{H_2^0ZZ'}H_2^0(x)Z_\mu(x)Z'^\mu(x), \qquad (8)$$

where

$$Z = Z_1\cos\phi + Z_2\sin\phi, \qquad Z' = Z_1\sin\phi + Z_2\cos\phi,$$

$$g'_{V_l} = -g'_{V_d} = \frac{ec_\beta}{c_W}\sqrt{\frac{5}{30}}, \qquad g'_{A_l} = g'_{A_d} = \frac{e}{2c_W}\sqrt{\frac{5}{3}}\left[\frac{c_\beta}{\sqrt{10}} + \frac{s_\beta}{\sqrt{6}}\right],$$

$$g'_{V_\nu} = g'_{A_\nu} = \frac{e}{2c_W}\sqrt{\frac{5}{3}}\left[\frac{3c_\beta}{\sqrt{40}} + \frac{s_\beta}{\sqrt{24}}\right], \qquad g'_{V_u} = \frac{e}{2c_W}\sqrt{\frac{5}{3}}\left[-\frac{c_\beta}{\sqrt{10}} + \frac{s_\beta}{\sqrt{6}}\right], \qquad g'_{A_u} = 0,$$

$$g_{H_2^0ZZ} = \frac{em_Z}{s_W c_W}, \qquad g_{H_2^0ZZ'} = \frac{em_Z}{3c_W}(c_\beta^2 - 4s_\beta^2), \qquad \tan\beta = \frac{v_2}{v_1}$$

$c_\beta = \cos\beta$ and $s_\beta = \sin\beta$ (the expressions for $g_{H_2^0ZZ}$ and $g_{H_2^0ZZ'}$ are given in the limit of large n and small neutral gauge boson mixing). β is treated as a free parameter and its particular values correspond to special models. The more popular models of this kind (effective rank-5 models — ER5M's) are as follows: the Z_χ model ($\beta = 0$), the Z_ψ model ($\beta = \pi/2$), the Z_η model ($\beta = \pi - \arctan\sqrt{5/3}$), the Z_I model — the inert model ($-\beta = \arctan\sqrt{3/5}$), the Z_N model — the neutral model ($\beta = \arctan\sqrt{15}$), the Z_N model — the secluded sector model ($\beta = \arctan\sqrt{15}/9$).

At present, the highest mass bounds on most extra neutral gauge bosons are obtained by searches at the LHC by the ATLAS and CMS experiments. The current direct limits of $m_{Z'}$ are ~ 3 TeV. For example, results based on dilepton resonance searches in $\mu^+\mu^-$ and e^+e^- final states have used data from the 7 TeV proton collisions collected in 2011 and the more recent 8 TeV data collected in 2012. ATLAS [**?**] has obtained exclusion limits at 95% C.L. of $m_{Z_\eta} > 2.44$ TeV, $m_{Z_\chi} > 2.54$ TeV and $m_{Z_\psi} > 2.38$ TeV from 8 TeV collisions with 20 fb^{-1} integrated luminosity, while CMS [13] has obtained 95% C.L. exclusion limits of $m_{Z_\psi} > 2.60$ TeV from 8 TeV collisions using 20 fb^{-1} of integrated luminosity.

3 The process $e^+e^- \to ZH$

Now we investigate the Higgsstrahlung (HS) that should be measured with precision at high-energy e^+e^- colliders. At present various options of such devices are being considered. The most nature concept is a linear e^-e^+ collider, and two projects are being developed, namely, the International Linear Collider (ILC) that would operate at energies up to 1 TeV, and Compact Linear Collider (CLIC) that could possibly operate at energies up to 3 TeV.

Let us start with the LRM. The Feynman diagram corresponding to the process $e^+e^- \to Z_1S_1$ are shown on Fig.1. The total cross section is defined by the expression

$$\sigma^{LRM}(e^+e^- \to Z_1S_1) = \frac{\beta_{Z_1}(3m^2_{Z_1}+\beta^2_{Z_1})}{18\pi m^2_{Z_1}\sqrt{s}}\Bigg\{\frac{g^2_{S_1Z_1Z_1}(g^2_{V_e}+g^2_{A_e})}{[(s-m^2_{Z_1})^2+m^2_{Z_1}\Gamma^2_{Z_1}}+$$

$$+\frac{2g_{S_1Z_1Z_1}g_{S_1Z_1Z_2}(g_{V_e}g^{(2)}_{V_e}+g_{A_e}g^{(2)}_{A_e})[(s-m^2_{Z_1})(s-m^2_{Z_2})+m_{Z_1}m_{Z_2}\Gamma_{Z_1}\Gamma_{Z_2}]}{[(s-m^2_{Z_1})^2+m^2_{Z_1}\Gamma^2_{Z_1}][(s-m^2_{Z_2})^2+m^2_{Z_2}\Gamma^2_{Z_2}]}+$$

$$+\frac{g^2_{S_1Z_1Z_2}[(g^{(2)}_{V_e})^2+(g^{(2)}_{A_e})^2]}{[(s-m^2_{Z_2})^2+m^2_{Z_2}\Gamma^2_{Z_2}]}\Bigg\}, \tag{9}$$

where

$$\beta_{Z_1} = \frac{\sqrt{s-(m^2_{Z_1}-m^2_{S_1})}\sqrt{s-(m^2_{Z_1}+m^2_{S_1})}}{2\sqrt{s}},$$

and s is the center-of-mass energy squared.

The first term in Eq. (9) which we shall symbolize as a $\sigma_{Z_1Z_1}$ is connected with the Feynman diagram for $e^-e^+ \to Z_1Z_1^* \to Z_1S_1$ and when $g_{S_1Z_1Z_1} = g_{HZZ} = 2em_Z/\sin 2\theta_W$ it coincides with the SM result [14]. The third term denoted by $\sigma_{Z_2Z_2}$ is associated with the Feynman diagram for $e^-e^+ \to Z_1Z_2^* \to Z_1S_1$. The second term $\sigma_{Z_1Z_2}$ describes the interference of the diagrams with the Z_1- and Z_2-boson in the virtual states. Since at the substitution

$$g_{S_1Z_1Z_2} \to g_{S_1Z_1Z_1}, \qquad g^{(2)}_{V_e} \to g_{V_e}, \qquad g^{(2)}_{A_e} \to g_{A_e}, \qquad m_{Z_2} \to m_{Z_1}, \quad \Gamma_{Z_2} \to \Gamma_{Z_1}, \tag{10}$$

the diagram describing $e^-e^+ \to Z_1Z_2^* \to Z_1S_1$ passes into the diagram describing $e^-e^+ \to Z_1Z_1^* \to Z_1S_1$, then the expression (25) admits the following simple checking. When we fulfill the replacement (10) then the third term must change over to the first one and the second term (interference term) divided by two converts to the first one too, that is to say,

$$\sigma_{Z_2Z_2} \to \sigma_{Z_1Z_1}, \qquad \frac{1}{2}\sigma_{Z_1Z_2} \to \sigma_{Z_1Z_1}. \tag{11}$$

In Refs.[15, 16] the HS has been considered within the $SU(2)_L \times U(1)_Y \times U(1)_{B-L}$ model with the extra Z' boson. The results of these works do not coincide even with each other. Moreover, in the cross section obtained in Ref. [15] the term σ_{ZZ} does not equal to the SM expression. In regard to the work [16], then here the obtained cross section holds the SM contribution but at the replacement of the type (10) the interference term divided by two does not change-over to the SM term as one should expect.

At e^+e^- collider the Higgsstrahlung process is dominant for moderate values of the ratio $m_H/\sqrt{s}$. At high energies it falls off according to the scaling law s^{-1}. When $\sqrt{s} = m_{Z_2}$ there is the resonance peak whose high crucially depends on the ratio $r = g_{S_1Z_1Z_2}/g_{S_1Z_1Z_1}$ and the total decay width of Z_2 boson. By virtue of the fact that in the LRM $r = -2\sqrt{c^2_W - s^2_W}$, the resonance effect will be important.

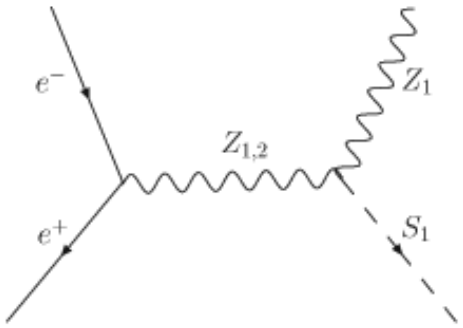


Figure 1: The Feynman diagrams for $e^+e^- \to Z_1S_1$.

If one assumes that $g_{S_1Z_1Z_1}$ takes the SM value, then we may directly compare the SM cross section with the LRM one. In this case the latter will look like

$$\sigma^{LRM} = \sigma^{SM} + \Delta\sigma, \tag{12}$$

where

$$\Delta\sigma = \frac{g_L^4\beta_{Z_1}(3m_{Z_1}^2+\beta_{Z_1}^2)}{72\pi c_W^4\sqrt{s}}\left\{\frac{(16s_W^4-6s_W^2)[(s-m_{Z_1}^2)(s-m_{Z_2}^2)+m_{Z_1}m_{Z_2}\Gamma_{Z_1}\Gamma_{Z_2}]}{[(s-m_{Z_1}^2)^2+m_{Z_1}^2\Gamma_{Z_1}^2][(s-m_{Z_2}^2)^2+m_{Z_2}^2\Gamma_{Z_2}^2]}-\right.$$

$$\left.-\frac{20s_W^4-12s_W^2+2}{[(s-m_{Z_2}^2)^2+m_{Z_2}^2\Gamma_{Z_2}^2]}\right\}.$$

Calculations show that in the case under consideration contributions coming from the second term in (12) are sizable

$$\left.\frac{\Delta\sigma}{\sigma^{SM}}\right|_{\sqrt{s}=0.5\ TeV} \simeq \begin{cases} 8.3\%, & \text{when } m_{Z_2}=2.5 \text{ TeV}, \\ 3.2\%, & \text{when } m_{Z_2}=4 \text{ TeV}, \\ 0.5\%, & \text{when } m_{Z_2}=10 \text{ TeV}, \end{cases} \tag{13}$$

and

$$\left.\frac{\Delta\sigma}{\sigma^{SM}}\right|_{\sqrt{s}=1\ TeV} \simeq \begin{cases} 35\%, & \text{when } m_{Z_2}=2.5 \text{ TeV}, \\ 13.5\%, & \text{when } m_{Z_2}=4 \text{ TeV}, \\ 2.1\%, & \text{when } m_{Z_2}=10 \text{ TeV}. \end{cases} \tag{14}$$

Once it will prove that the coupling constant $g_{S_1Z_1Z_1}$ does not equal to the SM value, then we have to examine the case of the polarized electron-positron beams. With this object in mind the following quantities are introduced

$$\lambda_\pm^{SM} = \frac{\sigma_\pm^{SM}}{\sigma_+^{SM}+\sigma_-^{SM}}, \qquad \lambda_\pm^{LRM}(s) = \frac{\sigma_\pm^{LRM}}{\sigma_+^{LRM}+\sigma_-^{LRM}}, \qquad \lambda_\pm^{LRM}(s) = \lambda_\pm^{SM} + \delta_\pm^{LRM}(s), \tag{15}$$

where σ_+^{SM} (σ_-^{SM}) is the SM cross section for the left- (right-) polarized electrons. Then we have

$$\left.\frac{\delta_+^{LRM}(s)}{\lambda_+^{SM}}\right|_{\sqrt{s}=0.5\ TeV} \simeq \begin{cases} 1.7\%, & \text{when } m_{Z_2} = 2.5 \text{ TeV}, \\ 0.6\%, & \text{when } m_{Z_2} = 4 \text{ TeV}, \\ 0.1\%, & \text{when } m_{Z_2} = 10 \text{ TeV}, \end{cases} \tag{16}$$

and

$$\left.\frac{\delta_+^{LRM}(s)}{\lambda_+^{SM}}\right|_{\sqrt{s}=1\ TeV} \simeq \begin{cases} 9\%, & \text{when } m_{Z_2} = 2.5 \text{ TeV}, \\ 2.8\%, & \text{when } m_{Z_2} = 4 \text{ TeV}, \\ 0.4\%, & \text{when } m_{Z_2} = 10 \text{ TeV}. \end{cases} \tag{17}$$

The total cross section of the Higgsstrahlung for the ER5M's follows from Eq. (9) under replacements like (10). In that event we shall also work with the polarized electron-positron beams. Analysis reveals that the maximal deviation from the SM prediction occurs for the η-model. But now the value of $\delta_+^{E_6}$ is one order of magnitude less than in the LRM case. To cite an example, we have

$$\left.\delta_+^{E_6}(s)\right|_{\sqrt{s}=1\ TeV} \simeq \begin{cases} -0.026, & \text{when } m_{Z_2} = 1.5 \text{ TeV} \\ -0.013, & \text{when } m_{Z_2} = 2 \text{ TeV}, \\ -0.003, & \text{when } m_{Z_2} = 4 \text{ TeV}. \end{cases} \tag{18}$$

At ILC one could achieve luminosity $\mathcal{L} \sim \mathcal{O}(\text{ab}^{-1})$ at $\sqrt{s} \sim \mathcal{O}(\text{TeV})$. In this range, for example, the SM cross section has the order of magnitude of $few \times 10^{-1}$ pb. Then, the number of the produced Higgs bosons which is predicted by the SM is as large as

$$n = \sigma^{SM}(e^+e^- \to ZH) \times \mathcal{L} \sim \text{few pb} \times 1 \text{ ab}^{-1} \sim \text{few} \times 10^5.$$

So, we have a good chance to establish whether deviations from the SM take place or not.

The cleanest channel for isolating the Higgsstrahlung from the background is provided by the $\mu^+\mu^-$ or e^+e^- decay mode for the Z boson and $b\bar{b}$ decay mode for H boson. The detail analysis of the background processes could be found in Ref. [14].

4 Conclusion

Two kinds of models arising in a GUT scenario, the LRM and ER5M's, have been considered. The associated Higgs production with Z boson at the electron positron annihilation (Higgsstrahlung)

$$e^+ + e^- \to Z + H$$

has been investigated. For the LRM two cases, $g_{S_1Z_1Z_1} = g_{HZZ}$ and $g_{S_1Z_1Z_1} \neq g_{HZZ}$, have been treated. In the former the total cross sections of the LRM and the SM have been compared directly. Analysis shows that even at the first stage of the ILC ($\sqrt{s} = 0.5$ TeV) one could get the lower bound on the extra neutral gauge boson mass equal to 7 TeV. To do this, one should measure the Higgsstrahlung with the precision of about 1%.

For the case $g_{S_1Z_1Z_1} \neq g_{HZZ}$ we have compared the ratios of the cross section with the left-polarized electrons to the cross section with the unpolarized electrons for the SM and LRM. If one supposes that the cross sections would be measured with the precision of about 1%, then the lower bound on the extra neutral gauge boson mass equal to 6.4 TeV would be obtained only at the second stage of the ILC ($\sqrt{s} = 1$ TeV).

The Higgsstrahlung investigation has shown that among the ER5M's the maximal deviations from the SM gives the η model. In that event the deviations from the SM predictions appear to be small than for the LRM, and, as a result, the 1% precision of the cross section measurement allows to obtain the bound $m_{Z'} \geq 2.5$ TeV only at $\sqrt{s} = 1$ TeV.

References

[1] Georges Aad *et al.*, Phys. Lett. **B716** (2012) 1.

[2] Serguei Chatrchyan *et al.*, Phys. Lett. **B716** (2012) 30.

[3] Jogesh C. Pati and Abdus Salam, Phys. Rev. **D10** (1974) 275.

[4] F. Pisano and V. Pleitez, Phys. Rev. **D46** (1992) 410; P. H. Frampton, Phys. Rev. Lett. **69** (1992) 2889.

[5] N. Arkani-Hamed, A. G. Cohen, E. Katz and A. E. Nelson, JHEP, **0207** (2002) 034.

[6] R. S. Chivukula, E. H. Simmons and J. Terning, Phys. Lett. B **331** (1994) 383.

[7] G. Brooijmans, M.-C.Chen, and B.A. Dobrescu, Chinese Physics C **38** (2014) 610 (The Review of Particle Physics).

[8] J. Erler, P. Langacker, S. Munir and E. Rojas, JHEP **1111** (2011) 076.

[9] O. M. Boyarkin, *Advanced Particles Physics, Volume II: The Standard Model and Beyond* (Taylor and Francis Group, London - New York, 2011), 555 pp.

[10] D. Chang, R.N. Mohapatra and M.K. Parida, Phys. Rev. Lett. **52** (1984) 1072.

[11] Sudhanwa Patra, Farinaldo S. Queiroz, Werner Rodejohann, arXiv:1506.03456 [hep-ph].

[12] E. Witten, Phys. Lett. **B155** (1985) 151; Nucl. Phys. **B258** (1985) 75. (2013) 081803; ATLAS Collaborations, Phys. Lett. **B726** (2013) 120.

[13] CMS Collaboration, CMS-PAS-EXO-12-061.

[14] V. Barger, K. Cheung, A. Djouadi, B. A. Kniehl, P. M. Zerwas, Phys. Rev. **D49**(1994) 79.

[15] J. Lorenzo Diaz-Cruz, Javier M. Hernandez-Lopez, Javier A. Orduz-Ducuara, arXiv:1304.0016 [hep-ph] (2013).

[16] A. Gutierrez-Rodriguez, M. A. Hernandez-Ruiz, arXiv:1506.07575 [hep-ph].

Neutrino mass matrix parameterization

V. V. Gilewsky and **I. S. Satsunkevich**
JIPNR-Sosny, NAS Belarus

Abstract

The restrictions on mass matrix of quarks and neutrino are analyzed. We compare known parameterizations and propose new one, based on Gibbs-Fedorov vector-parameter.

1 Introduction

We don't know what neutrinos will prepare for us even in the near future because the recent history of neutrino is full by sudden discoveries. For example, a rather new experimental discovery of neutrino oscillations [1] means the revealing of neutrino fields mixing. This phenomenon provides the solution for a "solar neutrino puzzle", which for a long time annoying many physicists. The idea of vacuum oscillations was proposed by B. Pontecorvo [2] to explain disappearance of neutrino in analogy with $K-$mesons behavior. This idea is more complex then K-meson oscillation because the neutrinos are spin 1/2 particle and could have Majorana field nature [3]. This fermion type is a special kind of truly neutral (self-conjugated) fermion field. It may be used for electrically neutral fermions. There is no experimental evidence up to now about how to consider neutrino as Dirac or Majorana fermion. Later the idea of vacuum oscillation was complemented by the mechanism of oscillation in the time of movement through the matter [4] and strong magnetic field. Now the deep analysis of possible descriptions of fields and manner of mass terms introducing is going forward. There are also a wide experimental program for investigation of neutrino properties on neutrino fluxes from nuclear reactors, from accelerators, from Sun , from atmosphere and from galactic sources [5].

2 Mass matrix and mixing matrix parameters

In Standard model of electroweak interactions the neutrino field is represented by kinetic part of free field and interaction part, which is the main experimental characteristic. In interaction part there are participated only left component of bi-spinor field ν_L, which is equivalent to the description by Weyl two-component spinor. There are two possibility to complement spinor to whole bi-spinor:

(a) to add right components of Dirac bi-spinor field ν_R, which appear do not interacting with any other Standard model fields ("sterile" components). These additional fields may be experimentally discovered in their gravitational interaction (as a part of dark

matter of Universe) or in vacuum oscillation of neutrino flux when field has some kind of mass terms (even diagonal on flavor terms produce the oscillation sterile to active components);

(b) to add right component, which is conjugated to left spinor component. This way do not increase the physical degree of freedom and lead to Majorana bi-spinor field [3]. In this case it is possible the reaction of $0\nu\beta\beta$ decay. The oscillation sterile$\leftrightarrow$active components is impossible (i.e. the partial disappearance of neutrino flux). The flavor oscillation is possible only for non-diagonal mass terms.

Because the absence of experimental prove of disappearance of neutrino flux the last variant is preferable. The another support is the implication of Occam's razor – "Entities must not be multiplied beyond necessity" (in Latin – Non sunt multiplicanda entia sine necessitate).

It is well known that neutrino appears in weak interaction in pair with lepton of one of three known lepton family (or flavor). So, neutrino belongs to one of weak lepton isotopic doublets $L_{L\alpha}$, where index $\alpha = e, \mu, \tau$ takes one of three value. The mass matrix may be non-diagonal:

$$L_M^0 = \frac{i}{4} \sum_{\alpha} \bar{\nu}_\alpha \overset{\leftrightarrow}{\partial}_\mu \gamma_\mu \nu_\alpha + \frac{i}{2} \sum_{\alpha,\alpha'} \bar{\nu}_\alpha M_{\alpha,\alpha'} \nu_{\alpha'}, \tag{1}$$

which means the violation of electron, muon and tau lepton charges separately for each flavor. From Hermitian properties of Lagrangian follows the Hermitian of mass matrix ($M^+ = M$). The existence of such matrix introduce in general 9 additional mass parameters in case of three family (3 are the real diagonal elements and 3 are complex off-diagonal ones). If mass matrix is real, the Hermitian condition becomes the condition of diagonal symmetry and the number of mass parameters will be 6.

The movement of neutrino to detector is motion free of interactions, therefore, it has to be described as motion of particle with definite inertial mass. So, to describe propagation of neutrino it is needed to switch on basis where mass matrix is diagonal:

$$M_{diag} = U^+ M U = diag(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}), \qquad U^+ U = 1, \tag{2}$$

the unitary property of the transformation matrix is follows from the demand of Lagrangian kinetic part invariance:

$$(\nu_\alpha) = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = (U_{\alpha i} \nu_i). \tag{3}$$

Unitary transformation has in general 9 parameters, which allow to transform mass matrix into diagonal form.

In case of Majorana fermions it is possible to treat them as real bi-spinors. This statement is evident in Majorana representation of Dirac matrices and are takes place in any other representation. From the definition of charge conjugation $\chi = \chi^C = C\chi^*$ follows the conclusion $\bar{\chi_a}\chi_b = \bar{\chi_b}\chi_a$. This property made mass matrix M_{ij} a symmetric (not only Hermitian) one, because the off-diagonal mass terms hold only one real parameter: $\bar{\chi_a}(\mu_{ab} + i\rho_{ab})\chi_b + \bar{\chi_b}(\mu_{ab} - i\rho_{ab})\chi_a = \mu_{ab}(\bar{\chi_a}\chi_b + \bar{\chi_b}\chi_a) + i\rho_{ab}(\bar{\chi_a}\chi_b - \bar{\chi_b}\chi_a)$ and last term is

zero for anti-commuting fermions. So the mass matrix of Majorana fermions $M^T = M$ is described by six parameters (3 are the real diagonal elements and 3 are real off-diagonal ones).

In case of real bi-spinors it is natural to demand – the Majorana real bi-spinors have to be transformed into the real Majorana ones. This condition made the unitary matrix U becomes orthogonal matrix O: $O^T O = 1$. This approach decreases the number of parameters in mixing matrix to three – rotation angles of orthogonal 3-dimensional group. Such approach also removes all questions connected with CP-violation in neutrino mixing.

3 Parameterizations of fermion mixing matrix

It is well known that for correct description of hadron processes it is needed to introduce the quark mixing matrix. This matrix produces superpositions of charged Dirac fermions (of the same charge) and called CKM mixing matrix (according to authors names N. Cabibbo, M. Kobayashi, T. Maskawa) [6]. This matrix has 4 parameters because 5 phases are absorbed by 6 quark fields (one global phase has no influence). The 3 parameters came from orthogonal subgroup – they are 3 angles of rotation around 3 independent axes. The last parameter is unabsorbed complex phase, which connected with small $CP-$violation:

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{4}$$

where $s_{ij}(c_{ij}) = \sin\theta_{ij}(\cos\theta_{ij})$. Besides of the number of parameters this parameterization seems as ansatz, which is a little modification of rotations. In case of Dirac neutrino the simplest way is the use of the same parameterization of mixing matrix.

For Majorana fermions, which have no phase symmetry and can not absorb arbitrary phases of mass matrix, the standard mixing matrix [7] has defined as matrix with 3 angle parameters and 3 phases. This parameterization is called PMNS mixing matrix (according to authors names – B. Pontecorvo, Z.Maki, M. Nakagawa, S. Sakata [8]):

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{23} & 0 \\ 0 & 0 & 1 \end{pmatrix} P(\alpha_1, \alpha_2) = \\ &= V_{CKM} P(\alpha_1, \alpha_2), \end{aligned} \tag{5}$$

where $P(\alpha_2, \alpha_3) = diag(1, e^{i\alpha_2/2}, e^{i\alpha_3/2}, 1)$ contains the unabsorbed by Majorana fields phase factors.

For quarks there is another widely used parameterization of mixing matrix which proposed by L. Wolfenstein [9] and takes into account the numerical values of elements. It is based on series on parameter λ equals to sin of Cabibbo angle:

$$V_{CKM} = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{6}$$

where $\lambda = s_{12}$, $A\lambda^2 = s_{23}$, $A\lambda^3(\bar{\rho} - i\bar{\eta}) = s_{13}e^{-i\delta}$. The experimental values of all parameters are of the same order: $\lambda = 0.2254$, $A = 0.814$, $\bar{\rho} = 0.117$, $\bar{\eta} = 0.353$ [7]. This parameterization is used for perturbation calculation.

These parameterizations are based on simple calculation the number of parameters for matrix U. General complex $n \times n$ matrix has $2n^2$ real parameters. The unitary condition $U^+U = 1$ gives n^2 real equations, so the unitary matrix has $2n^2 - n^2 = n^2$ parameters. Orthogonal subgroup $O(n)$ of real matrices is parametrized by anti-symmetric rotation generators, which is equal to number of the off-diagonal elements $(n^2 - n)/2$. Therefore, there are $n_{angle} = \frac{n(n-1)}{2}$ rotational parameters, defined by, for instance, Euler angles of orthogonal subgroup. The remaining parameters $n_{phases} = n^2 - n_{angle} = \frac{n(n+1)}{2}$ are phases.

To exclude unphysical phases it is used to take into account the form of charged current, which contain $2n$ fermion field. That is from all phases it is need to extract $2n-1$ phases absorbed by fields (one global phase has no influence). Finally, it is resulting $n_{phys\phi} = \frac{(n-1)(n-2)}{2}$ physical phases. For the mass case (not an interaction Lagrangian with transition between field) it may be absorbed only $n-1$ phases. So, the physical phases are $n = \frac{(n+1)(n-2)}{2}$. That is for 3 generation there is 2 phases (not 1 as in CKM-matrix and 3 as in PMNS-matrix). In Majorana case the mixing matrix is real orthogonal matrix without any phase parameters.

4 Vector parameterization of mixing matrix

For parameterization of mixing matrix of neutrino and quarks it is possible to use vector parameterization of rotation group $SO(3)$ by one three-dimensional vector $\vec{\rho}$, proposed by Gibbs, then reinvented and developed by F.I. Fedorov [10]. The basis the parameterization is grounded on a simple fact that every $SO(3)$ transformation may be presented as single rotation in respect to definite axis (or by choosing the basis where rotation matrix becomes diagonal). In this approach the every 3-vector $\vec{\rho}$ defines corresponding matrix from proper orthogonal group:

$$\begin{aligned} O(\vec{\rho}) &= \frac{1 - \vec{\rho}^2 + 2\vec{\rho}\cdot\vec{\rho} + 2\vec{\rho}^{\times}}{1 + \vec{\rho}^2} = \\ &= \frac{1}{1+\vec{\rho}^2}\begin{pmatrix} 1+\rho_1^2-\rho_2^2-\rho_3^2 & 2(\rho_1\rho_2-\rho_3) & 2(\rho_1\rho_3+\rho_2) \\ 2(\rho_2\rho_1+\rho_3) & 1+\rho_2^2-\rho_1^2-\rho_3^2 & 2(\rho_2\rho_3+\rho_1) \\ 2(\rho_3\rho_1+\rho_2) & 2(\rho_3\rho_2-\rho_1) & 1+\rho_3^2-\rho_1^2-\rho_2^2 \end{pmatrix}. \end{aligned} \quad (7)$$

And every rotation matrix from proper group O uniquely corresponds to 3-dimensional vector-parameter $\vec{\rho}$, obtained by following recipe:

$$\vec{\rho}^{\times} = (\varepsilon_{aib}\rho_i) = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{pmatrix} = \frac{O - O^T}{1 + Tr(O)}, \quad (8)$$

$$\rho_1 = (\vec{\rho}^{\times})_{32}, \quad \rho_2 = (\vec{\rho}^{\times})_{13}, \quad \rho_3 = (\vec{\rho}^{\times})_{21},$$

based on the relations

$$1 + Tr(O) = \frac{4}{1+\vec{\rho}^2}, \qquad O - O^T = \frac{4\vec{\rho}^{\times}}{1+\vec{\rho}^2}. \quad (9)$$

This parameterization has additional advantage — vector-parameter has physical sense. The direction of vector $\vec{\rho}$ is defined by the axis of rotation, the value of $|\vec{\rho}|$ is defined by the value of rotation angle α: $|\vec{\rho}| = \tan\frac{\alpha}{2}$.

For the mean experimental values of quark mixing matrix [7]:

$$V = (V_{ff'}) = \begin{pmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{pmatrix}, \tag{10}$$

where small CP-violation phase δ is omitted.

From the above experimental matrix it is easy to extract the numerical values of vector-parameter $\vec{\rho} = (-0.000022803, -0.00134538, -0.0000354714)$ or $-\vec{\rho} = (2.28 \times 10^{-5}, 1.31 \times 10^{-3}, 3.55 \times 10^{-5})$, $|\vec{\rho}| = \sqrt{\vec{\rho}} = 0.116056$ and $\alpha = 0.0269$.

In [7] the data are represented already in standard CKM-parameterization in terms of $\sin^2 \theta_{ij}$. The experimental data from [7] produce the next neutrino mixing matrix:

$$U = \begin{pmatrix} 0.8222 & 0.5485 & 0.153 \\ -0.5004 & 0.5679 & 0.6532 \\ 0.2714 & -0.6136 & 0.7412 \end{pmatrix}. \tag{11}$$

From this matrix according to (8) it is easy to obtain corresponding vector-parameter $\vec{\rho} = (-0.40456, -0.03781, -0.33497)$.

Previously widely used matrix, taking into account initial experimental date has more simple form:

$$U = \begin{pmatrix} c & s & s_{13} \\ -\frac{s}{\sqrt{2}} & \frac{c}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s}{\sqrt{2}} & -\frac{c}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{12}$$

where $s = \sin \theta_{sun} \approx \sqrt{0.3}$, $c = \cos \theta_{sun}$, $s_{13} = \sin \theta_{13} \approx 0$. Then corresponding vector-parameter was next form:

$$\vec{\rho} = \left(-\frac{1}{1+\sqrt{2}}, -\frac{s - s_{13}}{(1+\sqrt{2})(1+c)}, -\frac{s}{(1+c)} \right) \tag{13}$$
$$= (-0.414, -0.124 + 0.226 s_{13}, -0.298) \approx (-0.414, -0.124, -0.298).$$

The mixing parameters are not small, and this gives rise of maximal mixing hypothesis. There was proposed semi-empirical model of "Tri-Bimaximal Mixing" [11], where parameters are selected for maximal mixing of neutrino pairs compatible with experimental data $s = 1/\sqrt{3}$ and $s_{13} = 0$, then vector-parameter is equal to $\vec{\rho} = (-0.414, -0.132, -0.316)$.

Note that in case of quark field mixing the free masses of quark is not observable because the prohibition of existence of free quarks (confinement) and main attention of experimentalists focused on the mixing matrix elements, which are extracted from numerous particles decay channels. In case of neutrino mixing the masses itself (and the difference of mass values for description of oscillation) are the goals of experiments.

So, it is demonstrated that vector parametrization of rotation group, developed by F.I. Fedorov, may be used as alternative parameterization of quark and neutrino mixing. In case of Majorana fermions this parameterization is a complete description. In general this approach gives good fist approximation. The interpretation of these parameters is evident. For the subtle CP-violation effects this parametrization has to be extended by inclusion of phase parameters.

References

[1] Ahmed S.N. et.al. (SNO), arXiv:nucl-ex/0309004; Eguchi K. et al. (KamLAND), Phys.Rev.Lett., 2003, **90**,021802; Fukuda S. et al.(Super-Kamiokande), Phys. Lett. B, 2002, **539**, 179; Ahn M. H. et al. (K2K), Phys. Rev. Lett., 2003, **90**, 041801.

[2] Pontecorvo B., Zh. Eksp. Teor. Fiz., 1967, **53** (5), 1717 [Sov. Phys. JETP, 1968, **26**, 984].

[3] Majorana E., Nuovo Cimento, 1937, **14**, 171-184.

[4] Wolfenstein L., Phys.Rev. 1978, **D17**, 2369; S. P. Mikheev and A. Y. Smirnov, Sov. J. Nucl. Phys. 1985, **42**, 913; Nuovo Cimento 1986, **9C**, 17.

[5] Ardellier F. at al., arXiv:hep-ex/10606025; Xinheng Guo et al., arXiv:hep-ex/10701029; Beire E. et al., preprint FermiLab-PUB-07-072-D0, Dec.2007.

[6] Cabibbo N., Phys. Rev. Lett., 1963, **10**,531; Kobayashi M., Maskawa T., Prog. Theor. Phys., 1973, **49**, 2, 652-657.

[7] K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 090001 (2014).

[8] Pontecorvo B., Zh. Eksp. Teor. Fiz., 1957, **33**, 549 and 1958, **34**, 247; Maki Z., Nakagawa M., Sakata S., Prog. Theor. Phys. 1962, **28**, 870-880.

[9] Wolfenstein L., Phys. Rev. Lett. 1983, **51**, 1945.

[10] Fedorov F.I., Lorenz group, Moskaw 1978, 350 pp.

[11] Harrison P. F., Perkins D. H., Scott W.G., arXiv:hep-ph/0202074.

Dipole spin polarizabilities and gyrations of spin-1 particles in the Duffin-Kemmer-Petiau formalism

E.V. Vakulina
elvakulina@yandex.ru; Affiliated branch of Bryansk State University
n.a. Academy Fellow I.G. Petrovskiy, Novozybkov, Russia
N.V. Maksimenko
The F. Skorina Gomel State University, Gomel, Belarus
S.M. Kuchin
Affiliated branch of Bryansk State University
n.a. Academy Fellow I.G. Petrovskiy, Novozybkov, Russia

Abstract

In this paper relativistic-invariant phenomenological Lagrangians of interaction between spin-1 particles and electromagnetic field were obtained in the Duffin-Kemmer-Petiau formalism on the basis of the covariant model that takes into account both spin polarizabilities and gyrations of the above-mentioned particles. It was shown that in the suggested covariant model with regard to the crossing symmetry, spatial parity and gauge invariance conservation laws, definite spin polarizabilities and gyrations of spin-1 particles contribute to the expansion of Compton scattering amplitude, starting from the corresponding orders on energy of pfotons that is in the agreement with low-energy theorems for that process.

1 Introduction

With the development of the Standard Model of electroweak interaction, new electromagnetic properties of hadrons have been introduced recently. These properties, by analogy with gyration [1,2], are connected with parity violation [3,4]. In their turn, such electromagnetic characteristics as polarizabilities and gyrations are directly related to the inner structure of hadrons and the mechanism of electroweak photon-hadron interactions. For more reliable determination of polarizabilities and hadron characteristics connected with parity violation, a wide class of electrodynamic processes is used. These processes include real and virtual photons scattering, as wells as two-photon production in hadron-hadron interactions. In this context, the task of consistent relativistic-invariant determination of the contributions of polarizabilities and electroweak characteristics of particles to the electrodynamic processes' amplitudes and cross-sections is of great relevance.

The solution for this task can be found in the framework of relativistic theoretical and field approach to the description of interaction between electromagnetic field and hadrons with the account for polarizabilities (both electromagnetic and electroweak) of the latter. In papers [1, 5-9] covariant techniques describing the interaction between electromagnetic field and hadrons were presented. In such techniques the electromagnetic characteristics of particles are fundamental.

Effective covariant Lagrangian of interaction between electromagnetic field and spin-1/2 particles that takes into account the polarizabilities of the latter was introduced in [1, 10] and has been recently used for fitting the photon-proton scattering experimental data at the energies close to resonance production $\Delta(1232)$ [11]. Characterization of electrodynamic processes on

the basis of relativistic theoretical and field approaches, which are focused on the obtaining of phenomenological Lagrangians, equations that describe interaction of electromagnetic field with hadrons, as well as the calculation of electrodynamic processes amplitudes consistent with the Standard Model's low-energy theorems is one of the most effective methods of interaction processes investigation.

Currently there is a number of theoretical papers (see [12-16]) devoted to introduction and calculation of spin polarizabilities of spin-1/2 hadrons that contribute to the series expansion of Compton scattering amplitude at the energies of photons in the third expansion order. Along with the investigations of spin-1/2 hadrons polarizabilities, a number of papers present the results of determination and estimation of spin-1 particles polarizabilities [17-20]. Such particles are characterized by both dipole, spin and tensor polarizabilities.

Low-energy theorems play an important role in the understanding of interaction between electromagnetic field and hadrons. It is stipulated by the fact that they are based on the general concepts of quantum field theory and series expansion of Compton scattering amplitude in powers of photons energy. Currently, one of the most efficient methods of electrodynamic processes investigation is the technique that uses phenomenological Lagrangians obtained in the framework of theoretical and field approaches and consistent with the low-energy theorems that are specified by the Standard Model of electroweak interactions. Construction of such Lagrangians allows to obtain physical interpretation of electromagnetic and electroweak characteristics of hadrons.

In paper [19] low-energy theorems for Compton scattering on a spin-1 particle were obtained. On the basis of these and with the use of techniques for determination of the contribution of spin-1/2 particles polarizabilities to the amplitudes of electrodynamic processes, one can obtain relativistic-invariant effective Lagrangians and covariant spin structures of two-photon interaction amplitudes with consideration of polarizabilities and electroweak properties (gyrations) on spin-1 particles. The present paper is entirely devoted to the above-mentioned task.

In paper [21] the construction of the effective relativistic-invariant Lagrangian of interaction between electromagnetic field and particles with constant electric and magnetic dipole moments was performed with the help of dipole moments' anti-symmetric tensor that is independent of electromagnetic field tensor $F_{\mu\nu}$.

The present article uses quantum-field relativistic-invariant Lagrangian, in which a tensor of induced dipole moments is introduced. It means that, in contrast to paper [21], this tensor depends on $F_{\mu\nu}$[22]. In its turn, polarizabilities tensor [23, 24] is introduced to determine contributions of polarizabilities and gyrations to the low-energy Compton scattering amplitude with provision for particles' spin degrees of freedom. Moreover, we take into account hermiticity requirements, algebra of spin operators and the behavior of tensor components under space and time inversion.

Such phenomenological approach allows to determine the effective relativistic-covariant Lagrangian using the relativistic field consideration of the properties of C-, P- and T-transformations, as wells as the crossing symmetry. It also provides for conformance with the low-energy theorems for Compton scattering on spin-1 particles.

In the present paper the Lagrangian and the amplitude of Compton scattering on the spin-1 particles in the Duffin-Kemmer-Petiau formalism with consideration of their polarizabilities and gyrations were obtained in the framework of covariant theoretical and field approach. The technique presented in papers [5, 22, 25, 26] was used.

2 Determination of the spin structure of low-energy amplitude for spin-1 particle Compton scattering

We will follow the paper [27] in order to determine the contributions of polarizabilities and gyrations to the low-energy amplitude of electromagnetic field scattering on spin-1 particle. However, to calculate induced electric $\overrightarrow{d}$ and magnetic $\overrightarrow{m}$ moments in terms of the electric $\overrightarrow{E}$ and magnetic $\overrightarrow{H}$ vectors of electromagnetic field strength, we use the following formulas [2]:

$$\overrightarrow{d} = 4\pi\hat{\alpha}\overrightarrow{E} \tag{1}$$

$$\overrightarrow{m} = 4\pi\hat{\beta}\overrightarrow{H} \tag{2}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are matrices, matrix-elements of which are the tensors of electric and magnetic polarizabilities. Diagonal elements of these matrices are expressed through scalar electric and magnetic polarizabilities:

$$\alpha_{ij} = \alpha_1\delta_{ij}$$
$$\beta_{ij} = \beta_1\delta_{ij}$$

Low-energy amplitude of electromagnetic field scattering that was obtained using formulas (1) and (2) can be presented in the following way [26]:

$$M(\vec{n}_2) = 4\pi\omega^2\{(\vec{e}^{(\lambda_2)^*}\hat{\alpha}\vec{e}^{(\lambda_1)}) + (\vec{n}_2\vec{e}^{(\lambda_1)})(\vec{n}_1\hat{\beta}\vec{e}^{(\lambda_2)^*}) + \tag{3}$$
$$+(\vec{n}_1\vec{e}^{(\lambda_2)^*})(\vec{e}^{(\lambda_1)}\hat{\beta}\vec{n}_2) - (\vec{e}^{(\lambda_2)^*}\vec{e}^{(\lambda_1)})(\vec{n}_1\hat{\beta}\vec{n}_2) - (\vec{n}_1\vec{n}_2)(\vec{e}^{(\lambda_1)}\hat{\beta}\vec{e}^{(\lambda_2)^*}) +$$
$$+[(\vec{n}_2\vec{n}_1)(\vec{e}^{(\lambda_2)^*}\vec{e}^{(\lambda_1)}) - (\vec{n}_2\vec{e}^{(\lambda_1)})(\vec{n}_1\vec{e}^{(\lambda_2)^*})]Sp(\hat{\beta})\}.$$

Expression (3) includes the following designations: ω is the incident wave frequency, $\vec{n}_1 = \frac{\vec{k}_1}{|\vec{k}_1|}$, $\vec{e}^{(\lambda_1)}$ and $\vec{k}_1$ are correspondingly the polarization and wave vectors of the incident wave.

According to the definitions of $\vec{d}$ and $\vec{m}$ presented in (1) and (2), it follows that $\hat{\alpha}$ and $\hat{\beta}$ should satisfy the hermiticity requirement. Taking into account this requirement as well as the algebra of spin-1 operators $\hat{S}_i$ [19] we can obtain the following:

$$[\hat{S}_i, \hat{S}_j] = i\delta_{ijk}\hat{S}_k, \tag{4}$$

$$\hat{S}_i\hat{S}_j\hat{S}_k = i\delta_{ijk} + \frac{1}{2}(\hat{S}_i\delta_{jk} + \hat{S}_k\delta_{ij}) + \frac{i}{2}\delta_{ikl}(\hat{S}_j\hat{S}_l + \hat{S}_l\hat{S}_j) \tag{5}$$

$\hat{\alpha}$ and $\hat{\beta}$ operators can be presented in the following way [26]:

$$\alpha_{ij} = \alpha_1\delta_{ij} + i\alpha_2\delta_{ijk}\hat{S}_k + i\chi_E\delta_{ijk}\partial_k + \bar{\bar{\alpha}}(\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i), \tag{6}$$

$$\beta_{ij} = \beta_1\delta_{ij} + i\beta_2\delta_{ijk}\hat{S}_k + i\chi_M\delta_{ijk}\partial_k + \bar{\bar{\beta}}(\hat{S}_i\hat{S}_j + \hat{S}_j\hat{S}_i), \tag{7}$$

where i,j, k and l can take the value of 1, 2 or 3, while δ_{ijk} - is the three-dimensional Levi-Civita tensor.

In formulas (6) and (7) α_1 and β_1 are scalar dipole electric and magnetic polarizabilities correspondingly, $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are tensor polarizabilities, α_2 and β_2 are spin dipole polarizabilities, while χ_E and χ_M are correspondingly electric and magnetic gyrations. As a consequence of crossing symmetry, α_2, β_2 and χ_E, χ_M have non-zero contribution to the amplitude of Compton scattering in the third expansion order of the photons energy.

As it was shown in [26], by substituting formulas (6) and (7) into (3) and taking into account the contributions of α, β, $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ polarizabilities, one can obtain the scattering amplitude in the second expansion order of the photons energy. It coincides with beyond the Born part of the amplitude and is due to the low-energy theorem [19].

Let's determine relativistic-invariant spin structures of the effective Lagrangian and the amplitudes of Compton scattering on spin-1 particles with the help of covariant representation of (6) and (7) in the Duffin-Kemmer-Petiau (DKP) formalism following paper [26].

The DKP equations for an unbounded spin-1 particle have the following form [28]:

$$(\beta_\mu \vec{\partial}_\mu + m)\psi(x) = 0, \tag{8}$$

$$\bar{\psi}(x)(\beta_\mu \vec{\partial}_\mu - m) = 0, \tag{9}$$

where $\psi(x)$ and $\bar{\psi}(x) = \psi^+(x)\eta$ are ten-dimensional functions of particles, $\eta = 2(\beta_4^{(10)})^2 - I$, vectors over derivatives ∂_μ show the direction of their action, while four-dimensional vector is defined as $a_\mu\{\vec{a}, ia_0\}$. In formulas (8) and (9) β_μ are ten-dimensional DKP matrices that satisfy the following commutation rules:

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\rho + \delta_{\rho\nu}\beta_\mu.$$

In the framework of theoretical and field covariant approach the effective Lagrangian of interaction between electromagnetic field and spin-1 particle with provision for polarizabilities has the form [5, 8, 26]:

$$L = -\frac{\pi}{2m}\bar{\psi}[\beta_\nu \hat{L}_{\nu\sigma} \overset{\leftrightarrow}{\partial_\sigma} + \hat{L}_{\nu\sigma}\beta_\nu \overset{\leftrightarrow}{\partial_\sigma}]\psi \tag{10}$$

where $\overset{\leftrightarrow}{\partial_\sigma} = \overrightarrow{\partial}_\sigma - \overleftarrow{\partial}_\sigma$.

The formula (10) for the Lagrangian includes tensor $\hat{L}_{\nu\sigma}$, which is expressed in terms of polarizabilities and gyrations as:

$$\hat{L}_{\nu\sigma}(\alpha, \chi_E) = \hat{L}_{\nu\sigma}(\alpha_1) + \hat{L}_{\nu\sigma}(\bar{\alpha}) + \hat{L}_{\nu\sigma}(\alpha_2) + \hat{L}_{\nu\sigma}(\chi_E), \tag{11}$$

$$\hat{L}_{\nu\sigma}(\beta, \chi_M) = \hat{L}_{\nu\sigma}(\beta_1) + \hat{L}_{\nu\sigma}(\bar{\bar{\beta}}) + \hat{L}_{\nu\sigma}(\beta_2) + \hat{L}_{\nu\sigma}(\chi_M), \tag{12}$$

In order to determine the influence of crossing symmetry on the contributions of spin polarizabilities and gyrations to the Compton scattering amplitude in dipole representation we will transform tensors (11) as (see [22]):

$$\hat{L}_{\nu\sigma}(\alpha_1) + \hat{L}_{\nu\sigma}(\bar{\alpha}) = F_{\nu\mu}\hat{\alpha}^{\mu\rho}(\alpha_1)F_{\rho\sigma} + F_{\nu\mu}\hat{\alpha}^{\mu\rho}(\bar{\alpha})F_{\rho\sigma}, \tag{13}$$

$$\hat{L}_{\nu\sigma}(\alpha_2) + \hat{L}_{\nu\sigma}(\chi_E) = F_{\nu\mu} \overset{\leftrightarrow}{\partial_\lambda} F_{\rho\sigma}\hat{k}_{\mu\rho\lambda}(\alpha_2) + F_{\nu\mu} \overset{\leftrightarrow}{\partial_\lambda} F_{\rho\sigma}\hat{k}_{\mu\rho\lambda}(\chi_E). \tag{14}$$

Derivatives in equation (14) operate only on the tensors of electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Tensors $\hat{\alpha}^{\mu\rho}(\alpha_1)$ and $\hat{\alpha}^{\mu\rho}(\bar{\alpha})$, as well as $\hat{k}_{\mu\rho\lambda}(\alpha_2)$ and $\hat{k}_{\mu\rho\lambda}(\chi_E)$ are the covariant generalization of tensors that appear in the right part of formula (6). They have the following form:

$$\hat{\alpha}_{\mu\rho} = \alpha_1\delta_{\mu\rho} + \bar{\bar{\alpha}}(\widehat{W}_\mu\widehat{W}_\rho + \widehat{W}_\rho\widehat{W}_\mu), \tag{15}$$

$$\hat{k}_{\mu\rho\lambda} = \frac{i\alpha_2}{2m}\delta_{\mu\rho\lambda k}\widehat{W}_k + \frac{i\chi_E}{2m}\delta_{\mu\rho\lambda k}\overleftrightarrow{\partial_k} \tag{16}$$

In equations (15) and (16) the definition of covariant spin vector is used. This vector can be expressed in terms of β_ν matrices (see [28]):

$$W_\mu = -\frac{i}{4m}\delta_{\mu\chi\delta\eta}\hat{J}^{[\delta\eta]}\overleftrightarrow{\partial_\chi},$$

where $\hat{J}^{[\delta\eta]} = \beta_\delta\beta_\eta - \beta_\eta\beta_\delta$. All derivatives found in (15) and (16) operate on wave functions ψ and $\bar{\psi}$.

Tensor (12) is defined in a similar way. One just needs to introduce constants β_1, β_2, $\bar{\bar{\beta}}$ and χ_M, in formulas (13)-(14) and make a replacement

$$F_{\nu\mu} \to \widetilde{F}_{\nu\mu},$$

where

$$\widetilde{F}_{\mu\nu} = \frac{i}{2}\delta_{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

Let's now determine the spin structures of the amplitude of Compton scattering on spin-1 particle with provision for polarizabilities and gyrations. We will take Lagrangian (10) as a basis and follow the procedure presented in paper [28]:

$$< k_2, p_2|\hat{S}|k_1, p_1 >= \frac{im\delta(k_1 + p_1 - k_2 - p_2)}{(2\pi)^2\sqrt{4\omega_1\omega_2E_1E_2}}M, \tag{17}$$

here M is the Compton scattering amplitude that represents the sum of polarizabilities and gyrations contributions according to formulas (11) and (12).

As it was shown in [26], the contribution of α, β and $\bar{\alpha}$, $\bar{\bar{\beta}}$ is expressed as a sum of amplitudes

$$M_1 = M_1(\alpha, \beta) + M_1(\bar{\alpha}, \bar{\bar{\beta}}). \tag{18}$$

Spin structure $M(\alpha, \beta)$ in equation (18) has the following form:

$$M_1(\alpha, \beta) = \left(-\frac{2\pi i}{m}\right)\Big\{\alpha[F^{(2)}_{\nu\mu}F^{(1)}_{\mu\sigma} + F^{(1)}_{\nu\mu}F^{(2)}_{\mu\sigma}]+ \tag{19}$$

$$+\beta[\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)}_{\mu\sigma} + \tilde{F}^{(1)}_{\nu\mu}\tilde{F}^{(2)}_{\mu\sigma}]\Big\}P_\sigma\bar{\psi}^{(r_2)}(p_2)\beta_\nu\psi^{(r_1)}(p_1).$$

In its turn, structure $M(\bar{\alpha}, \bar{\bar{\beta}})$ is determined as:

$$M_1(\bar{\alpha}, \bar{\bar{\beta}}) = \left(-\frac{\pi i}{m}\right)\Big\{\bar{\alpha}[F^{(2)}_{\nu\mu}F^{(1)}_{\mu\sigma} + F^{(1)}_{\nu\mu}F^{(2)}_{\mu\sigma}]+ \tag{20}$$

$$+\bar{\bar{\beta}}[\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)}_{\mu\sigma} + \tilde{F}^{(1)}_{\nu\mu}\tilde{F}^{(2)}_{\mu\sigma}]\Big\}P_\sigma\bar{\psi}^{(r_2)}(p_2)[\beta_\nu\{\widehat{W}_\mu, \widehat{W}_\rho\} + \{\widehat{W}_\mu, \widehat{W}_\rho\}\beta_\nu]\psi^{(r_1)}(p_1).$$

Equations (19) and (20) include the following designations:

$$F^{(2)}_{\nu\mu} = k_{2\nu}e^{(\lambda_2)^*}_\mu - k_{2\mu}e^{(\lambda_2)^*}_\nu,$$

$$F^{(1)}_{\mu\sigma} = k_{1\mu}e^{(\lambda_1)}_\sigma - k_{1\sigma}e^{(\lambda_1)}_\mu,$$

where $\widetilde{F}^{(2)}_{\nu\mu} = \frac{i}{2}\delta_{\nu\mu\chi\delta}F^{(2)}_{\chi\delta}$, $P_\sigma = \frac{1}{2}(p_1 + p_2)_\sigma$, p_1 and p_2 are the momenta of initial and final spin-1 particles correspondingly.

Ten-dimensional wave functions in the DKP formalism are introduced using complete matrix algebra elements ε^{AB} [28]

$$\psi^{(r)}(p) = \psi^{(r)}_{\mu}(p)\varepsilon^{\mu 1} + \frac{1}{2}\psi^{(r)}_{[\mu\nu]}(p)\varepsilon^{[\mu\nu]1}.$$

In this formula

$$\psi^{(r)}_{\mu}(p) = \frac{i}{\sqrt{2}}\lambda^{(r)}_{\mu},$$

$$\psi^{(r)}_{[\mu\nu]}(p) == -\frac{1}{\sqrt{2m}}\Big(p_{\mu}\lambda^{(r)}_{\nu} - \lambda^{(r)}_{\mu}p_{\nu}\Big),$$

$\lambda^{(r)}_{\mu}$ are the components of polarization vectors of spin-1 particle, while ε^{AB} are the elements of complete matrix algebra [28]:

$$(\varepsilon^{AB})_{CD} = \delta_{AC}\delta_{BD}, \;\; \varepsilon^{AB}\varepsilon^{CD} = \delta_{BC}\varepsilon^{AD},$$

where for spin-1 particle indices A, B,C, $D = \mu, [\rho\sigma]$, while square brackets stand for the anti-symmetry with respect to indices ρ and σ.

Wave functions $\bar{\psi}^{(r)}(p)$ that are conjugate with respect to $\psi^{(r)}(p)$ are expressed in the following way (taking into account η matrix):

$$\bar{\psi}^{(r)}(p) = \psi^{+}(p)\eta = \Big(-\frac{i}{\sqrt{2}}\Big)[\dot{\lambda}^{(r)}_{\mu}\varepsilon^{1\mu} + \frac{i}{2m}\varepsilon^{1[\mu\nu]}(p_{\mu}\dot{\lambda}^{(r)}_{\nu} - p_{\nu}\dot{\lambda}^{(r)}_{\mu})],$$

where $\dot{\lambda}^{(r)}_{\mu}\Big\{\lambda^{(r)^*}_{i}, \lambda^{(r)}_{4}\Big\}$.

Let's now determine the spin structures of the amplitudes with provision for the contributions of spin polarizabilities α_2, β_2 and gyrations χ_E, χ_M, i.e.

$$M_2 = M_2(\alpha, \beta) + M_2(\chi_E, \chi_M).$$

Using the summands $\hat{L}_{\nu\sigma}(\alpha_2)$, $\hat{L}_{\nu\sigma}(\chi_E)$, $\hat{L}_{\nu\sigma}(\beta_2)$ (11) and $\hat{L}_{\nu\sigma}(\chi_M)$ (12), of the Lagrangian, as well as the previous technique for determination of polarizations contributions to the Compton scattering amplitude, one can find:

$$M_2(\alpha_2, \beta_2) = \frac{\pi}{m}(k_1 + k_2)_{\lambda}\delta_{\mu\rho\lambda k}\Big\{\alpha_2[F^{(2)}_{\nu\mu}F^{(1)}_{\rho\sigma} - F^{(1)}_{\nu\mu}F^{(2)}_{\rho\sigma}] + \tag{21}$$

$$+\beta_2[\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)}_{\rho\sigma} - \tilde{F}^{(1)}_{\nu\mu}\tilde{F}^{(2)}_{\rho\sigma}]\Big\}\bar{\psi}^{(r_2)}(p_2)[\beta_{\nu}\widehat{W}_k + \widehat{W}_k\beta_{\nu}]P_{\sigma}\psi^{(r_1)}(p_1).$$

Amplitude (21) in the target's rest frame and with the neglect of the target particle's recoil can be expressed as:

$$M_2(\alpha_2, \beta_2) = 4i\pi(\omega_1 + \omega_2)\omega_1\omega_2\vec{\lambda}^{(r_2)^*}\Big\{\alpha_2(\vec{S}[\vec{e}^{(\lambda_2)^*}\vec{e}^{(\lambda_1)}]) + \tag{22}$$

$$+\beta_2\vec{S}[\vec{n}_2\vec{e}^{(\lambda_2)^*}][\vec{n}_1\vec{e}^{(\lambda_1)}]\Big\}\vec{\lambda}^{(r_1)}.$$

Formulas (21) and (22) imply that dipole spin polarizabilities α_2 and β_2 contribute to the amplitude of Compton scattering on spin-1 particle in the third expansion order (series expansion in the energy of photons), while the crossing symmetry requirements and parity conservation (with respect to space inversion) rules are satisfied.

Using the above-introduced technique for constructing covariant blocks of the effective Lagrangian with provision for the crossing symmetry and parity violation, we can obtain the second summand of the amplitude that depends on the contributions of electric and magnetic gyrations:

$$M_2(\chi_E,\chi_M)=\frac{2i\pi}{m^2}(k_1+k_2)_\lambda\delta_{\mu\rho\lambda k}\Big\{\chi_E[F^{(2)}_{\nu\mu}F^{(1)}_{\rho\sigma}-F^{(1)}_{\nu\mu}F^{(2)}_{\rho\sigma}]+ \tag{23}$$

$$+\chi_M[\tilde{F}^{(2)}_{\nu\mu}\tilde{F}^{(1)}_{\rho\sigma}-\tilde{F}^{(1)}_{\nu\mu}\tilde{F}^{(2)}_{\rho\sigma}]\Big\}P_kP_\sigma\bar{\psi}^{(r_2)}(p_2)\beta_\nu\psi^{(r_1)}(p_1).$$

If we use approximation $\vec{P}=0$, in equation (23), i.e. we consider the particle to be at rest and neglect its recoil momentum, the formula (23) can be rewritten in the following way:

$$M_2(\chi_E,\chi_M)=4\pi\omega_1\omega_2(\vec{\lambda}^{(r_2)^*}\vec{\lambda}^{(r_1)})\Big\{\chi_E(\vec{k}_1+\vec{k}_2)[\vec{e}^{(\lambda_2)^*}\vec{e}^{(\lambda_1)}])+ \tag{24}$$

$$+\chi_M(\vec{k}_1+\vec{k}_2)[\vec{\Sigma}_2\vec{\Sigma}_1]\Big\},$$

where $\vec{\Sigma}_2=[\vec{n}_2\vec{e}^{(\lambda_2)^*}]$, $\vec{\Sigma}_1=[\vec{n}_1\vec{e}^{(\lambda_1)}]$.

3 Conclusion

Hence, we determined the contributions of polarizabilities to the low-energy Compton scattering amplitude with provision for the spin degrees of freedom of particles by transforming the polarizabilities tensor that satisfies both hermiticity requirement and spin algebra. This tensor is also invariant with respect to space inversion transformations.

The relativistic-covariant form of contributions of spin and tensor polarizabilities, as well as gyrations to the Compton scattering amplitude in the DKP formalism was found.

The effective Lagrangian that takes into account the changes of spin structures during space inversion transformations and considers the crossing symmetry of Compton scattering amplitude on spin-1 particle was obtained in the DKP formalism using theoretical and field relativistic generalization. The coordination of this amplitude with the low-energy theorems was performed as well.

References

[1] Maksimenko, N. V. Phenomenological description of the polarizability of elementary particles in the field theory / Maksimenko, N. V. and Moroz, L. G. // Proc. of the XI-th International School of Young Scientists in High Energy Physics and Relativistic Nuclear Physics [in Russian], D2-11707, Joint Institute for Nuclear Research, Dubna (1979), pp. 533–543.

[2] Fedorov, F. I. Theory of Gyrotropy [in Russian], Nauka i Tekhnika, Minsk (1976).

[3] Bedaque, P. F. Parity violation in ?p Compton Scattering / P. F. Bedaque, M. J. Savage // Phys. Rev. C. – 2000 – V. 62 – P. 018501 – 1–6.

[4] Gorchtein, M. Forward Compton Scattering with neutral current: constraints from sum rules / M. Gorchtein, X. Zhang [Electronic resourse]. – 2015. Mode of access: http://nucl-th/1501.0535v1. – Date of access: 22.01.2015.

[5] Maksimenko, N. V. Covariant definition polarizability of hadrons with spin one // Doklady Akad. Of Sciences BSSR, Ser. Fiz.-Mat. Navuk, No. 6, 508–510 (1992).

[6] Levchuk, M. I. The nucleon gyration as one of nucleon electromagnetic structure characteristics / M. I. Levchuk, L. G. Moroz // Proc. Academy of Sciences of BSSR. Ser. fiz.-mat.navuk. – 1985. – No. 1. – P. 45–54.

[7] Bogush, A. A., Kisel, V. V., and Moroz, L. G. On the description of the polarizability of scalar particles in the theory of relativistic wave equations in: Covariant Methods in Theoretical Physics. Physics of Elementary Particles and Theory of Relativity, [in Russian], Minsk (1981), pp. 81–90.

[8] Andreev, V. V., Maksimenko, N. V. Polarizability of elementary particles in the theoretical-field approach // Problems of Physics, Mathematics and Technics. – 2011. – No. 4 (9). – pp. 7–11.

[9] Maksimenko, N. V., Deryuzhkova, O. M. Covariant gauge-invariant Lagrangian formalism taking into account polarizability of the particle // Vestsi NAN Belarusi, Ser. Fiz.-Mat. Navuk, No. 2, pp. 27–30 (2011).

[10] Ilyichev, A. Static polarizability vertex and its applications / A. Ilyichev, S. Lukashevich, N. Maksimenko // [Electronic resource]. – 2006. – Mode of access:arXiv: //hep-ph/0611327v1. – Date of access: 27.11.2006.

[11] Zhang, Y. Proton Compton scattering in a unified proton - $\triangle^+$ Model / Y. Zhang, K. Savvidy // Phys. Rev. C. – 2013. – Vol. 88. – P. 064614-1–12.

[12] Holstein, B. R. Hadron polarizabilities / B. R. Holstein, S. Scherer. [Electronicresourse]. – 2013. – Mode of access: http: // hep-ph/1401.0140v1. – Date of access: 31.12.2013].

[13] Carlson, C. E. Constraining off-shell effects using low-energy Compton scattering / C. E. Carlson, M. Vanderhaeghen // [Electronic resource]. – 2011. – Mode of access: http://physics.atom-ph/1109.3779. – Date of access: 04.10.2011.

[14] Raguza, S. Third-order spin polarizabilities of the nucleon: I / S. Raguza // Phys. Rev. D. – 1993. – Vol. 47. - No. 9. – P. 3757–3767.

[15] Raguza, S. Third-order spin polarizabilities of the nucleon: II / S. Raguza // Phys. Rev. D. – 1994. – Vol. 49. - No. 7. – P. 3157–3159.

[16] Low-energy Compton scattering of polarized photons on polarized nucleons / D. Babusci [et. al.] // Rev. C. – 1998. – Vol. 58. – P. 1013–1041.

[17] Chen, J. W. The polarizability of the deuteron / J. W. Chen [et. al.] // Nucl. Phys. A. – 1998. – Vol. 644. – P. 221–234.

[18] Friar, J. L. Deuteron dipole polarizability and sum rules / J. L. Friar, G. L. Payne // Rev. C. – 2005. – Vol. 72. – P. 014004–1–014004–6.

[19] Lin, K. Y. Forward dispersion relation and low-energy theorems for Compton scattering on spin – 1 targets / K. Y. Lin, J. C. Chen // J. Phys. G: Nucl. Phys. – 1975 – Vol.1 – P. 394–399.

[20] F. Hagelstein. Sum Rules for Electromagnetic Moments and Polarizabilities of Spin-1 Particles in Massive Yang-Mills QED / Masterarbeit in Physik vorgelegt dem Fachbereich Physik, Mathematik und Informatik (FB 08) der Johannes Gutenberg-Universit.at Mainz am 11. Ma ?rz 2014.

[21] Anandan, J. S. Classical and quantum interaction of the dipole / J. S. Anandan // Phys. Rev. Lett. – 2000. – Vol.85. – P. 1354–1357

[22] Andreev, V. V., Deryuzhkova, O. M., and Maksimenko, N. V. The covariant representation spin polarizability of the nucleon // Problems of Physics, Mathematics and Technics. – 2014. – No. 3(20). – pp. 7–12.

[23] Galynskii, M. V., Fedorov, F. I. Transformation of the beam tensor during a light-medium interaction // Journal of Applied Spectroscopy, 1986, Vol. 44, No. 2, pp. 200–203.

[24] Baryshevskii, V. G. Nuclear Optics of Polarized Media [in Russian], Energoatomizdat, Moscow (1995).

[25] Vakulina, E. V., Maksimenko, N. V. Polarizability of the pion in the formalism of Duffin – Kemmer // Problems of Physics, Mathematics and Technics. – 2013. – No. 3. – pp. 16–18.

[26] Maksimenko, N.V. Spin 1 Particle Polarizability in the Duffin-Kemmer-Petiau Formalism // N.V. Maksimenko, E.V. Vakulina, S.M. Kuchin // Physics of Paticles and Nuclei Letters.- 2015. - Vol. 12, No.7, P. 807–812.

[27] Landau, L. D., Lifshitz, E. M. Course of Theoretical Physics. Volume 2. The Classical Theory of Fields 4 Edition. – Butterworth-Heinemann, 1975. – 402 p. – Series: Course of Theoretical Physics Series (Book 2).

[28] Bogush, A. A. Introduction to Gauge Field Theory of Electroweak Interactions [in Russian], Nauka i Tekhnika, Minsk (1987).

Периоды во временных рядах флуктуаций скорости радиоактивного распада и возможный механизм их образования

В.А. Панчелюга, **М.С. Панчелюга**
panvic333@yahoo.com;
Институт теоретической и экспериментальной биофизики РАН, Пущино, Россия,
НИИ Гиперкомплексных систем в геометрии и физике, Фрязино, Россия

Аннотация

В настоящей работе кратко рассмотрены известные к настоящему времени периоды, найденные в флуктуациях скорости радиоактивного распада. Показано, что все периоды могут быть разделены на две группы: связанные с движениями в системе Солнце-Земля-Луна и периоды связанные с собственными колебаниями Земли. Предложено единое объяснение наблюдаемой феноменологии, основанное на предположении о том, что наблюдаемые периоды могут быть обусловлены частицами странного излучения.

Практически с момента открытия радиоактивности Беккерелем были предприняты активные исследования возможности внешних воздействий на скорость радиоактивного распада. Полученные при этом, преимущественно школой Резерфорда, отрицательные результаты на долгие годы закрепили в мировом научном мнении устойчивое представление о том, что радиоактивный распад, если мы рассматриваем его мгновенные значения, является пуассоновским процессом. Если же рассматриваются его средние значения, то он может служить примером ультрастабильного процесса, определеяемого только константой распада и не зависящего от каких-либо внешних воздействий.

Развитие экспериментальной техники и дальнейшее изучение возможности внешних влияний на скорость радиоактивного распада показали, что экстремальные воздействия (сверхсильные магнитные поля, давление, изменение химического окружения, степень ионизации атома) все-таки могут приводить к изменению скорости различных типов распада.

Исследования, связанные с низкоэнергетичными ядерными реакциями (LENR), сделало влияние на скорость радиоактивного распада, практически, повседневной экспериментальной практикой [1–3]. В большинстве таких исследований внешние воздействия на процесс радиоактивного распада также можно отнести к экстремальным. Хотя, необходимо отметить, энергия таких <экстремальных> воздействий значительно меньше, чем необходимо для преодоления кулоновского барьера для инициации наблюдаемых ядерных реакций.

Наряду с упомянутыми, существует другая, к настоящему времени уже достаточно многочисленная, группа работ в которых некоторые неизвестные внешние и, по всей вероятности, чрезвычайно слабые воздействия приводят к появлению периодов во временных рядах флуктуаций скорости радиоактивного распада. В этих работах было показано наличие периодов в многолетних рядах измерений скорости радиоактивного распада в диапазоне от лет до месяцев. Необходимо отметить, что эти периоды были обнаружены по большей части традиционными методами спектрального анализа. Их относительная амплитуда уменьшается с уменьшением их длительности. Так годовые и сезонные периоды обычно имеют амплитуды $\sim 10^{-1} \dots 10^{-3}$, а околомесячные и суточные $\sim 10^{-3} \dots 10^{-5}$. Возможно, что данное

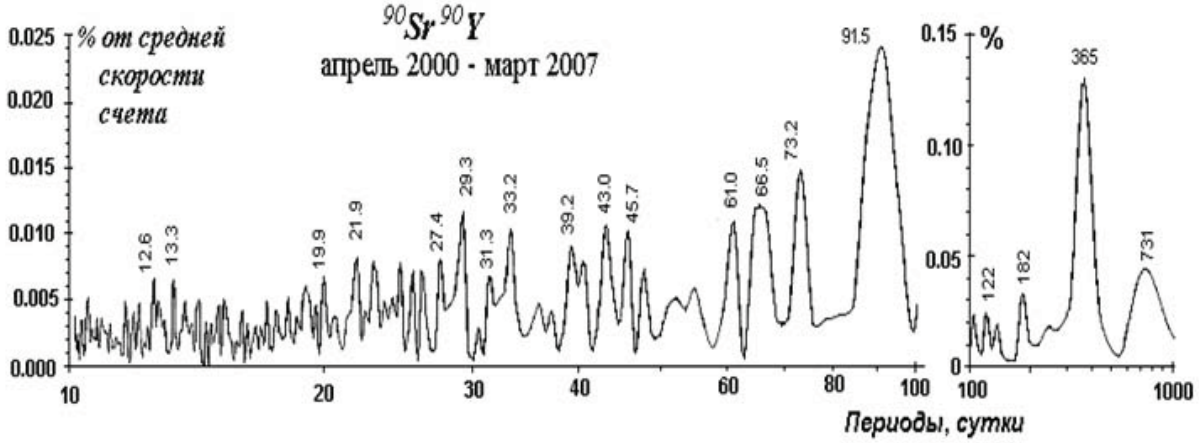


Рис. 1: Периодограмма вариаций скорости счета β-источника ^{90}Sr-^{90}Y со счетчиком Гейгера. Анализируемый промежуток времени 7 лет. Показаны периоды, соответствующие вершинам пиков [8].

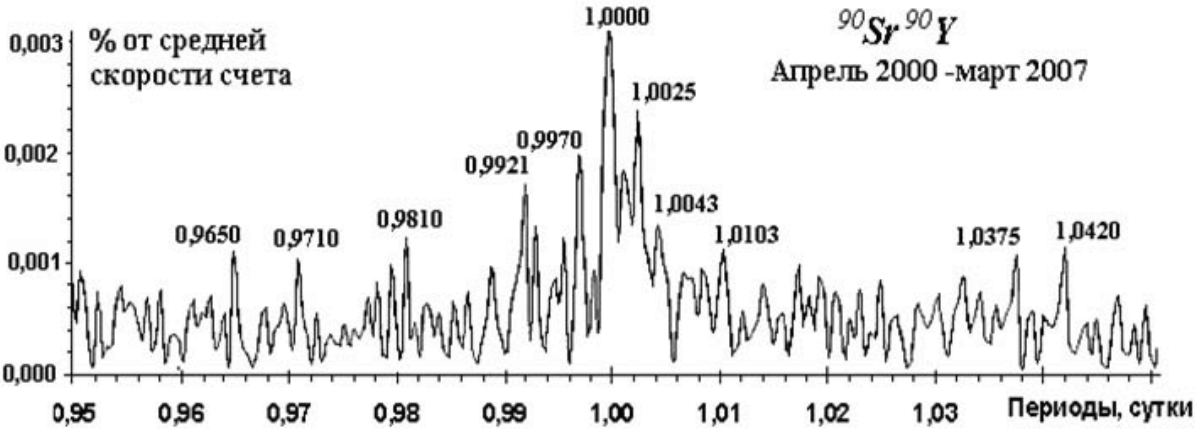


Рис. 2: Околосуточные периоды в рядах флуктуаций скорости бета-распада. Помимо пика солнечно-суточного (1,0000) периода, видны пики звездно-суточного (0,9970) и лунно-суточного периода (1,0375 и 1,0420) [8].

обстоятельство является причиной того, что более короткие периоды не обнаруживаются упомянутыми классическими методами анализа временных рядов. Хотя, согласно классической <резерфордовской> точке зрения таких периодов быть не должно.

Изменение мирового научного мнения на возможность существования упомянутых периодов, произошло с появлением серии работ, инициированной публикациями Е. Фишбаха с соавторами [4–6], в которых было продемонстрировано наличие годового периода в многолетних рядах измерений скорости β-распада, коррелирующего с ходом зависимости $1/R^2$, где R – расстояние между Землей и Солнцем.

Но важно отметить, что задолго до Е. Фишбаха, о годовых периодах в радиоактивном распаде сообщалось в работах А.Г. Пархомова, начатых еще в 90-годы и в результате которых уже к 2004 г были получены убедительные экспериментальные доказательства наличия годовых и других периодов в β-радиоактивности [7]. Его исследования суммированы в книге [8].

На рис. 1 и рис. 2 приведены спектры периодов, полученные в работах А.Г. Пархомо-

ва [8]. Видно, отмеченное выше уменьшение амплитуды, найденных периодов с уменьшением их величины. Необходимо отметить, что периоды короче суточного методами спектрального анализа, использованными в [4–8], не обнаружены. Такие периоды (от единиц минут до суточных) удалось обнаружить в серии наших работ [9,10], основанной на разработанной в [9] методике локального фрактального анализа методом всех сочетаний (МВС). Показано, что данный метод позволяет обнаруживать периодичности необнаружимые спектральными методами анализа временных рядов.

Так анализ МВС-методом 329-суточного массива флуктуаций скорости α-распада обнаружил устойчивый набор периодов в диапазоне 1-115 мин [10]. На рис. 3 приведены спектры периодов, полученные с использованием МВС-метода для параметризации исходного временного ряда и с последующим исследованием содержащихся в нем периодов четырьмя разными методами, в результате чего получены, представленные на рис. 3 спектры: а) сумма 329 двухсуточных распределений интервалов для диапазона периодов 1-115 мин, б) результаты перемножения 329 двухсуточных нормированных распределений интервалов для диапазона периодов 1-115 мин, в) распределение частот встречаемости пиков для диапазона периодов 1-115 мин, построеное на основе 329 двухсуточных распределений интервалов, содержащих 5695 пиков, г) распределение частот встречаемости пиков для диапазона периодов 1-115 мин, построенное на основе 329 двухсуточных нормированных по амплитуде распределений интервалов, содержащих 5695 пиков (см. детальное описание использованных процедур в [10]).

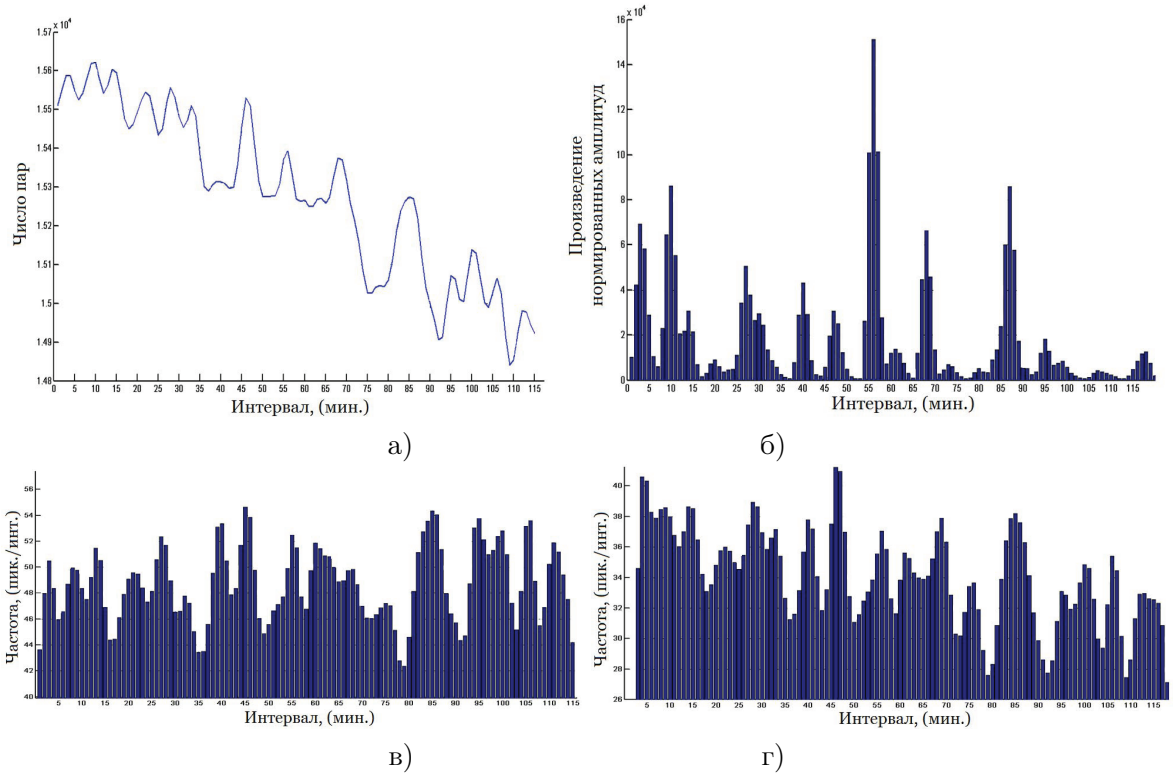


Рис. 3: Спектры периодов, полученные на основе 329 2-суточных временных рядов, параметризованных МВС-методом: а) сумма 329 2-суточных распределений интервалов; б) произведение 329 2-суточных распределений интервалов; в) сумма пиков на 329 2-суточных распределениях интервалов; г) сумма нормированных амплитуд пиков на 329 2-суточных распределениях интервалов.

Таблица 1: Периоды, найденные на Рис. 3 а) - г) и соответствующие им периоды собственных колебаний Земли и сейсмогравитационных колебаний Земли. Все значения периодов приведены в минутах.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
Рис.3а)	3	9	13	22	28	33	40	46	56	–	68	–	–	–	85	95	100	106	113
Рис.3в)	3	8	14	21	27	32	40	45	55	60	68	–	75	–	85	95	100	106	111
Рис.3г)	4	9	14	22	28	33	40	46	56	61	69	–	76	–	85	95	100	106	113
Рис.3б)	3	10	14	20	27	30	40	47	56	61	68	73	–	80	87	95	99	107	–
	Собственные колебания Земли									Сейсмогравитационные колебания									
МОДА	$_8S_5$ ... $_{15}S_2$	$_1S_7$ ... $_5S_2$	$_0T_6$	$_0T_4$	$_0T_3$	$_0S_3$	$_2S_1$	$_0T_2$	$_0S_2$	59.2 60.8	67.8	72	77	82	86.7	95.2	103	107	–

Таблица 2: Периоды длинноволновых собственных колебаний Земли и периоды, найденные в флуктуациях скорости альфа-распада МВС-методом.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
СКЗ	2.02	2.18	2.41	2.67	3.03	3.45	4.01	4.8	6.01	7.83	8.02
МВС-метод	2.06	2.15	2.35	2.64	3.08	3.44	4.02	4.68	6.03	7.82	7.96

Продолжение Таблицы 2

	12	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
СКЗ	8.43	11.3	12.02	12.4	12.78	20.02	21.66	22.02	23.45	24.01	24.49
МВС-метод	2.06	2.15	2.35	2.64	3.08	3.44	4.02	4.68	6.03	7.82	7.96

Одним из интересных результатов, полученных в [10], является то, что найденные в данной работе периоды совпадают с периодами собственных колебаний Земли [11]. Это совпадение проявляется лучше всего для т.н. сейсмогравитационных или длинноволновых колебаний Земли [11]. Присутствующие в спектрах на рис. 2 периоды приведены в первых четырех строках Табл. 1. Периоды собственных колебаний Земли даны в последней строке Табл. 1. Данная строка делиться на две части. Первая – т.н. <классические> собственные колебания Земли [11] (приведены моды собственных колебаний Земли, соответствующие найденным частотам), вторая – длинноволновые или сейсмогравитационные собственные колебания Земли [12]. Видно, что, с одной стороны, найденные на рис. 2 координаты пиков хорошо совпадают между собой, а с другой стороны – соответствуют приведенным значениям периодов собственных колебаний Земли.

В ходе дальнейших исследований нами были получены устойчивые внутрисуточные периоды в диапазоне 2-24 часа, которые суммированы в Табл. 2. В этой таблице (во второй строке) приведены, описанные в [13], периоды собственных колебаний Земли, характерные для данного диапазона. Периоды, найденные с использованием МВС-метода, приведены в третьей строке Табл. 2. Как можно видеть, в этом случае, также наблюдается хорошее совпадение МВС-периодов с периодами собственных колебаний Земли.

Говоря об описанных выше спектрах периодов, их можно разделить на две группы. Первая – периоды от года до суток, которым можно сопоставить определенные движения, происходящие в системе Солнце-Земля-Луна. Вторая группа - периоды, связанные с собственными колебаниями Земли. Периоды первой группы находят свое объяснение, если предположить, что Солнце (и, возможно, удаленные звезды) являются источниками кор-

пускулярного излучения, которое может эффективно взаимодействовать с ядрами исследуемых в эксперименте изотопов и индуцировать их деление. Подобное излучение впервые было описано в [14] и получило название <странного излучения>, а в [1-3] экспериментально показано, что оно может изменять скорость радиоактивного распада.

Экспериментальное исследование странного излучения показывает, что оно состоит из легких, электрически нейтральных частиц, обладающих выраженными магнитными свойствами. В силу наличия массы у частиц странного излучения, рядом авторов высказывается предположение, что оно, с одной стороны, может накапливаться у поверхности Земли, а также высвобождаться в ходе упругих деформаций, сопровождающих собственные колебания Земли.

В случае реализации подобного сценария собственные колебания Земли могут модулировать концентрацию частиц странного излучения, которые, в свою очередь, как показано в [1–3], могут обуславливать появление соответствующих периодов в флуктуациях скорости радиоактивного распада.

Т.о. странное излучение может оказаться универсальным посредником, обуславливающим известный к настоящему времени спектр периодов скорости радиоактивного распада.

Список литературы

[1] Ивойлов Н.Г., Бикчантаев М.М., Стребков О.А. Магнитно-стимулированный скачок активности β-распада Co^{57} // Прикладная физика, 2009, №1, с.36-40.

[2] Н.Г.Ивойлов, Л.И.Уруцкоев Влияние <странного> излучения на мессбауэровские спектры Fe57 в металлических фольгах // Прикладная физика, 2004, №5, с. 20 - 25.

[3] Adamenko S.V., Selleri F., A. van der Merve Controlled nucleosynthesis. Breakthroughs in experiment and theory. Springer Netherlands, 2007 - 782 p.

[4] Jere H. Jenkins, Ephraim Fischbach, John B. Buncher, John T. Gruenwald, Dennis E. Krause, and Joshua J. Mattes Evidence for Correlations Between Nuclear Decay Rates and Earth-Sun Distance // arXiv:0808.3283v1 [astro-ph] 25 Aug 2008.

[5] Jere H. Jenkins, Ephraim Fischbach, John B. Buncher, John T. Gruenwald, Dennis E. Krause, Joshua J. Mattes Evidence of correlations between nuclear decay rates and Earth-Sun distance // Astroparticle Physics 32 (2009) 42-46.

[6] E. Fischbach, J.B. Buncher, J.T. Gruenwald, J.H. Jenkins, D.E. Krause, J.J. Mattes, J.R. Newport Time-Dependent Nuclear Decay Parameters: New Evidence for New Forces? // Space Sci Rev (2009) 145: 285-335.

[7] Пархомов А.Г., Макляев Е.Ф. Исследование ритмов и флуктуаций при длительных измерениях радиоактивности, частоты кварцевых резонаторов, шума полупроводников, температуры и атмосферного давления // Физическая мысль России, №1, 2004, с. 1-12.

[8] Пархомов А.Г. Космос. Земля. Человек. Новые грани науки. М., Наука, 2009 - 272 с.

[9] Панчелюга В.А., Панчелюга М.С. Фрактальная размерность и гистограмный метод: методика и некоторые предварительные результаты анализа шумоподобных временных рядов // Биофизика, 2013, том. 58, вып. 2, с. 377-384.

[10] Панчелюга В.А., Панчелюга М.С. Локальный фрактальный анализ шумоподобных временных рядов методом всех сочетаний в диапазоне периодов 1-115 мин // Биофизика, 2015, том. 60, вып. 2, с. 395-410.

[11] T.G.Masters, R.Widmer Free oscillations: frequencies and attenuations // Global Earth physics: a handbook of physical constants / Ed. by Thomas J. Ahrens, American Geophysical Union, 1995, pp.104-125.

[12] Е.М. Линьков Сейсмические явления Л., Изд-во Ленингр. ун-та, 1987 - 248 с.

[13] Баркин Ю.В. Свободные трансляционные колебания системы «ядро-мантия» Земли и вариации природных процессов с часовыми периодами // Нелинейный мир, №1-2, т.5, 2007, с.101-110.

[14] Уруцкоев Л.И., Ликсонов В.И., Циноев В.Г Экспериментальное обнаружение «странного» излучения и трансформации химических элементов // Прикладная физика, 2000, №4, с.83-100.

Анализ теста прототипа калориметра эксперимента COMET

Д. В. Шёлковый
shoukavy@ifanbel.bas-net.by, Институт Физики НАН Беларуси

Аннотация

Проведена обработка данных эксперимента по тестированию прототипа электромагнитного калориметра на основе кристаллов GSO и LYSO на пучке электронов с импульсами от 65 до 145 МэВ/с. Показано, что исходя из требования эксперимента COMET предъявляемому к энергетическому разрешению калориметра меньше 5% (для электрона с энергией 105 МэВ) оптимальным выбором будет калориметр на основе кристаллов LYSO.

1 Введение

Стандартная модель (СМ) [1] хорошо описывает все экспериментальные факты физики высоких энергий. Однако Стандартную модель нельзя считать полностью удовлетворительной схемой несмотря на триумф связанный с открытием бозона Хиггса [2], [3], который являлся единственной неоткрытой частицей в СМ, из-за следующих недостатков. Во-первых, как стало очевидно после 1998 г., у нейтрино есть небольшие массы покоя, и нейтрино разных поколений могут, смешиваясь, осциллировать, т.е. превращаться друг в друга. В изначальной версии СМ нейтрино предполагаются безмассовыми. Во-вторых, в СМ нет полностью удовлетворительных кандидатов на "темную материю"Вселенной, а последняя, согласно физическим и астрофизическим фактам, составляет почти 0,23 плотности массы Вселенной. СМ также не может пока объяснить 0.73 доли темной энергии Вселенной, нет объяснения наблюдаемой ассиметрии вещества и антивещества. Кроме того, СМ содержит большой набор свободных параметров (массы частиц, константы взаимодействий, элементы матрицы смешивания) и оставляет без рассмотрения гравитационное взаимодействие. Следовательно, несмотря на успех в описании экспериментов, СМ не может считаться окончательной теорией элементарных частиц, даже несмотря на ее триумф связанный с открытием бозона Хиггса на Большом адронном коллайдере [2], [3]. Предполагается, что СМ является низкоэнергетическим пределом некоторой общей теории, учитывающей гравитационное взаимодействие, включающей дополнительные механизмы нарушения симметрий, новые частицы. Поэтому одной из важнейших задач современной физики частиц является получить хотя бы первые намеки на то, что это за более глубокая теория. Основной задачей эксперимента COMET [4] является поиск новой физики за рамками СМ через обнаружение процесса протекающего с несохранение аромата заряженных лептонов. Планируется осуществить поиск когерентной безнейтринной конверсии мюона в электрон $\mu- + N(A,Z) \rightarrow e^- + N(A,Z)$ ($\mu-e$ конверсия). Этот процесс, как известно, запрещен в СМ физики элементарных частиц. Однако, модели, которые выходят за рамки СМ предсказывают этот процесс существует с относительной вероятностью распада начиная с 10^{-15}. Цель эксперимента направлена на достижение чувствительности 10^{-17}, следовательно, будет иметь возможность наблюдать

конверсия мюонов в электроны в ядрах. Отрицательные мюоны останавливаются в тонкой мишени и впоследствии захватываются ядром. Энергия возможного конвертировавшего электрона равна массе покоя мюона минус мюонная энергия связи. На данный момент два типа кристаллов LYSO и GSO рассматриваются в качестве кандидатов для электромагнитного калориметра. Требования эксперимента накладывают следующие ограничения на калориметр детектора COMET:

1. энергетическое разрешение $< 5\%$ (для электрона с энергией 105 МэВ);
2. временная характеристика < 100 нс;
3. пространственное разрешение < 1 см.

2 Описание эксперимента

В качестве прототипа электромагнитного калориметра установки COMET тестировались следующие сборки. Для кристалла GSO ($Gd_2SiO_5 : Ce$) прототип представлял собой сборку из 49 кристаллов в виде квадрата 7x7, в центре которого был расположен блок 3x3 с длиной кристаллов 15 см, остальные кристаллы имели длину 12 см. Поперечное сечение кристаллов - квадрат 2x2 см. В случае кристаллов LYSO ($Lu_{1.8}Y_{.2}SiO_5 : Ce$) все аналогично, за исключением того, что все кристаллы имели длину 12 см. Это связано с тем, что у LYSO радиационная длина 1,14 см., а у GSO - 1,38 см. Поэтому для GSO в случае длины 15 см., мы имеем 10,9 радиационных длин, а для LYSO величине 10,9 радиационных длин соответствует длина кристалла 12 см. Каждый кристалл был обернут в 2 слоя фольгой из алитиованного майлара. Пучок электронов с импульсами от 65 МэВ/с до 145 МэВ/с с шагом 20 был сфокусирован в центр сборки перпендикулярно плоскости кристаллов.

Сигнал в кристаллах GSO и LYSO может быть аппроксимирован следующей функцией

$$f(x) = \begin{cases} A\frac{p_3}{p_1\sqrt{2\pi}}g(x) + b(x) & \text{for } p_2X(x) > -1, \\ b(x), \end{cases} \tag{1}$$

где

$$A = \frac{\sinh\left(p_2 \log\left(4\right)\right)}{p_2 \log(4)},\ b(x) = p_4,$$

$$g(x) = \exp\left(-\frac{1}{2}\left(\left(\frac{\log\left(1 + p_2X(x)\right)}{p_2}\right)^2 + p_2^2\right)\right),\ X(x) = A\frac{x - p_0}{p_1},$$

p_0 - положение максимума, нс, p_1 - ширина, нс, p_2 - ассиметрия, p_3 - интегральный заряд, p_5 - основная линия; p_6 - угол наклона.

3 Анализ и результаты

Для обработки и анализа экспериментальных данных на отклик прототипа электромагнитного калориметра на основе кристаллов LYSO (GSO) на электроны была реализована следующая схема, которая в первом приближение может быть описана следующими шагами:

1.Так как исходные экспериментальные данные – это сырые данные (RAW data) необходимо перекодировать эти данные в формат пригодный для анализа с помощью ROOT [3]. ROOT – объектно-ориентированная (C++) программная оболочка (библиотека), предоставляющая большое количество инструментов, необходимых для обработки данных. Таким образом, первый шаг это перекодировка из формата raw data в формат root в специальный класс TTree предназначен для сохранения и анализа большого количества однородных объектов.

2.Написание кода для анализа TTree по следующиму алгоритму. Для каждого события (под событием понимается попадание одного электрона в сборку) аппроксимируем форму колебаний сигнала с помощью функции (1) и находим соответствующий интегральный заряд, выделившийся в каждом кристалле сборке и записываем в root файл. Таким образом, пробегаем по всем событиям сеанса. В результате можем получить новый TTree файл, где одна из ветвей будет суммарный интегральный заряд, выделившийся в каждом кристалле сборки для всех событий конкретного сеанса.
3. Отбросить "нефизические события" (это такие события, когда есть сигнал в большинстве значительно превышает среднее значение интегрального заряда), так называемые ring noise events.
4. Определение коэффициентов перевода величины интегрального заряда в энергию для каждого кристалла. Для получения коэффициентов перевода использовалась калибровка по космическим мюонам.
5. Финальный шаг построение гистограммы, где по оси X откладывается энергия, а по оси Y – число событий энергия и определение энергетического разрешения прототипа калориметра путем апроксимации распределением Гаусса.

Более подробно остановимся на ключевых моментах анализа.

Для того, чтобы отбросить "нефизические события" хорошим требованием является наложение следующих условий на величину интегрального заряда:

$$cut_1 = \frac{40}{3} \cdot (ave_{height} - 200) < max_{height} - \text{для GSO}, \tag{2}$$

$$cut_2 = \frac{2000}{175} \cdot (ave_{height} - 125) < max_{height} - \text{для LYSO}, \tag{3}$$

где ave_{height} - средняя величина амплитуды сигнала, max_{height} - максимальная величина амплитуды сигнала.

В качестве примера, приведен Рис.1, на котором показано правильность выбора условий (2) и (3) на примере кристаллов LYSO. На Рис.1 представлены гистограммы распределения интегрального заряда по кристаллам сборки на основе кристаллов LYSO для событий, которые были отфильтрованы условием (3) в случае бомбардировки сборки пучком электронов с энергий 105 МэВ. Общее число событий для сеансов с энергий 105 МэВ было - 174154, условие (3) отбросило 2508 события.

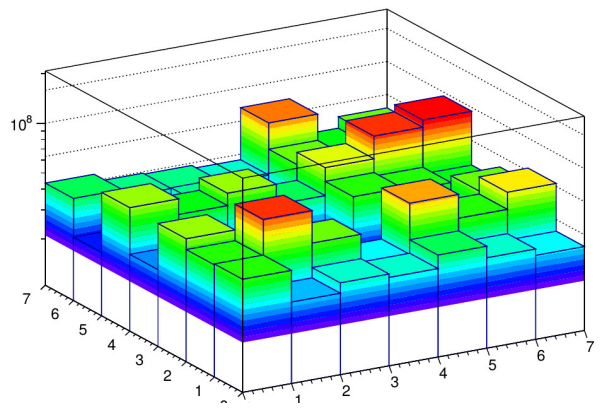


Рис. 1: Распределение выделившегося заряда по кристаллам сборки на основе LYSO для пучка электронов с энергией 105 МэВ для событий, отброшенных условием (3)

Стоить отметить, что у нас общее число событий для космических мюонов используемых для калибровка порядка 17500 для обоих типов кристаллов. В результате у нас на каждый кристалл приходится 1/7 часть, т.е. около 2500, а на самом деле еще меньше из-за накладывания условий (2) и (3) соответственно. В результате у нас низкая статистика, и для улучшения результата фитирования распределением Ландау гистограммы, где по оси X откладывается - интегральный заряд, а по оси Y - число событий, использовался метод максимального правдоподобия, который имеет преимущество над хи-квадрат методом в случае низкой статистики. Потери энергии космическим мюоном при прохождение прототипа калориметра оценивались на основе Geant4 моделирования. В результате было найдено, что средняя выделившая энергия на кристалл E_μ при прохождение космического мюона составила около 19.3 МэВ для LYSO, и порядка 18.2 МэВ для GSO. Коэффициент перевода величины интегрального заряда в энергию для каждого кристалла оценивался как отношение E_μ к наиболее вероятному значению заряда полученному из аппроксимации аналитическим приближением распределения Ландау.

Следующий шаг обработка root файлов для сеансов с энергий пучка 65 - 105 МэВ для прототипа электромагнитного на основе кристаллов LYSO и GSO с помощью специально написанной программы. Далее строим гистограммы распределения выделившегося заряда по кристаллам сборки для двух типов кристаллов.

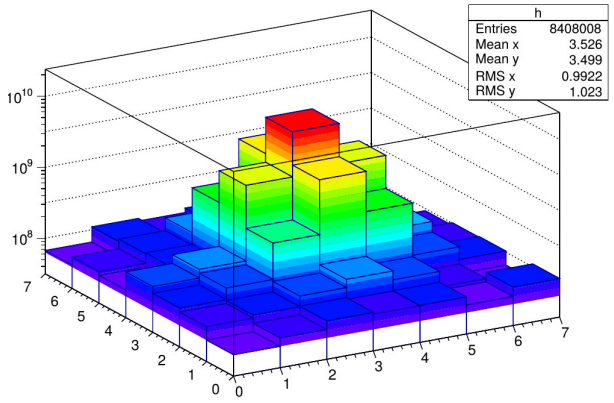


Рис. 2: Распределение выделившегося заряда по кристаллам сборки на основе LYSO для пучка электронов с энергией 105 МэВ. Для событий прошедших условие (3)

Как видно максимум выделившего заряда приходится в центр сборки, куда и был направлен пучок электронов. Также отчетливо видно из Рис. 2. , что практически вся энергия выделилась в 9 центральных кристаллах калориметра. Поэтому целесообразно в первом приближение, для оценки энергетического разрешения калориметра взять в рассмотрение 25 кристаллов, т.е. центральную выборку 5 на 5 кристаллов. В дальнейшем необходимо будет внедрить алгоритм, который будет определять, в какой конкретный кристалл попал электрон и брать в рассмотрение только те кристаллы, где сигнал от этого события больше некоторого минимального. Дальнейший шаг это аппроксимация полученных энергетических спектров для пучков электронов с энергиями 65-145 МэВ распределением Гаусса и нахождение энергетического разрешения сборок на основе LYSO и GSO.

Энергетическое разрешение определяем как

$$R = \frac{\sigma}{E_{mean}} \cdot 100\%, \tag{4}$$

где σ- среднее стандартное отклонение E_{mean} - средняя величина выделившийся энергии

в калориметре, обе эти величины берем из аппроксимации.

График энергетического разрешения калориметра для двух типов кристаллов от энергии электронов приведен на Рис.3

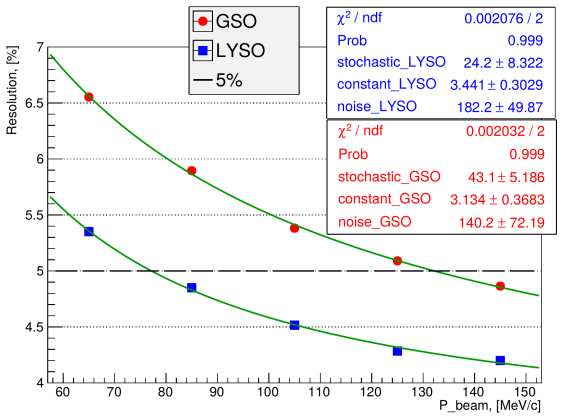


Рис. 3: Энергетическое разрешение прототипа калориметра на основе кристаллов GSO и LYSO в зависимости от энергии пучка.

Точки соответствуют энергетическому разрешению сборки при энергии пучка 65 МэВ, 85 МэВ, 105 МэВ, 125 МэВ и 145 МэВ, полеченные путем обработки эксперимента (Tohoku Beam Test). Линия - аппроксимация экспериментальных точек формулой

$$R = \sqrt{\frac{a^2}{E_{beam}} + b^2 + \frac{c}{E_{beam}^2}}, \tag{5}$$

где a так называемый ″стохастический″коэффициент отражает статистический, как правило, пуассоновский, характер процесса преобразования поглощаемой энергии в измеряемый эффект (в нашем случае заряд ФЭУ), b – ″константный″член отражает, неопределённости калибровки, неоднородности сигнала, материала детектора, температуры, утечки части ливня, – ″шумовой″вклад шумов электроники, энергия пучка.

Важнейшим требованием, предъявляемым к калориметру установки COMET является, условие того, что энергетическое разрешения $< 5\%$ для электрона с энергией 105 МэВ. Калориметр на основе кристаллов GSO выполняет это требование, как следует из Рис.3, только начиная с энергии электронов 145 МэВ. Калориметр на основе кристаллов LYSO выполняет основное условие: энергетическое разрешение меньше 5% уже начиная с энергии электронов 85 МэВ.

4 Заключение

Выполнен тестирование прототипа калориметра эксперимента COMET на основе двух кристаллов – кандидатов GSO и LYSO (март 2014 г., г. Тохоку). Показано, что, исходя из требований, накладываемых на калориметр эксперимента COMET, из двух кристаллов – кандидатов GSO и LYSO только кристалл LYSO удовлетворяет основному критерию энергетического разрешению – быть меньше 5% для энергии 105 МэВ.

Список литературы

[1] В. М. Емельянов, Стандартная модель и ее расширения. М. Физматлит, 2007, 584 с.

[2] G. Aad et al. (Atlas Collab.), Phys. Lett. B **716**, 1–29 (2012).

[3] S. Chatrchyan et al. (CMS Collab.), Phys. Lett. B **716**, 30–61 (2012).

[4] Y. Kuno, A search for muon-to-electron conversion at J-PARC the COMET experiment, Prog. Theor. Exp. Phys., 2013, 022C01 (43 pages).

Сжатые и перепутанные состояния глюонов в КХД

В.А. Шапоров, В.И. Кувшинов
shaparau@sosny.bas-net.by; v.kuvshinov@sosny.bas-net.by;
Joint Institute for Power and Nuclear Research - Sosny,
National Academy of Sciences of Belarus
220109 Minsk, Belarus

Аннотация

Получены условия возникновения состояний обобщенной одномерной квантовой системы, минимизирующих соотношение неопределенностей как для дисперсий операторов координаты и импульса, так и для соответствующих обобщенных операторов безразмерных величин.

В данной работе получен явный вид условия существования многомодовых сжатых состояний виртуальных глюонов. Аналитически установлено сосуществования эффектов двухмодового сжатия и перепутанности для виртуальных глюонов с учетом их поляризации. Показано, что условие перепутанности накладывает более сильные ограничения на начальные амплитуды и фазы глюонных полей.

Путем исследования нормированной корреляционной функции второго порядка показано наличие эффектов группировки и антигруппировки цветных частиц в зависимости от фаз калибровочных полей глюонов.

1 Введение

В настоящее время процессы множественного рождения адронов (ПМРА) активно исследуются на экспериментальной установке LHC, поскольку являются одним из основных источников информации о строении вещества на самых малых расстояниях, о возможных новых состояниях материи, о новых свойствах сильных взаимодействий, о конфайнменте и т.п. Изучение ПМРА привело к открытию струй адронов, чьи импульсные характеристики в основном определялись свойствами породивших их партонов. Поэтому исследование таких характеристик, как распределение по множественности, корреляции и флуктуации частиц в струйных событиях дает соответствующую информацию о свойствах их прародителей – кварках и глюонах и представляет большой интерес как для теоретиков, так и для экспериментаторов.

Несмотря на большое количество теоретических работ, посвященных описанию ПМРА, остается ряд экспериментальных фактов, в частности, ширина распределения по множественности и поведение корреляционной функции, которые неудовлетворительно описываются в рамках стандартной пертурбативной квантовой хромодинамики (КХД) с учетом поправок по бегущей константе связи. Поэтому полагают, что необходимо правильно учесть вклад непертурбативной части развития кварк-глюонного каскада, когда константа связи становится большой, и стандартная теория возмущений становится неприменимой. Поскольку из экспериментов получено, что среднее число рождающихся глюонов превалирует над средним числом кварков и, следовательно, роль глюонных полей растет с ростом константы связи, поэтому учет их непертурбативной эволюции становится необходимым.

Главным недостатком большого количества существующих феноменологических непертурбативных моделей является их оторванность от КХД. Поэтому в данной работе учет непертурбативных свойств глюонов основан на стандартном лагранжиане КХД, который содержит в себе самодействие глюонов в отличие от фотонов. Это приводит к важным физическим следствиям.

Одним из них является квантовый эффект сжатия одной из квадратурных компонент поля [1–3]. Квантовые флуктуации в сжатом состоянии не равны и зависят от фазы: неопределенность по одной из квадратурных компонент поля в этом квантовом состоянии может быть меньше, чем в когерентном. Благодаря этому свойству стало возможным понизить квантовый шум и тем самым уменьшить квантово-механический предел измерения той или иной характеристики. В частности, использование сжатого света повышает чувствительность интерферометра без увеличения мощности лазера [4].

В то время, как когерентный свет имеет функцию распределения фотонов, описываемую стандартным распределением Пуассона, для сжатых состояний распределение проявляет осцилляции [5, 6]. Поэтому таким состояниям соответствуют либо субпуассоновская, либо суперпуассоновская статистики, что может говорить об антигруппировке или группировке фотонов соответственно.

Если сжатые состояния фотонов образуются в результате взаимодействия с нелинейной средой, то сжатые состояния глюонов могут возникать из-за их самодействия благодаря цветовому заряду. Поэтому характерные особенности сжатых состояний фотонов могут проявляться и для глюонов. Следовательно, сжатые состояния глюонов могут возникать на непертурбативном этапе эволюции кварк-глюонного каскада. В отличие от ряда наших ранних работ [7–9] здесь рассматривается не квантово-механическая модельная система КХД струи, а квантово-полевая общая система.

В квантовой оптике (КО) многомодовые сжатые состояния фотонов неразрывно связаны с перепутанными состояниями [10]. Поэтому можно полагать, что сжатые состояния глюонов в КХД при определенных условиях одновременно могут быть перепутанными. Исследование данного вопроса представляет интерес по следующей причине: в конце непертурбативного каскада развития КХД-струи каждый глюон дает $q\bar{q}$-пару, эффект перепутанности переходит на мультикварковые состояния, т.е. к моменту начала обесцвечивания кварки могут находиться в перепутанном состоянии. Взаимодействие перепутанных состояний кварков со стохастическим вакуумом (аналог квантового измерения) приводит к следующему эффекту: как только "измерение" проецирует данное состояние кварков на состояние с определенным цветом, так мгновенно находится кварк с противоположным цветом (процесс обесцвечивания), между ними натягивается струна, и происходит адронизация.

2 Сжатые состояния, как представители класса состояний, минимизирующих соотношение неопределенности

В квантовой физике термин "сжатие" неразрывно связан с квантовой неопределенностью измеряемой на эксперименте какой-либо физической величины. Если в соотношении неопределенностей Шредингера-Робертсона для любых двух некоммутирующих эрмитовых операторов[1] $\hat{A}$ и $\hat{B}$

$$\langle(\widehat{\Delta A})^2\rangle\langle(\widehat{\Delta B})^2\rangle \geqslant \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\hat{A},\hat{B}\}\rangle|^2 \qquad (1)$$

[1]Обобщенное соотношение неопределенностей для любых двух некоммутирующих неэрмитовых операторов приведено в работе [11].

достигается знак равенства, тогда квантовая система находится в состоянии, описываемом вектором $|\psi\rangle$, отвечающему минимальным дисперсиям наблюдаемых:

$$\langle(\widehat{\Delta A})^2\rangle = \langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle = (-i)\frac{|\gamma|^2}{2\gamma_{\mathrm{R}}}\,\langle[\hat{A},\hat{B}]\rangle, \tag{2а}$$

$$\langle(\widehat{\Delta B})^2\rangle = \langle(\hat{B} - \langle\hat{B}\rangle)^2\rangle = \frac{i}{2\gamma_{\mathrm{R}}}\,\langle[\hat{A},\hat{B}]\rangle^*, \qquad \gamma_{\mathrm{R}} \neq 0 \tag{2б}$$

и удовлетворяющему уравнению

$$\left(\hat{A} + i\,\gamma\hat{B}\right)|\psi\rangle = \left(\langle\hat{A}\rangle + i\gamma\langle\hat{B}\rangle\right)|\psi\rangle. \tag{3}$$

Здесь γ – произвольный параметр, в зависимости от которого ковариация физических величин A и B отлична (γ – комплексный) либо равна нулю (γ – вещественный), их дисперсии равны ($|\gamma| = 1$) или нет ($|\gamma| \neq 1$). Вектор $|\psi\rangle$, удовлетворяющий (3), описывает более близкое к классическому состояние системы. Очевидно, что при $|\gamma| = 1$ среднеквадратичные отклонения операторов $\hat{A}$ и $\hat{B}$ (2а), (2б) равны, и тогда мы имеем дело с когерентным состоянием системы [1–3].

Если же дисперсия одного из рассматриваемых операторов меньше его неопределенности в когерентном состоянии, а квантовая неопределенность измерения другой физической величины возрастает ($|\gamma| \neq 1$), в этом случае система находится в идеальном сжатом состоянии [1]. В таких квантовых состояниях порог квантовых измерений определенной величины уменьшается, что позволяет проводить измерения той или иной физической характеристики данной системы с большей точностью, чем в когерентном состоянии.

Для выявления истоков эффекта сжатия рассмотрим дисперсии фазочувствительных эрмитовых операторов:

$$\hat{Y}_\vartheta = \hat{X}_1\cos\frac{\vartheta}{2} + \hat{X}_2\sin\frac{\vartheta}{2}, \qquad\qquad \hat{Y}_{\vartheta+\pi} = -\hat{X}_1\sin\frac{\vartheta}{2} + \hat{X}_2\cos\frac{\vartheta}{2}, \tag{4}$$

которые при определенных значениях вещественного параметра ϑ ($\pi n, n = 0, 1, 2, \ldots$) с точностью до множителя совпадают с операторами $\hat{X}_1, \hat{X}_2$:

$$\hat{X_1} = \frac{\hat{a} + \hat{a}^+}{2}, \quad \hat{X_2} = \frac{\hat{a} - \hat{a}^+}{2\,i}, \tag{5}$$

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}, \tag{6}$$

связанными с операторами координаты и импульса обобщенной одномерной квантовой системы[2]:

$$\hat{X}_1 = (\lambda_{\mathrm{R}}\hat{x} + \delta_{\mathrm{I}}\hat{p})/\sqrt{2\hbar}, \qquad \hat{X}_2 = (-\lambda_{\mathrm{I}}\hat{x} + \delta_{\mathrm{R}}\hat{p})/\sqrt{2\hbar}. \tag{7}$$

Очевидно, что коммутатор $\left[\hat{Y}_\vartheta, \hat{Y}_{\vartheta+\pi}\right]$ совпадает с соответствующим выражением для операторов $\hat{X}_1, \hat{X}_2$ (6), из которого вытекает связь между произвольными комплексными числами λ и δ:

$$\mathrm{Re}(\delta^*\lambda) = \lambda_{\mathrm{R}}\delta_{\mathrm{R}} + \lambda_{\mathrm{I}}\delta_{\mathrm{I}} = 1. \tag{8}$$

Чтобы найти вектор состояния, который минимизирует соотношение неопределенностей для дисперсий $\langle(\widehat{\Delta Y}_\vartheta)^2\rangle, \langle(\widehat{\Delta Y}_{\vartheta+\pi})^2\rangle$, необходимо решить уравнение, аналогичное (3),

$$\left(\hat{Y}_\vartheta + i\gamma\hat{Y}_{\vartheta+\pi}\right)|\psi_\vartheta\rangle = \beta|\psi_\vartheta\rangle, \qquad \gamma \in \Re, \tag{9}$$

[2] $\hat{X}_1$ и $\hat{X}_2$ представляют собой операторы безразмерных величин координаты и импульса при $\lambda_{\mathrm{I}} = \delta_{\mathrm{I}} = 0$ или $\lambda_{\mathrm{R}} = \delta_{\mathrm{R}} = 0$.

которое в координатном представлении имеет вид:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\psi_\vartheta(x) = \left\{-\frac{\xi x}{\hbar} + \tilde{\beta}\right\}\psi(x), \tag{10}$$

где

$$\beta = \langle\hat{Y}_\vartheta\rangle + i\gamma\langle\hat{Y}_{\vartheta+\pi}\rangle,\ \xi = \frac{\lambda_{\mathrm{R}}\left(\cos\frac{\vartheta}{2} - i\gamma\sin\frac{\vartheta}{2}\right) - \lambda_{\mathrm{I}}\left(\sin\frac{\vartheta}{2} + i\gamma\cos\frac{\vartheta}{2}\right)}{\delta_{\mathrm{R}}\left(\gamma\cos\frac{\vartheta}{2} - i\sin\frac{\vartheta}{2}\right) - \delta_{\mathrm{I}}\left(\gamma\sin\frac{\vartheta}{2} + i\cos\frac{\vartheta}{2}\right)},\ \tilde{\beta} = \frac{\xi\,\langle\hat{x}\rangle}{\hbar} + \frac{i}{\hbar}\langle\hat{p}\rangle. \tag{11}$$

Решением уравнения (10) в рассматриваемом представлении является следующая волновая функция:

$$\psi_\vartheta(x) = C\exp\left\{-\frac{\xi\,x}{2\hbar}\big(x - 2\langle\hat{x}\rangle\big) + \frac{i}{\hbar}\langle\hat{p}\rangle\,x\right\}, \tag{12}$$

где C – нормировочный множитель:

$$C = \left(\frac{\mathrm{Re}(\xi)}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{\mathrm{Re}(\xi)\,\langle\hat{x}\rangle^2}{2\hbar}\right\}, \tag{13}$$

Вектор $|\psi_\vartheta\rangle$, удовлетворяющий уравнению (9), минимизирует соотношение неопределенностей для дисперсий операторов $\hat{Y}_\vartheta$ и $\hat{Y}_{\vartheta+\pi}$:

$$\langle(\widehat{\Delta Y}_\vartheta)^2\rangle = \frac{\gamma}{4},\quad \langle(\widehat{\Delta Y}_{\vartheta+\pi})^2\rangle = \frac{1}{4\gamma}, \tag{14}$$

и значения $\hat{Y}_\vartheta$ и $\hat{Y}_{\vartheta+\pi}$ некоррелируют. Очевидно, что только при $\gamma = 1$ дисперсии равны, и данный вектор будет описывать когерентные состояния.

В то же время как видно из дисперсий операторов $\hat{X}_1, \hat{X}_2$ и $\hat{x}, \hat{p}$:

$$\langle(\widehat{\Delta X}_1)^2\rangle = \frac{\gamma}{4}\cos^2\frac{\vartheta}{2} + \frac{1}{4\gamma}\sin^2\frac{\vartheta}{2},\qquad \langle(\widehat{\Delta X}_2)^2\rangle = \frac{\gamma}{4}\sin^2\frac{\vartheta}{2} + \frac{1}{4\gamma}\cos^2\frac{\vartheta}{2}, \tag{15а}$$

$$\langle(\widehat{\Delta X}_1)^2\rangle\langle(\widehat{\Delta X}_2)^2\rangle = \frac{1}{16}\left[1 + \frac{(\gamma^2-1)^2}{4\gamma^2}\sin^2\vartheta\right], \tag{15б}$$

их значения коррелируют.

$$\langle(\widehat{\Delta x})^2\rangle\langle(\widehat{\Delta p})^2\rangle = \frac{\hbar^2}{4\cos^2(\varphi-\zeta)} + \frac{\hbar^2(\gamma^2-1)}{16\,\gamma^2\,\cos^2(\varphi-\zeta)}\Big\{\gamma^2\Big[\sin(\zeta-\varphi) + \sin(\zeta+\varphi+\vartheta)\Big]^2 - \\ - \Big[\sin(\zeta-\varphi) - \sin(\zeta+\varphi+\vartheta)\Big]^2\Big\}, \tag{16}$$

где ζ, φ – фазы комплексных величин λ, δ соответственно.

Из соотношений (15б), (16) следует, что произведение дисперсий операторов $\hat{X}_1, \hat{X}_2$ и $\hat{x}, \hat{p}$ будут одновременно минимальными и некоррелированными при выполнении следующих условий[3]:

$$1)\ \gamma = 1, \varphi - \zeta = \pi\,n_1 \quad (n_1 = 0, \pm2, \pm4, \ldots), \tag{17а}$$

$$2)\ \gamma \neq 1, \varphi - \zeta = \pi\,n_1, \varphi + \zeta = \pi\,n_2, \vartheta = \pi\,n_3\ (n_2, n_3 = 0, \pm1, \pm2, \ldots). \tag{17б}$$

[3]При этом учитывается ограничение (8), накладываемое на фазы комплексных параметров λ, δ.

Если представить параметр γ в виде функции e^{-r}, получим идеальные сжатые состояния системы [1] с условием сжатия в виде

$$\langle(\widehat{\Delta Y}_{\vartheta})^2\rangle = \frac{\mathrm{e}^{-2r}}{4}, \qquad \langle(\widehat{\Delta Y}_{\vartheta+\pi})^2\rangle = \frac{\mathrm{e}^{2r}}{4}. \tag{18}$$

Наглядно их можно представить в виде контура функции Вигнера, представляющего собой эллипс малая и большая полуоси которого

$$\sqrt{\langle(\widehat{\Delta Y}_{\vartheta})^2\rangle} = \mathrm{e}^{-r}/2, \qquad \sqrt{\langle(\widehat{\Delta Y}_{\vartheta+\pi})^2\rangle} = \mathrm{e}^{r}/2 \tag{19}$$

повернуты на угол $(\vartheta/2)$ относительно оси X_1 (рис.(1)), параметр ϑ задает направление малой полуоси эллипса, а, следовательно, определяет направление максимального сжатия, а r – степень сжатия.

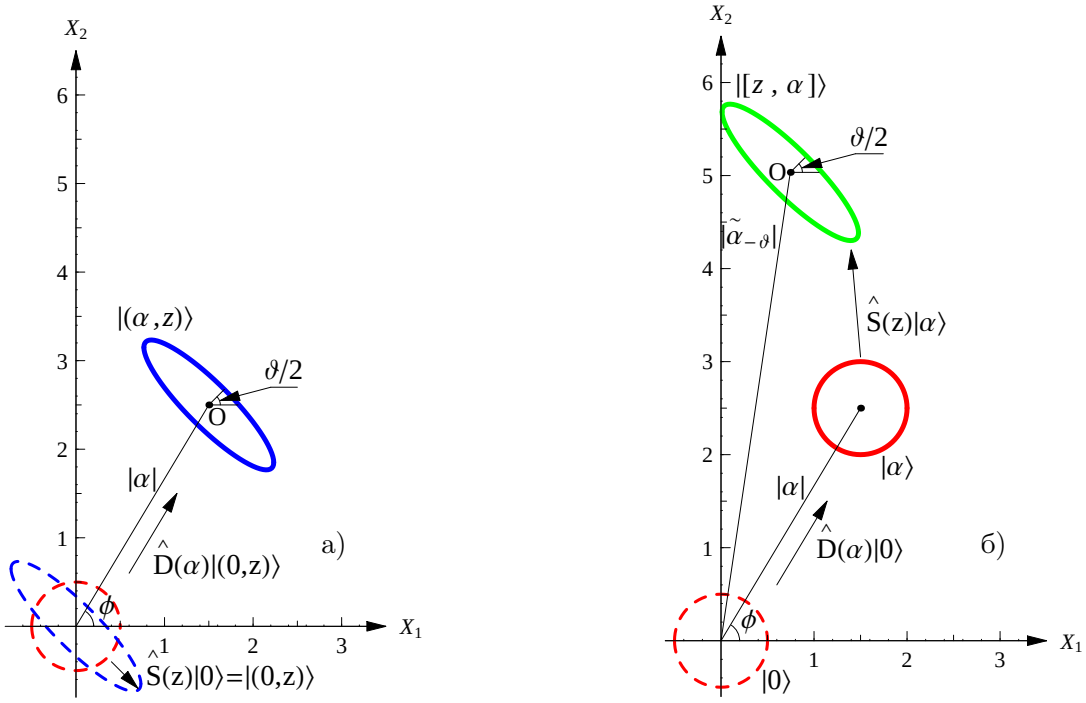


Рис. 1: Квантовые неопределенности идеальных сжатых состояний: а) $|(\alpha,z)\rangle = \hat{D}(\alpha)\,\hat{S}(z)|0\rangle$, б) $|[z,\alpha]\rangle = \hat{S}(z)\hat{D}(\alpha)|0\rangle$, $z = r\mathrm{e}^{i\vartheta}$.

3 Сжатые и перепутанные состояния глюонов в КХД

Для исследования эффекта многомодового (в частности, p-модового) сжатия по различным цветовым зарядам необходимо по аналогии с КО [2] ввести следующие фазово-чувствительные эрмитовы операторы:

$$(\hat{X}_{\lambda}^{i_1,\ldots,i_p})_1 = \frac{1}{2\sqrt{p}}\sum_{j=1}^{p}\left[\hat{b}_{\lambda}^{i_j} + (\hat{b}_{\lambda}^{i_j})^{+}\right], \tag{20а}$$

$$(\hat{X}_{\lambda}^{i_1,\ldots,i_p})_2 = \frac{1}{2i\sqrt{p}}\sum_{j=1}^{p}\left[\hat{b}_{\lambda}^{i_j} - (\hat{b}_{\lambda}^{i_j})^{+}\right] \tag{20б}$$

и затем проверить условия, при которых дисперсия одного из них будет меньше, чем в когерентном состоянии. Здесь $\hat{b}_\lambda^{i_j}, (\hat{b}_\lambda^{i_j})^+$ – операторы уничтожения (рождения) глюонов с цветами $i_1, \ldots, i_p = \overline{1,8}$ и поляризацией $\lambda = \overline{1,3}$.

Условие сжатия по цветовым зарядам $i_1, \ldots, i_p$ выражается в виде неравенства

$$\left\langle \hat{N} \left(\Delta(\hat{X}_\lambda^{i_1,\ldots,i_p})_{\frac{1}{2}} \right)^2 \right\rangle < 0. \tag{21}$$

Здесь $\hat{N}$ – оператор нормального упорядочивания такой, что

$$\left\langle \hat{N} \left(\Delta(\hat{X}_\lambda^{i_1,\ldots,i_p})_{\frac{1}{2}} \right)^2 \right\rangle = \pm \frac{1}{4p} \sum_{j,k=1}^{p} \Big\{ \left\langle \Delta\big(\hat{b}_\lambda^{i_j}, \hat{b}_\lambda^{i_k}\big) \right\rangle + \left\langle \Delta\big(\hat{b}_\lambda^{i_j +}, \hat{b}_\lambda^{i_k +}\big) \right\rangle \pm$$
$$\pm \left\langle \Delta\big(\hat{b}_\lambda^{i_j +}, \hat{b}_\lambda^{i_k}\big) \right\rangle \pm \left\langle \Delta\big(\hat{b}_\lambda^{i_k +}, \hat{b}_\lambda^{i_j}\big) \right\rangle \Big\}, \tag{22}$$

$\left\langle \Delta\big(\hat{b}_\lambda^{i_j}, \hat{b}_\lambda^{i_k}\big) \right\rangle = \left\langle \hat{b}_\lambda^{i_j} \hat{b}_\lambda^{i_k} \right\rangle - \left\langle \hat{b}_\lambda^{i_j} \right\rangle \left\langle \hat{b}_\lambda^{i_k} \right\rangle$. Здесь усреднение проводится по вектору $|\mathrm{f}\rangle$, который описывает эволюцию виртуального глюонного поля в течение малого промежутка времени Δt в представлении взаимодействия:

$$|\mathrm{f}\rangle \simeq |\mathrm{in}\rangle - i\,\Delta t\,\hat{H}_{\text{вз}}(t_0)\,|\mathrm{in}\rangle, \tag{23}$$

где $|\mathrm{in}\rangle$ – начальный вектор состояния виртуального глюонного поля, $\Delta t = t - t_0$ (в дальнейшем мы будем считать $t_0 = 0$, и, следовательно, $\Delta t = t$), $\hat{H}_{\text{вз}}(t_0) = \hat{H}^{(3)}_{\text{вз}}(t_0) + \hat{H}^{(4)}_{\text{вз}}(t_0)$ – гамильтониан глюонного самодействия, состоящий из гамильтонианов трехглюонного $\hat{H}^{(3)}_{\text{вз}}$ и четырехглюонного $\hat{H}^{(4)}_{\text{вз}}$ самодействия, которые в импульсном представлении записываются как

$$\hat{H}^{(3)}_{\text{вз}}(t) = \frac{(-i)g}{(2\pi)^{3/2}} f_{abc} \sum_{\lambda_1,\lambda_2,\lambda_3} \int \prod_{i=1}^{3} \frac{d^3k_i}{\sqrt{2k_{0i}}} \Big[\vec{k}_1 \vec{\varepsilon}_{\lambda_2}(k_2) + \vec{k}_3 \vec{\varepsilon}_{\lambda_2}(k_2) \Big] \Big(\varepsilon^{\nu}_{\lambda_1}(k_1) \varepsilon^{\lambda_3}_{\nu}(k_3) \Big) \times$$
$$\times \Big[\hat{b}^{a+}_{\lambda_1}(k_1) \hat{b}^{b}_{\lambda_2}(k_2) \hat{b}^{c}_{\lambda_3}(k_3)\, e^{2i(k_{01}-k_{02}-k_{03})t}\, \delta(\vec{k}_1 - \vec{k}_2 - \vec{k}_3) +$$
$$+ \hat{b}^{a+}_{\lambda_1}(k_1) \hat{b}^{b+}_{\lambda_2}(k_2) \hat{b}^{c}_{\lambda_3}(k_3)\, e^{2i(k_{01}+k_{02}-k_{03})t}\, \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3) \Big], \tag{24}$$

$$\hat{H}^{(4)}_{\text{вз}}(t) = \frac{g^2}{4(2\pi)^3} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \int \prod_{i=1}^{4} \frac{d^3k_i}{\sqrt{2k_{0i}}} \Big\{ f_{abc} f_{ade}\, \varepsilon^{\mu}_{\lambda_1}(k_1) \varepsilon^{\lambda_3}_{\mu}(k_3) \varepsilon^{\nu}_{\lambda_2}(k_2) \varepsilon^{\lambda_4}_{\nu}(k_4) \times$$
$$\times \Big[\Big(\hat{b}^{b}_{\lambda_1}(k_1) \hat{b}^{c}_{\lambda_2}(k_2) \hat{b}^{d}_{\lambda_3}(k_3) \hat{b}^{e}_{\lambda_4}(k_4)\, e^{-2i(k_{01}+k_{02}+k_{03}+k_{04})t} +$$
$$+ \hat{b}^{b+}_{\lambda_1}(k_1) \hat{b}^{c+}_{\lambda_2}(k_2) \hat{b}^{d+}_{\lambda_3}(k_3) \hat{b}^{e+}_{\lambda_4}(k_4)\, e^{2i(k_{01}+k_{02}+k_{03}+k_{04})t} \Big) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) +$$
$$+ 4\hat{b}^{b+}_{\lambda_1}(k_1) \hat{b}^{c}_{\lambda_2}(k_2) \hat{b}^{d}_{\lambda_3}(k_3) \hat{b}^{e}_{\lambda_4}(k_4)\, e^{2i(k_{01}-k_{02}-k_{03}-k_{04})t}\, \delta(\vec{k}_1 - \vec{k}_2 - \vec{k}_3 - \vec{k}_4) +$$
$$+ 4\hat{b}^{b+}_{\lambda_1}(k_1) \hat{b}^{c+}_{\lambda_2}(k_2) \hat{b}^{d+}_{\lambda_3}(k_3) \hat{b}^{e}_{\lambda_4}(k_4)\, e^{2i(k_{01}+k_{02}+k_{03}-k_{04})t}\, \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 - \vec{k}_4) \Big] +$$
$$+ 2\hat{b}^{b+}_{\lambda_1}(k_1) \hat{b}^{c+}_{\lambda_2}(k_2) \hat{b}^{d}_{\lambda_3}(k_3) \hat{b}^{e}_{\lambda_4}(k_4)\, e^{2i(k_{01}+k_{02}-k_{03}-k_{04})t}\, \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \times$$
$$\times \Big[(f_{abc} f_{ade} + f_{abe} f_{adc})\, \varepsilon^{\mu}_{\lambda_1}(k_1) \varepsilon^{\lambda_3}_{\mu}(k_3) \varepsilon^{\nu}_{\lambda_2}(k_2) \varepsilon^{\lambda_4}_{\nu}(k_4) +$$
$$+ f_{abd} f_{ace}\, \varepsilon^{\mu}_{\lambda_1}(k_1) \varepsilon^{\lambda_2}_{\mu}(k_2) \varepsilon^{\nu}_{\lambda_3}(k_3) \varepsilon^{\lambda_4}_{\nu}(k_4) \Big] \Big\}. \tag{25}$$

Как и при исследовании одномодового эффекта сжатия из физических соображений в качестве начального вектора состояния выбирается следующий вектор:

$$|\mathrm{in}\rangle \;\equiv\; |\,\alpha\rangle \;=\; \prod_{\lambda=1}^{3}\prod_{b=1}^{8} |\alpha_\lambda^b\rangle.$$

По аналогии с КО вектор когерентного глюонного поля $|\alpha_\lambda^b\rangle$ является собственным вектором оператора уничтожения $\hat{b}_\lambda^b$ с собственным значением $\alpha_\lambda^b = |\alpha_\lambda^b| e^{i\,\gamma_\lambda^b}$. Здесь $|\alpha_\lambda^b|$ и γ_λ^b – амплитуда и фаза данного глюонного поля. В когерентном состоянии число глюонов с фиксированным цветом b и поляризацией λ может быть произвольным, фиксированы среднее число глюонов, равное квадрату амплитуды рассматриваемого поля, и фаза γ_λ^b.

Проводя усреднение операторов уничтожения и рождения $\hat{b}_\lambda^{i_j}, (\hat{b}_\lambda^{i_j})^+$ в выражении (22) по вектору $|\mathrm{f}\rangle$, который определяется согласно (23), и принимая во внимание свойство начального вектора, мы можем записать многомодовое условие сжатия в виде следующего неравенства

$$\Big\langle \hat{N}\Big(\Delta(\hat{X}_\lambda^{i_1,\ldots,i_p})_{\frac{1}{2}}\Big)^2\Big\rangle = \mp\frac{i\,t}{4p}\sum_{j,k=1}^{p}\Big\{\langle\alpha|\big[\hat{b}_\lambda^{i_k},\big[\hat{b}_\lambda^{i_j},\hat{H}_{\text{вз}}(0)\big]\big]|\alpha\rangle - \\ - \langle\alpha|\big[\big[\hat{H}_{\text{вз}}(0),\hat{b}_\lambda^{i_j+}\big],\hat{b}_\lambda^{i_k+}\big]|\alpha\rangle\Big\} < 0. \qquad (26)$$

Здесь учли, что $t_0 = 0$, и, следовательно, $\Delta t = t$.

Можно показать, что трехглюонное самодействие, описываемое гамильтонианом $\hat{H}_{\text{вз}}^{(3)}$, не дает вклада в эффект многомодового сжатия, как и для одномодового, поскольку следующие коммутаторы равны нулю:

$$\big[\big[\hat{H}_{\text{вз}}^{(3)}(0),\hat{b}_\lambda^{i_j+}\big],\hat{b}_\lambda^{i_k+}\big] = 0, \quad \big[\hat{b}_\lambda^{i_k},\big[\hat{b}_\lambda^{i_j},\hat{H}_{\text{вз}}^{(3)}(0)\big]\big] = 0. \qquad (27)$$

Таким образом, только четырехглюонное самодействие может приводить к появлению многомодового эффекта сжатия.

В явном виде неравенство (26) можно записать как

$$\Big\langle \hat{N}\Big(\Delta(\hat{X}_\lambda^{i_1,\ldots,i_p})_{\frac{1}{2}}\Big)^2\Big\rangle = \pm\frac{i\,t}{8\,p\,k_0}\frac{g^2}{(2\pi)^3}\sum_{j,k=1}^{p}\sum_{\lambda_1,\lambda_2=1}^{3}\int\prod_{i=1}^{2}\frac{d^3k_i}{\sqrt{2k_{0i}}}\times \\ \times\Big[\Big(\varepsilon_\mu^{\lambda_1}(k_1)\,\varepsilon_{\lambda_2}^{\mu}(k_2)\Big)\Big(\varepsilon_\nu^{\lambda}(k)\,\varepsilon_\lambda^{\nu}(k)\Big) - \Big(\varepsilon_\mu^{\lambda_1}(k_1)\varepsilon_\lambda^{\mu}(k)\Big)\Big(\varepsilon_\nu^{\lambda_2}(k_2)\varepsilon_\lambda^{\nu}(k)\Big)\Big]\times \\ \times\Big\{ f_{ai_jb}f_{ai_kc}\Big[\langle\alpha|\hat{b}_{\lambda_1}^{b+}(k_1)\hat{b}_{\lambda_2}^{c+}(k_2)|\alpha\rangle - \langle\alpha|\hat{b}_{\lambda_1}^{b}(k_1)\hat{b}_{\lambda_2}^{c}(k_2)|\alpha\rangle\Big]\times \\ \times\Big[\delta(2\vec{k}-\vec{k_1}-\vec{k_2}) - \delta(2\vec{k}+\vec{k_1}+\vec{k_2})\Big] + \big(f_{ai_jb}f_{ai_kc} + f_{ai_kb}f_{ai_jc}\big)\times \\ \times\langle\alpha|\hat{b}_{\lambda_1}^{b+}(k_1)\hat{b}_{\lambda_2}^{c}(k_2)|\alpha\rangle\Big[\delta(2\vec{k}-\vec{k_1}+\vec{k_2}) - \delta(2\vec{k}+\vec{k_1}-\vec{k_2})\Big]\Big\} < 0. \qquad (28)$$

Конкретные ограничения на амплитуды и фазы начальных глюонных полей зависят от эксперимента.

Для коллинеарных глюонов с равными импульсами данное условие сжатия (28) можно записать в виде

$$\Big\langle \hat{N}\Big(\Delta(X_\lambda^{i_1,\ldots,i_p})_{\frac{1}{2}}\Big)^2\Big\rangle = \pm\frac{g^2\,t}{8\,p\,(2\pi)^3\,k_0^2}\sum_{j,k=1}^{p}\sum_{\lambda_1\neq\lambda} f_{ai_kb}f_{ai_jc}|\alpha_{\lambda_1}^b||\alpha_{\lambda_1}^c|\sin(\gamma_{\lambda_1}^b+\gamma_{\lambda_1}^c) < 0, \qquad (29)$$

который будет справедлив независимо от экспериментальных условий. При этом было учтено, что $\alpha^b_{\lambda_1} = |\alpha^b_{\lambda_1}|e^{i\,\gamma^b_{\lambda_1}}$ и $\alpha^c_{\lambda_1} = |\alpha^c_{\lambda_1}|e^{i\,\gamma^c_{\lambda_1}}$. Условие многомодового сжатия (29) выполняется всегда за исключением случая, когда $\gamma^b_{\lambda_1} + \gamma^c_{\lambda_1} = 0, \pi$. В частности, если все начальные когерентные состояния глюонных полей с различными цветовыми и поляризационными индексами являются вещественными или мнимыми, тогда отсутствуют эффекты как одномодового сжатия, так и многомодового.

Из неравенства (29) следует, что эффект сжатия будет проявляться сильнее в двух случаях: 1) когда амплитуды начальных когерентных состояний глюонных полей достаточно велики, 2) константа связи $\alpha_s \sim g^2$ больше единицы (непертурбативная область эволюции).

Таким образом, аналитически было показано, что четырехглюонное самодействие приводит к возникновению многомодового эффекта сжатия глюонов в области большой константы связи.

В связи с дальнейшим рассмотрением перепутанных состояний особый интерес представляет случай двухмодового сжатия, условием которого является выполнение неравенства (28) при $p = 2$. Выводы, сделанные для многомодового сжатия, естественно будут справедливыми и для аналогичного двухмодового эффекта. В частности, для коллинеарных глюонов из выражения (29) следует неравенство

$$\begin{aligned}\left\langle \hat{N}\left((\widehat{\Delta X}^{h,g}_{\lambda})_{\frac{1}{2}}\right)^2\right\rangle &= \pm\frac{g^2\,t}{16\,(2\pi)^3\,k_0^2}(f_{ahb}f_{ahc} + f_{agb}f_{agc} + f_{ahb}f_{agc} + f_{agb}f_{ahc}) \times \\ &\times \sum_{\lambda_1\neq\lambda} |\alpha^b_{\lambda_1}||\alpha^c_{\lambda_1}|\sin(\gamma^b_{\lambda_1} + \gamma^c_{\lambda_1}) < 0, \end{aligned} \tag{30}$$

из которого следует, что эффект двухмодового сжатия всегда будет присутствовать за исключением случая, когда $\gamma^b_{\lambda_1} + \gamma^c_{\lambda_1} = 0, \pi$.

С условием двухмодового (многомодового) сжатия неразрывно связано условие перепутанности. Действительно, мерой перепутанности может выступать безразмерная величина y [12], значение которой должно лежать в интервале от 0 до 1: $0 \leq y < 1$ (перепутывание отсутствует при $y = 0$). В случае виртуальных глюонов мера перепутывания по двум цветовым зарядам запишется в виде:

$$y \simeq 2\sqrt{2}\left|\left\langle \hat{N}\left((\widehat{\Delta X}^{h,g}_{\lambda})_{\frac{1}{2}}\right)^2\right\rangle\right|, \tag{31}$$

где $\left\langle \hat{N}\left((\widehat{\Delta X}^{h,g}_{\lambda})_{\frac{1}{2}}\right)^2\right\rangle$ определяется согласно (28) при $p = 2$. Очевидно, что конкретные ограничения на амплитуды и фазы начальных глюонных полей зависят от эксперимента.

Используя (30) и (31), можно получить независимое от типа эксперимента ограничение на меру перепутанности коллинеарных виртуальных глюонов

$$\begin{aligned} 0 < \Bigg| &\frac{g^2\,t}{4\sqrt{2}(2\pi)^3\,k_0^2}(f_{ahb}f_{ahc} + f_{agb}f_{agc} + f_{ahb}f_{agc} + f_{agb}f_{ahc})\times \\ &\times \sum_{\lambda_1\neq\lambda} |\alpha^b_{\lambda_1}||\alpha^c_{\lambda_1}|\sin(\gamma^b_{\lambda_1} + \gamma^c_{\lambda_1})\Bigg| < 1. \end{aligned} \tag{32}$$

Очевидно, что полученное условие перепутанности более строгое, чем условие сжатия, и накладывает дополнительные ограничения на амплитуды начальных когерентных полей: их сумма должна нивелироваться суммой структурных констант цветовой группы.

Одним из критериев существования сжатых состояний глюонов может служить поведение корреляционной функции. Нормированную корреляционную функцию второго порядка

принято определять следующим образом [13]:

$$K_{(2)}(k_1,k_2) \;=\; \frac{C_{(2)}(k_1,k_2)}{\rho_1(k_1)\rho_1(k_2)}, \tag{33}$$

где $C_{(2)}(k_1,k_2)=\rho_2(k_1,k_2)-\rho_1(k_1)\rho_1(k_2)$, $\rho_2(k_1,k_2)\,(\rho_1(k)\,)$ — двухчастичное (одночастичное) инклюзивное распределение. Поэтому для глюонов с произвольным цветом h и поляризацией λ мы можем записать

$$K^h_{\lambda(2)}(k_1,k_2) \;=\; \frac{\rho^h_{\lambda(2)}(k_1,k_2)}{\rho^h_{\lambda(1)}(k_1)\rho^h_{\lambda(1)}(k_2)}-1. \tag{34}$$

Выражения для одночастичного и двухчастичного инклюзивных распределений можно выразить через операторы рождения и уничтожения глюонов:

$$\left.\begin{array}{l}\rho_1(k) \;=\; \langle f|\hat b^{h+}_\lambda(k)\hat b^h_\lambda(k)|f\rangle,\\ \rho_2(k_1,k_2) \;=\; \langle f|\hat b^{h+}_\lambda(k_2)\hat b^{h+}_\lambda(k_1)\hat b^h_\lambda(k_1)\hat b^h_\lambda(k_2)|f\rangle.\end{array}\right\} \tag{35}$$

Проводя усреднение по конечному вектору $|\mathrm{f}\rangle$ (23), получаем следующее выражение для нормированной корреляционной функции второго порядка

$$\begin{aligned}K^h_{\lambda(2)}(k_1,k_2) = -i\,t|\alpha^h_\lambda(k_1)|\,|\alpha^h_\lambda(k_2)|\Big\{&e^{-i(\gamma_1+\gamma_2)}\big\langle\alpha\big|\big[\hat b^h_\lambda(k_1),\big[\hat b^h_\lambda(k_2),\hat H_{\text{вз}}(0)\big]\big]\big|\alpha\big\rangle-\\ &-\;e^{i(\gamma_1+\gamma_2)}\big\langle\alpha\big|\big[\big[\hat H_{\text{вз}}(0),\hat b^{h+}_\lambda(k_2)\big],\hat b^{h+}_\lambda(k_1)\big]\big|\alpha\big\rangle\Big\}\Big/\Big(\rho^h_{\lambda(1)}(k_1)\rho^h_{\lambda(1)}(k_2)\Big),\end{aligned} \tag{36}$$

где

$$\begin{aligned}&\rho^h_{\lambda(1)}(k_1)\rho^h_{\lambda(1)}(k_2)=|\alpha^h_\lambda(k_1)|^2\,|\alpha^h_\lambda(k_2)|^2-i\,t\Big\{|\alpha^h_\lambda(k_1)|\,|\alpha^h_\lambda(k_2)|^2\times\\ &\times\Big[e^{-i\gamma_1}\big\langle\alpha\big|\big[\hat b^h_\lambda(k_1),\hat H_{\text{вз}}(0)\big]\big|\alpha\big\rangle-e^{i\gamma_1}\big\langle\alpha\big|\big[\hat H_{\text{вз}}(0),\hat b^{h+}_\lambda(k_1)\big]\big|\alpha\big\rangle\Big]+\\ &+|\alpha^h_\lambda(k_2)|\,|\alpha^h_\lambda(k_1)|^2\Big[e^{-i\gamma_2}\big\langle\alpha\big|\big[\hat b^h_\lambda(k_2),\hat H_{\text{вз}}(0)\big]\big|\alpha\big\rangle-e^{i\gamma_2}\big\langle\alpha\big|\big[\hat H_{\text{вз}}(0),\hat b^{h+}_\lambda(k_2)\big]\big|\alpha\big\rangle\Big]\Big\},\end{aligned} \tag{37}$$

$|\alpha^h_\lambda(k_1)|,|\alpha^h_\lambda(k_2)|$ и γ_1,γ_2 – амплитуды и фазы исследуемых глюонных полей.

Используя полученные ранее гамильтонианы глюонного самодействия (24), (25) можно показать, что вклад в исследуемую корреляционную функцию, как и в эффект сжатия, дает только четырехглюонное самодействия в силу (27).

Из полученного выражения для $K^h_{\lambda(2)}(k_1,k_2)$ (36) следует, что нормированная корреляционная функция может принимать как отрицательные значения (эффект антигруппировки цветных частиц), так и положительные значения (группировка цветных частиц) в зависимости от фаз исследуемых калибровочных полей глюонов.

4 Заключение

Приведена волновая функция сжатых состояний для обобщенной одномерной квантовой системы. Получены условия возникновения состояний данной системы, минимизирующих соотношение неопределенностей как для дисперсий операторов $\hat x$ и $\hat p$, так и для обобщенных операторов безразмерных величин координаты и импульса.

В данной работе получен явный вид условия существования многомодовых сжатых состояний виртуальных глюонов. Аналитически установлено сосуществования эффектов двухмодового сжатия и перепутанности для виртуальных глюонов с учетом их поляризации. Показано, что условие перепутанности накладывает более сильные ограничения на начальные амплитуды и фазы глюонных полей.

Путем исследования нормированной корреляционной функции второго порядка показано наличие эффектов группировки и антигруппировки цветных частиц в зависимости от фаз калибровочных полей глюонов.

Список литературы

[1] Килин, С.Я. Квантовая оптика: поля и их детектирование / С.Я. Килин. – Минск: Навука і тэхніка, 1990. – 176 с. – Berlin: Springer-Verlag, 1991. – 570 p.

[2] Walls, D.F. Quantum optics / D.F. Walls, G.J. Milburn. – Berlin: Springer-Verlag, 1995. – 351 p.

[3] Мандель, Л. Оптическая когерентность и квантовая оптика / Л. Мандель, Э. Вольф. : пер. с англ.; под ред. В.В. Самарцева. – М.: Физматлит, 2000. – 896 с.

[4] Caves, C.M. Qiantum - mechanical noise in an interterometer / C.M. Caves // Phys. Rev. D. – 1981. – Vol. 23, № 8. – P. 1693–1708.

[5] Schleich, W. Oscillations in photon distribution of squeezed states / T.A. Whecler, W. Schleich // T. Opt. Soc. Am. B. – 1987. – Vol.4, № 10. – P. 1715–1722.

[6] Vourdas, A. Photon -countig distribution of squeezed states / A. Vourdas, R.M. Weiner // Phys. Rev. A. – 1987. – Vol.36, № 12. – P. 5866–5869.

[7] V.I. Kuvshinov, V.A. Shaporov. Gluon Squeezed States in QCD Jet. Acta Physica Polonica B, **30**, no. 1, 59-68, (1999).

[8] V.I. Kuvshinov, V.A. Shaparau. Fluctuations and Correlations of Soft Gluons at the Nonperturbative Stage of Evolution of QCD Jets. Physics of Atomic Nuclei, **65**, no.2, 309-314 (2002).

[9] V.I. Kuvshinov, V.A. Shaparau. Two-Mode Squeezed and Entangled Gluon States. Acta Physica Polonica B, **35**, no. 1, 443-446 (2004).

[10] Kilin, S.Ya. Quanta and information / S.Ya. Kilin // Progress in Optics. – 2001. – Vol. 42. – P. 1–91.

[11] Dodonov, V.V. Generalized uncertainty relation and correlated coherent states / V.V. Dodonov, E.V. Kurmyshev, V.I. Man'ko // Phys. Lett. A. – 1987. – Vol.79, № 2,3. – P. 150–152.

[12] Dodonov V.V. Covariance entanglement measure for two-mode continuous variable system / V.V. Dodonov, A.S.M. de Castro, S.S. Mizrahi // Phys. Lett. A. – 2002 – Vol. 296, №2,3. – P. 73–81.

[13] Wolf, E.A.De. Scaling laws for density correlations and fluctuations in multiparticle dynamics / E.A.De Wolf, I.M. Dremin, W. Kittel // Phys. Rep. – 1996. – Vol.270, № 1, 2. – P. 1–141.

Tensor Fields in Relativistic Quantum Mechanics

Valeriy V. Dvoeglazov
UAF, Universidad de Zacatecas
Apartado Postal 636, Suc. 3, Zacatecas 98061 Zac., México
Web: http://fisica.uaz.edu.mx/˜valeri/
E-mail: valeri@fisica.uaz.edu.mx

Abstract

We re-examine the theory of antisymmetric tensor fields and 4-vector potentials. We discuss corresponding massless limits. We analize the quantum field theory taking into account the mass dimensions of the notoph and the photon. Next, we deduced the gravitational field equations from relativistic quantum mechanics.

In this presentation we re-examine the theory of the 4-potential field, the antisymmetric tensor fields of the second ranks and the spin-2 fields, coming from the modified Bargmann-Wigner formalism. In the series of the papers [1, 2, 3, 4] we tried to find connection between the theory of the quantized antisymmetric tensor (AST) field of the second rank (and that of the corresponding 4-vector field) with the $2(2s+1)$ Weinberg-Tucker-Hammer formalism [5, 6]. Several previously published works [7, 8, 9, 10, 11] introduced the concept of the *notoph* (the Kalb-Ramond field) which is constructed on the basis of the antisymmetric tensor "potentials". It represents itself the non-trivial spin-0 field. See also Refs. [12, 13, 14]. We try to discuss these problems on the basis of the generalized Bargmann-Wigner formalism [15, 16]. A field of the rest mass m and the spin $s \geq \frac{1}{2}$ is represented by a completely symmetric multispinor of the rank $2s$.

The spin-0 and spin-1 field particles can be constructed by taking the direct product of 4-spinors [15, 16]. One can choose the following definitions (p. 209 of Ref. [12])

$$\epsilon^\mu(\mathbf{0},+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}\,,\ \epsilon^\mu(\mathbf{0},0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\,,\ \epsilon^\mu(\mathbf{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}\,, \tag{1}$$

and ($\widehat{p}_i = p_i/\mid \mathbf{p} \mid,\ \gamma = E_p/m$), p. 68 of Ref. [12],

$$\epsilon^\mu(\mathbf{p},\sigma) = L^\mu_\nu(\mathbf{p})\epsilon^\nu(\mathbf{0},\sigma)\,, \tag{2}$$

$$L^0_0(\mathbf{p}) = \gamma\,,\ L^i_0(\mathbf{p}) = L^0_i(\mathbf{p}) = \widehat{p}_i\sqrt{\gamma^2-1}\,, \tag{3}$$

$$L^i_k(\mathbf{p}) = \delta_{ik} + (\gamma-1)\widehat{p}_i\widehat{p}_k \tag{4}$$

for the 4-vector potential field:

$$A^\mu(x^\mu) = \sum_{\sigma=0,\pm1}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\left[\epsilon^\mu(\mathbf{p},\sigma)a(\mathbf{p},\sigma)e^{-ip\cdot x} + (\epsilon^\mu(\mathbf{p},\sigma))^c b^\dagger(\mathbf{p},\sigma)e^{+ip\cdot x}\right]\,. \tag{5}$$

After renormalizing the potentials, *e. g.*, $\epsilon^\mu \to u^\mu \equiv m\epsilon^\mu$ we come to the field functions in the momentum representation, which do not diverge in the massless limit. Thus, the change

of the normalization can lead to the removal of divergent behaviour of classical polarization vectors in $m \to 0$. It is straightforward to find $\mathbf{B}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}\mathbf{p} \times \mathbf{u}(\mathbf{p},\sigma)$, $\mathbf{E}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}p_0\mathbf{u}(\mathbf{p},\sigma) - \frac{i}{2m}\mathbf{p}u^0(\mathbf{p},\sigma)$, and the corresponding negative-energy strengths for the $F_{\mu\nu}$ field operator (in general, complex-valued).

We begin with the Lagrangian, including, in general, mass term. Here we use the notation $A_{\mu\nu}$ for the AST due to different "mass dimension" of the fields. The massless ($m = 0$) Lagrangian is connected with the Lagrangians used in the conformal field theories by adding the total derivative:

$$\mathcal{L}_{CFT} = \mathcal{L} + \frac{1}{2}\partial_\mu \left(A_{\nu\alpha}\partial^\nu A^{\mu\alpha} - A^{\mu\alpha}\partial^\nu A_{\nu\alpha}\right) , \tag{6}$$

$$\mathcal{L} = \frac{1}{4}(\partial_\mu A_{\nu\alpha})(\partial^\mu A^{\nu\alpha}) - \frac{1}{2}(\partial_\mu A^{\mu\alpha})(\partial^\nu A_{\nu\alpha}) - \frac{1}{2}(\partial_\mu A_{\nu\alpha})(\partial^\nu A^{\mu\alpha}) + \frac{1}{4}m^2 A_{\mu\nu}A^{\mu\nu} . \tag{7}$$

The Lagrangian leads to the equation of motion in the following form (provided that the appropriate antisymmetrization procedure has been taken into account):

$$\frac{1}{2}(\Box + m^2)A_{\mu\nu} + (\partial_\mu A^{\ \ ,\alpha}_{\alpha\nu} - \partial_\nu A^{\ \ ,\alpha}_{\alpha\mu}) = 0 . \tag{8}$$

Following the variation procedure one can obtain the energy-momentum tensor:

$$\Theta^{\lambda\beta} = \frac{1}{2}\left[(\partial^\lambda A_{\mu\alpha})(\partial^\beta A^{\mu\alpha}) - 2(\partial_\mu A^{\mu\alpha})(\partial^\beta A^\lambda_{\ \alpha}) - 2(\partial^\mu A^{\lambda\alpha})(\partial^\beta A_{\mu\alpha})\right] - \mathcal{L}g^{\lambda\beta} . \tag{9}$$

One can also obtain that for rotations

$$\mathbf{J}^k = \frac{1}{2}\epsilon^{ijk}J^{ij} = \epsilon^{ijk}\int d^3\mathbf{x}\left[A^{0i}(\partial_\mu A^{\mu j}) + \ + A_\mu{}^j(\partial^0 A^{\mu i} + \partial^\mu A^{i0} + \partial^i A^{0\mu})\right] . \tag{10}$$

Now it becomes obvious that the application of the generalized Lorentz conditions (which are the quantum versions of the free-space dual Maxwell's equations) leads in such a formalism to the absence of electromagnetism in conventional sense. The resulting Kalb-Ramond field is longitudinal (helicity $h = 0$). We agree with the previous authors, e. g., Ref. [17], see Eq. (4) therein, about the gauge *non*-invariance of the separation of the angular momentum of the electromagnetic field into the "orbital" and "spin" part (10). We proved again that for the antisymmetric tensor field $\mathbf{J} \sim \int d^3\mathbf{x}\,(\mathbf{E} \times \mathbf{A})$.

For the spin 1 one can also start from the Weinberg-Tucker-Hammer equation:

$$[\gamma_{\alpha\beta}p^\alpha p^\beta - Ap^\alpha p_\alpha + Bm^2]\Psi = 0 , \tag{11}$$

where $p_\mu = -i\partial_\mu$, and $\gamma_{\alpha\beta}$ are the Barut-Muzinich-Williams covariantly-defined 6×6 matrices. One can recover the Proca theory from it:

$$\partial_\alpha\partial^\mu F_{\mu\beta} - \partial_\beta\partial^\mu F_{\mu\alpha} + m^2 F_{\alpha\beta} = 0 , \tag{12}$$

as a particular case.

Bargmann and Wigner claimed explicitly that they constructed $(2s+1)$ states (the Weinberg-Tucker-Hammer theory has essentially $2(2s+1)$ components). Therefore, we now apply

$$\Psi_{\{\alpha\beta\}} = (\gamma^\mu R)_{\alpha\beta}(c_a m A_\mu + c_f F_\mu) + (\sigma^{\mu\nu}R)_{\alpha\rho}(c_A m(\gamma^5)_{\rho\beta}A_{\mu\nu} + c_F I_{\rho\beta}F_{\mu\nu}) . \tag{13}$$

Thus, A_μ, $A_{\mu\nu}$ and F_μ, $F_{\mu\nu}$ have different mass dimension. The $\gamma^\mu R$, $\sigma^{\mu\nu}R$ and $\gamma^5\sigma^{\mu\nu}R$ are the *symmetrical* matrices. After the suitable choice of the dimensionless coefficients c_i, the Lagrangian density for the photon-notoph field can be proposed:

$$\mathcal{L} = \mathcal{L}^{Proca} + \mathcal{L}^{Notoph} = -\frac{1}{8}F_\mu F^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + + \frac{m^2}{2}A_\mu A^\mu + \frac{m^2}{4}A_{\mu\nu}A^{\mu\nu} . \tag{14}$$

The limit $m \to 0$ may be taken for dynamical variables, in the end of calculations only. In Ref. [18] a very interesting model has been proposed. It is based on gauging the Dirac field on using the coordinate-dependent parameters $\alpha_{\mu\nu}(x)$ in

$$\psi(x) \to \psi'(x') = \Omega\psi(x) \ , \ \Omega = \exp\left[\frac{i}{2}\sigma^{\mu\nu}\alpha_{\mu\nu}(x)\right] \ . \tag{15}$$

Thus, the second "photon" was introduced. The compensating 24-component field $B_{\mu,\nu\lambda}$ reduces to the 4-vector field as follows:

$$B_{\mu,\nu\lambda} = \frac{1}{4}\epsilon_{\mu\nu\lambda\sigma}a^{\sigma}(x) \, . \tag{16}$$

As readily seen after comparison of these formulas with those of Refs. [8, 9, 10] , the second photon is nothing more than the Ogievetskiĭ-Polubarinov *notoph* within the normalization.

Marques and Spehler obtained tensor equations [19]:

$$\frac{2}{m}\partial_{\mu}T_{\kappa}^{\mu\nu} = -G_{\kappa}^{\nu} \, , \tag{17}$$

$$\frac{2}{m}\partial_{\mu}R_{\kappa\tau}^{\mu\nu} = -F_{\kappa\tau}^{\nu} \, , \tag{18}$$

$$T_{\kappa}^{\mu\nu} = \frac{1}{2m}\left[\partial^{\mu}G_{\kappa}^{\nu} - \partial^{\nu}G_{\kappa}^{\mu}\right] \, , \tag{19}$$

$$R_{\kappa\tau}^{\mu\nu} = \frac{1}{2m}\left[\partial^{\mu}F_{\kappa\tau}^{\nu} - \partial^{\nu}F_{\kappa\tau}^{\mu}\right] \, . \tag{20}$$

As opposed to them we shall modify the formalism in the spirit of Ref. [14]. The field function is now presented as

$$\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma_{\mu}R)_{\alpha\beta}\Psi_{\gamma\delta}^{\mu} + \alpha_2(\sigma_{\mu\nu}R)_{\alpha\beta}\Psi_{\gamma\delta}^{\mu\nu} + \alpha_3(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}\widetilde{\Psi}_{\gamma\delta}^{\mu\nu} \, , \tag{21}$$

with

$$\Psi_{\{\gamma\delta\}}^{\mu} = \beta_1(\gamma^{\kappa}R)_{\gamma\delta}G_{\kappa}^{\mu} + \beta_2(\sigma^{\kappa\tau}R)_{\gamma\delta}F_{\kappa\tau}^{\mu} + \beta_3(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{F}_{\kappa\tau}^{\mu} \, , \tag{22}$$

$$\Psi_{\{\gamma\delta\}}^{\mu\nu} = \beta_4(\gamma^{\kappa}R)_{\gamma\delta}T_{\kappa}^{\mu\nu} + \beta_5(\sigma^{\kappa\tau}R)_{\gamma\delta}R_{\kappa\tau}^{\mu\nu} + \beta_6(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{R}_{\kappa\tau}^{\mu\nu} \, , \tag{23}$$

$$\widetilde{\Psi}_{\{\gamma\delta\}}^{\mu\nu} = \beta_7(\gamma^{\kappa}R)_{\gamma\delta}\widetilde{T}_{\kappa}^{\mu\nu} + \beta_8(\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{D}_{\kappa\tau}^{\mu\nu} + \beta_9(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}D_{\kappa\tau}^{\mu\nu} \, . \tag{24}$$

Hence, the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ can be expressed as a sum of nine terms. The essential constraints can be found in Ref. [20]. One should contract the field function with six antisymmetric matrices in order to ensure symmetrization. As a discussion, we note that in such a framework we have the physical content because only certain combinations of field functions are equal to zero. Furthermore, from the new system of equations one obtains the *second*-order equation for the symmetric traceless tensor of the second rank ($\alpha_1 \neq 0$, $\beta_1 \neq 0$):

$$\frac{1}{m^2}\left[\partial_{\nu}\partial^{\mu}G_{\kappa}^{\nu} - \partial_{\nu}\partial^{\nu}G_{\kappa}^{\mu}\right] = G_{\kappa}^{\mu} \, . \tag{25}$$

After the contraction in indices κ and μ this equation is reduced to:

$$\partial_{\mu}G_{\kappa}^{\mu} = F_{\kappa} \, , \tag{26}$$

$$\frac{1}{m^2}\partial_{\kappa}F^{\kappa} = 0 \, , \tag{27}$$

i. e., to the equations which connect the analogue of the energy-momentum tensor and the analogue of the 4-vector field.

The most general relativistic-invariant Lagrangian for the symmetric 2nd-rank tensor is

$$\mathcal{L} = -\alpha_1(\partial^\alpha G_{\alpha\lambda})(\partial_\beta G^{\beta\lambda}) - \alpha_2(\partial_\alpha G^{\beta\lambda})(\partial^\alpha G_{\beta\lambda}) - \alpha_3(\partial^\alpha G^{\beta\lambda})(\partial_\beta G_{\alpha\lambda}) + m^2 G_{\alpha\beta}G^{\alpha\beta}\,. \quad (28)$$

in the Minkowski space. It leads to the equation

$$\Big[\alpha_2(\partial_\alpha\partial^\alpha) + m^2\Big]\, G^{\{\mu\nu\}} + (\alpha_1+\alpha_3)\partial^{\{\mu|}\,(\partial_\alpha G^{\,\alpha|\nu\}}) = 0\,. \quad (29)$$

In the case $\alpha_2 = 1 > 0$ and $\alpha_1 + \alpha_3 = -1$ it coincides with Eq. (25). The mass dimension of $G^{\mu\nu}$ is $[energy]^1$. We now present possible relativistic interactions of the symemtric 2nd-rank tensor. The simplest ones should be the following ones: $\mathcal{L}^{int}_{(1)} \sim G_{\mu\nu}F^\mu F^\nu$, $\mathcal{L}^{int}_{(2)} \sim (\partial^\mu G_{\mu\nu})F^\nu$, $\mathcal{L}^{int}_{(3)} \sim G_{\mu\nu}(\partial^\mu F^\nu)$. On using

$$V(F) = \lambda_1(F_\mu F^\mu) + \lambda_2(F_\mu F^\mu)(F_\nu F^\nu) \quad (30)$$

we can give the mass to the G_{00} component of the spin-2 field. This is due to the possibility of the analogue of the Higgs spontaneous symmetry breaking, $v^2 = (F_\mu F^\mu) = -\lambda_1/2\lambda_2 > 0$.

Next, since the interaction of fermions with notoph are that of the order $\sim e^2$ since the beginning (as opposed to the fermion current interaction with the 4-vector potential A_μ, which is proportional to e) in the Lagrangian, it is more difficult to observe the notoph field.

We considered the Bargmann-Wigner formalism in order to derive the equations for the AST fields, and for the symmetric tensor of the 2nd rank. This problem is connected with the problem of the observability of the gauge. We introduced the additional symmetric matrix in the Bargmann-Wigner expansion, namely $(\gamma^5\sigma^{\mu\nu}R)$, in order to account for the dual fields. The consideration was similar to Ref. [21]. The interaction may give the mass to spin-2 particles in the way which is reminiscent to the spontaneous-symmetry-breaking Higgs formalism. I acknowledge discussions with participants of recent conferences on Symmetries and Gravitation.

References

[1] D. V. Ahluwalia and D. J. Ernst, Int. J. Mod. Phys. E**2** (1993) 397.

[2] D. V. Ahluwalia and M. Sawicki, Phys. Rev. D**47** (1993) 5161.

[3] V. V. Dvoeglazov, Int. J. Theor. Phys. **37** (1998), 1915; hep-th/9611068.

[4] V. V. Dvoeglazov, Hadronic J. Suppl. **12** (1997) 221.

[5] S. Weinberg, Phys. Rev. B**133** (1964) 1318; ibid. B**134** (1964) 882; ibid. **181** (1969) 1893.

[6] R. H. Tucker and C. L. Hammer, Phys. Rev. D**3** (1971) 2448.

[7] E. Durandin and A. Erschow, Phys. Z. Sowjet. **12** (1937) 466; F. Belinfante, Physica **6** (1939) 887.

[8] V. I. Ogievetskiĭ and I. V. Polubarinov, Yadern. Fiz. **4** (1966) 216 [English translation: Sov. J. Nucl. Phys. **4** (1967) 156].

[9] K. Hayashi, Phys. Lett. B**44** (1973) 497.

[10] M. Kalb and P. Ramond, Phys. Rev. D**9** (1974) 2273.

[11] L. V. Avdeev and M. V. Chizhov, Phys. Lett. B**321** (1994) 212; hep-th/9407067; M. V. Chizhov, Phys. Part. Nucl. **42** (2011) 93.

[12] S. Weinberg, *The Quantum Theory of Fields. Vol. I. Foundations.* (Cambridge University Press, 1995), Ch. 5.

[13] V. V. Dvoeglazov, Czech. J. Phys. **50** (2000) 225; , hep-th/9712036.

[14] V. V. Dvoeglazov, Phys. Scripta **64** (2001) 201, physics/9804010.

[15] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (USA) **34** (1948) 211.

[16] D. Luriè, *Particles and Fields* (Interscience Publishers, 1968). Ch. 1.

[17] H. C. Ohanian, Am. J. Phys. **54** (1986) 500.

[18] P. C. Naik and T. Pradhan, J. Phys. A**14** (1981) 2795; T. Pradhan, R. P. Malik and P. C. Naik, Pramana J. Phys. **24** (1985) 77.

[19] G. Marques and D. Spehler, Mod. Phys. Lett. A**13** (1998) 553.

[20] V. V. Dvoeglazov, Adv. Appl. Clifford Algebras **10** (2000) 7, math-ph/9805017.

[21] V. V. Dvoeglazov, Int. J. Mod. Phys. CS **3** (2011) 121.

Об интерпретации электрослабого поля в теории релятивистских волновых уравнений

В.А. Плетюхов
erphys@brsu.brest.by; Брестский государственный университет имени А.С.Пушкина

Аннотация

Построено релятивистское волновое уравнение, описывающее векторное поле с четырьмя типами квантов – одним безмассовым и тремя массивными. При этом в теории неизбежно возникает скалярная частица с ненулевой массой. Скалярное и векторное поля описываются совместно в рамках одной не распадающейся по группе Лоренца системы уравнений и, следовательно, представляют собой единый физический объект. Его векторная составляющая может интерпретироваться как "электрослабое" поле, а скалярная – как аналог бозона Хиггса. В предлагаемой модели масса скалярного бозона совпадает с массой одного из промежуточных векторных бозонов (Z^0). Ненулевые массы всех частиц выступают в качестве произвольных параметров теории.

1 Введение

В предыдущих работах (см., напр., [1, 2]) показано, что при использовании расширенного набора неприводимых представлений группы Лоренца теория релятивистских волновых уравнений (РВУ) позволяет описывать совместно микрообъекты с ненулевой и нулевой массой на основе не распадающихся в релятивистски-инвариантном смысле уравнений.

Возьмём, например, схему зацеплений

$$\begin{array}{ccc} & (0,1) & \\ \swarrow & & \searrow \\ (\frac{1}{2},\frac{1}{2}) & & (\frac{1}{2},\frac{1}{2})' \\ \searrow & & \swarrow \\ & (1.0) & \end{array} \qquad (1)$$

где представления

$$(1/2,1/2),...[(0,1)\oplus(1,0)],(1/2,1/2)'$$

сопоставляются вектору ψ_μ, антисимметричным тензорам второго $\psi_{[\mu\nu]}$ и третьего $\psi_{[\mu\nu\alpha]}$ рангов соответственно. Рассмотрим систему линейных однородных тензорных уравнений первого порядка

$$\partial_\nu \psi_{[\mu\,\nu]} = 0\,, \qquad (2)$$

$$\partial_\mu \psi_{[\nu\,\alpha]} + \partial_\alpha \psi_{[\mu\,\nu]} + \partial_\nu \psi_{[\alpha\,\mu]} + m\,\psi_{[\mu\,\nu\,\alpha]} = 0\,, \qquad (3)$$

$$-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + \partial_\alpha \psi_{[\mu\,\nu\,\alpha\,]} + m\,\psi_{[\mu\,\nu\,]} = 0\,, \qquad (4)$$

базирующуюся на наборе представлений (1). Путём несложных преобразований из уравнений (2)–(4) можно получить уравнения второго порядка

$$\psi_\mu - \partial_\mu \partial_\nu \psi_\nu = 0\,, \tag{5}$$

$$\left(-m^2\right) \psi_{[\mu\,\nu\,\alpha]} = 0\,, \quad \varepsilon_{\mu\,\nu\,\alpha\,\beta}\, \psi_{[\mu\,\nu\,\alpha]} = 0\,, \tag{6}$$

означающие, что система (2)–(4) описывает свободное векторное поле, квантами которого являются безмассовая частица (ψ_μ) и частица с ненулевой массой m $\left(\psi_{[\mu\,\nu\,\alpha]}\right)$. При этом, поскольку система (2)–(4) не распадается по группе Лоренца, указанное поле с точки зрения теории РВУ представляет собой единый физический объект.

Для дальнейшего целесообразно рассмотреть также основанную на схеме зацеплений (1) систему первого порядка

$$\alpha\, \partial_\nu\, \psi_{[\mu\,\nu]} + m\, \psi_\mu = 0\,, \tag{7}$$

$$\beta\, \left(\partial_\mu \psi_{[\nu\,\alpha]} + \partial_\alpha \psi_{[\mu\,\nu]} + \partial_\nu \psi_{[\alpha\,\mu]}\right) + m\, \psi_{[\mu\,\nu\,\alpha]} = 0\,, \tag{8}$$

$$\alpha^* \left(-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu\right) + \beta^* \partial_\alpha \psi_{[\mu\,\nu\,\alpha]} + m\, \psi_{[\mu\,\nu]} = 0\,, \tag{9}$$

где α, β – произвольные комплексные параметры. Переходя опять от системы (7)–(9) к эквивалентным ей уравнениям второго порядка, будем иметь

$$\left(\, - \frac{m^2}{|\alpha|^2}\right) \psi_\mu = 0\,, \qquad \partial_\mu \psi_\mu = 0\,; \tag{10}$$

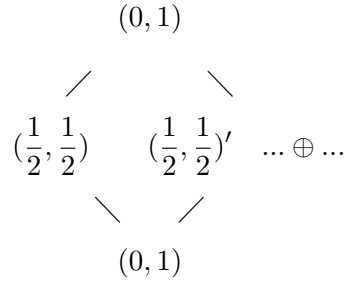
$$\left(\, - \frac{m^2}{|\beta|^2}\right) \psi_{[\mu\,\nu\,\alpha]} = 0\,, \qquad \varepsilon_{\mu\,\nu\,\alpha\,\beta}\, \partial_\beta \psi_{[\mu\,\nu\,\alpha]} = 0\,. \tag{11}$$

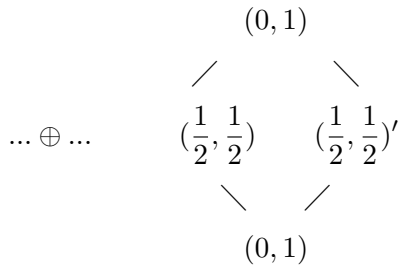
Уравнения (10), (11) показывают, что система (7)–(9) даёт совместное описание частиц (полей) со спином 1 и ненулевыми массами

$$m_1 \,=\, \frac{m}{|\alpha|}\,, \qquad m_2 \,=\, \frac{m}{|\beta|}\,. \tag{12}$$

2 Тензорная формулировка массивно-безмассового векторного поля

Как известно, в Стандартной модели три типа квантов “электрослабого” поля – промежуточные векторные бозоны – обладают ненулевыми массами и один (фотон) – нулевой. Очевидно, что при совместном рассмотрении уравнений (2)–(4) и (7)–(9) мы как раз получим систему первого порядка, описывающую массивно-безмассовое векторное поле с тремя массивными и одним безмассовым квантами. Однако при таком механическом объединении это поле не будет представлять единый физический объект, каковым является “электрослабое” поле, поскольку соответствующая ему схема зацеплений

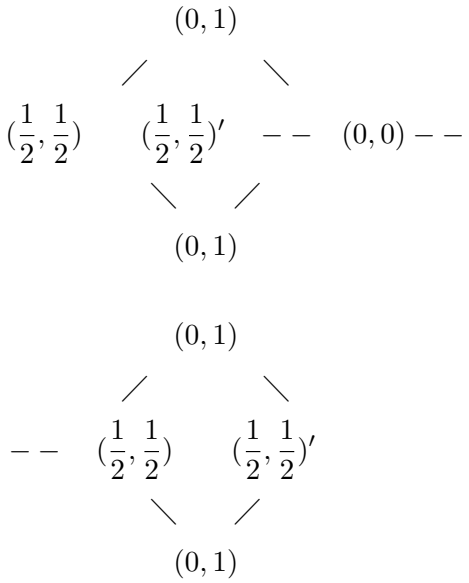




(13)

является распадающейся на два самостоятельных фрагмента вида (1).

Самый естественный и простой способ получения нераспадающейся схемы зацеплений, приводящей к описанию того же набора векторных частиц, заключается во введении в рассмотрение дополнительного скалярного представления (0, 0), обеспечивающего зацепление указанных фрагментов. В результате получается схема зацеплений



(14)

Соответствующая (14) система уравнений первого порядка имеет общий вид

$$\partial_\nu \, \psi_{[\mu\,\nu]} = 0 \,, \tag{15}$$

$$\alpha \, \left(\partial_\mu \psi_{[\nu\,\alpha]} + \partial_\alpha \psi_{[\mu\,\nu]} + \partial_\nu \psi_{[\alpha\,\mu]} \right) + \beta \, \varepsilon_{\mu\,\nu\,\alpha\,\beta} \, \partial_\beta \psi_0 + m \, \psi_{[\mu\,\nu\,\alpha]} = 0 \,, \tag{16}$$

$$-\partial_\mu \, \psi_\nu + \partial_\nu \, \psi_\mu + \alpha^* \partial_\alpha \, \psi_{[\mu\,\nu\,\alpha]} + m \, \psi_{[\mu\,\nu]} = 0 \,, \tag{17}$$

$$\beta^* \varepsilon_{\mu\,\nu\,\alpha\,\beta} \, \partial_\beta \, \psi_{[\mu\,\nu\,\alpha]} + \gamma \, \partial_\mu \varphi_\mu + m \, \psi_0 = 0 \,, \tag{18}$$

$$\rho \, \partial_\nu \, \varphi_{[\mu\,\nu]} + \gamma^* \, \partial_\mu \, \psi_0 + m \, \varphi_\mu = 0 \,, \tag{19}$$

$$\delta \, \left(\partial_\mu \varphi_{[\nu\,\alpha]} + \partial_\alpha \, \varphi_{[\mu\,\nu]} + \partial_\nu \varphi_{[\alpha\,\mu]} \right) + m \, \varphi_{[\mu\,\nu\,\alpha]} = 0 \,, \tag{20}$$

$$\rho^* \, \left(-\partial_\mu \, \varphi_\nu + \partial_\nu \, \varphi_\mu \right) + \delta^* \, \partial_\alpha \, \varphi_{[\mu\,\nu\,\alpha]} + m \, \varphi_{[\mu\,\nu]} = 0 \,, \tag{21}$$

где скаляр ψ_0 сопоставляется представлению (0, 0).

Найдём уравнения второго порядка, к которым приводит система (15)–(21). Подействуем на уравнение (17) оператором ∂_ν. Учитывая (1), получим

$$\psi_\mu - \partial_\mu \partial_\nu \psi_\nu = 0\,. \tag{22}$$

Применяя к уравнению (16) оператор $\varepsilon_{\mu\nu\alpha\gamma}\partial_\gamma$, имеем

$$\beta\,\varepsilon_{\mu\nu\alpha\gamma}\varepsilon_{\mu\nu\alpha\beta}\partial_\gamma\partial_\beta\psi_0 + m\,\varepsilon_{\mu\nu\alpha\gamma}\partial_\gamma\psi_{[\mu\nu\alpha]} = 0\,,$$

$$-6\,\beta\,\delta_{\gamma\beta}\partial_\gamma\partial_\beta\psi_0 + m\,\varepsilon_{\mu\nu\alpha\gamma}\partial_\gamma\psi_{[\mu\nu\alpha]} = 0$$

$$\varepsilon_{\mu\nu\alpha\gamma}\partial_\gamma\psi_{[\mu\nu\alpha]} = \frac{6\beta}{m}\,\psi_0\,. \tag{23}$$

Подставляем (23) в (18):

$$\frac{6\,|\beta|}{m}\;\psi_0 + \gamma\,\partial_\mu\varphi_\mu + m\,\psi_0 = 0\,. \tag{24}$$

Действуя на уравнение (19) оператором ∂_μ, находим:

$$\gamma^*\,\psi_0 + m\;\partial_\mu\;\varphi_\mu = 0$$

Сравнивая (24) и (25), получаем уравнение

$$\left(|\gamma|^2 - 6\,|\beta|^2\right)\;\psi_0 + m^2\,\psi_0 = 0,$$

или

$$\psi_0 - \frac{m^2}{|\gamma|^2 - 6\,|\beta|^2}\psi_0 = 0. \tag{25}$$

На параметры γ и β накладывается условие

$$|\gamma|^2 - 6\,|\beta|^2 > 0 \tag{26}$$

Комбинируя уравнения (20) и (21), нетрудно получить уравнение

$$|\delta|^2\left(\partial_\mu\partial_\gamma\,\varphi_{[\nu\alpha\gamma]} + \partial_\alpha\,\partial_\gamma\varphi_{[\mu\nu\gamma]} + \partial_\nu\,\partial_\gamma\,\varphi_{[\alpha\mu\gamma]}\right) - m^2\,\varphi_{[\mu\nu\alpha]} = 0\;. \tag{27}$$

Кроме того, из уравнения (20) вытекает соотношение

$$\partial_\mu\,\partial_\gamma\,\varphi_{[\nu\alpha\gamma]} + \partial_\alpha\,\partial_\gamma\,\varphi_{[\mu\nu\gamma]} + \partial_\nu\,\partial_\gamma\,\varphi_{[\alpha\mu\gamma]} = \;\;\varphi_{[\mu\nu\alpha]}, \tag{28}$$

с учетом которого уравнение (27) примет вид

$$\varphi_{[\mu\nu\alpha]} - \frac{m^2}{|\delta|^2}\;\varphi_{[\mu\nu\alpha]} = 0. \tag{29}$$

Теперь применим оператор ∂_ν к уравнению (21). Это даёт

$$\rho^*\,(-\,\partial_\mu\,\partial_\nu\,\varphi_\nu + \;\;\varphi_\mu) + m\,\partial_\nu\,\varphi_{[\mu\nu]} = 0. \tag{30}$$

Выразим из (19) член $\partial_\nu\varphi_{[\mu\nu]}$ и подставим в (30):

$$\varphi_\mu - \frac{m^2}{|\rho|^2\,\varphi_\mu} = \frac{\gamma^* m}{|\rho|^2}\;\partial_\mu\left(\psi_0 + \frac{m}{\gamma^*\,|\rho|^2}\,\varphi_\mu\right). \tag{31}$$

В свою очередь, из уравнения (19) можно получить соотношение

$$\partial_\nu \varphi_\nu = -\frac{\gamma^*}{m}\ \psi_0. \tag{32}$$

Подставляя (32) в (31), находим

$$\varphi_\mu - \frac{m^2}{|\rho|^2}\,\varphi_\mu = \frac{\gamma^*}{m\,|\rho|^2}\,\partial_\mu\left(-\,\psi_0 + m^2\,\psi_0\right). \tag{33}$$

Конкретизируем условие (26), положив

$$|\gamma|^2 - 6\,|\beta|^2 = 1. \tag{34}$$

В результате уравнения (25), (33) принимают соответственно вид

$$\psi_0 - m^2\,\psi_0 = 0, \tag{35}$$

$$\psi_\mu - \frac{m^2}{|\rho|^2}\,\varphi_\mu = 0. \tag{36}$$

С целью получения уравнения второго порядка для тензора $\psi_{[\mu\nu\alpha]}$ возьмём производную ∂_α от уравнения (16):

$$\alpha\,\left(\partial_\alpha\,\partial_\mu\,\psi_{[\nu\,\alpha]} + \ \psi_{[\mu\,\nu]} + \partial_\alpha\,\partial_\nu\,\psi_{[\alpha\,\mu]}\right) + m\,\partial_\alpha\,\psi_{[\mu\,\nu\,\alpha]} = 0\,. \tag{37}$$

Отсюда с учётом (15) находим

$$\partial_\alpha\,\psi_{[\mu\,\nu\,\alpha]} = -\frac{\alpha}{m}\ \psi_{[\mu\,\nu]}. \tag{38}$$

Подставляя (38) в (17), будем иметь

$$-\partial_\mu\,\psi_\nu + \partial_\nu\,\psi_\mu - \frac{|\alpha|^2}{m}\ \psi_{[\mu\,\nu]} + m\,\psi_{[\mu\,\nu]} = 0, \tag{39}$$

откуда

$$\psi_{[\mu\,\nu]} - \frac{m^2}{|\alpha|^2}\ \psi_{[\mu\,\nu]} + \frac{m}{|\alpha|^2}\,(\partial_\mu\,\psi_\nu - \partial_\nu\,\psi_\mu) = 0. \tag{40}$$

Из (16) вытекает очевидное соотношение

$$\alpha\,\left(\partial_\mu\,\psi_{[\nu\,\alpha]} + \partial_\alpha\,\psi_{[\mu\,\nu]} + \partial_\nu\,\psi_{[\alpha\,\mu]}\right) + \beta\,\varepsilon_{\mu\,\nu\,\alpha\,\beta}\,\partial_\beta\ \psi_0 + m\ \psi_{[\mu\,\nu\,\alpha]} = 0\,. \tag{41}$$

Выражая из (40) член $\psi_{[\mu\,\nu]}\left(\psi_{[\nu\,\alpha]}\,,\ \ \psi_{[\alpha\,\mu]}\right)$ и подставляя в (41), придём к уравнению

$$\psi_{[\mu\,\nu\,\alpha]} - \frac{m^2}{|\alpha|^2}\,\psi_{[\mu\,\nu\,\alpha]} + \beta m\left(1 - \frac{1}{|\alpha|^2}\right)\varepsilon_{\mu\,\nu\,\alpha\,\beta}\partial_\beta\,\psi_0 = 0. \tag{42}$$

Выбирая в (42)

$$|\alpha|^2 = 1, \tag{43}$$

получаем окончательно

$$\psi_{[\mu\,\nu\,\alpha]} - m^2\,\psi_{[\mu\,\nu\,\alpha]} = 0. \tag{44}$$

Уравнение (35) описывает скалярный бозон с массой m. Уравнения (22), (29), (36) и (44) показывают, что в системе (15)–(21) содержится также описание четырёх векторных частиц, одна из которых имеет нулевую массу, а три других – массы $\frac{m}{|\delta|}\,,\quad \frac{m}{|\rho|}$ и m соответственно.

3 Матричная интерпретация теории

Тензорная система (15)–(21) может быть записана в стандартной матричной форме

$$(\Gamma_\mu \partial_\mu + \Gamma_0)\ \psi = 0\,, \tag{45}$$

где проективная матрица Γ_0 в базисе

$$\psi = \left(\psi_\mu,\ \psi_{[\mu\,\nu\,\alpha]},\ \psi_{[\mu\,\nu]},\ \psi_0,\ \varphi_\mu,\ \varphi_{[\mu\,\nu\,\alpha]},\ \varphi_{[\mu\,\nu]}\right)$$

имеет вид

$$\Gamma_0 = \begin{pmatrix} 0_4 & \\ & m\,\mathrm{I}_{25} \end{pmatrix}\ , \tag{46}$$

где 0_4– нулевой блок размерности 4×4, I_{25} – единичная матрица 25×25.

Как известно (см., напр., [3]), основную роль в РВУ (45) наряду с матрицей Γ_0 играет матрица Γ_4; напомним, что матрицы $\Gamma_i\,(i=1,\,2,\,3)$ выражаются через Γ_4 и "бусты" I^{i4} лоренцевских преобразований в пространстве представления волновой функции ψ.

Для установления спектра возможных значений массы и спиновых характеристик микрообъекта, описываемого уравнением (45) со схемой зацеплений (14), удобно использовать так называемый канонический базис (базис Гельфанда–Яглома [3]). В этом базисе компоненты волновой функции $\psi = \psi^\tau_{s\,j}$ описывают состояния с определённым значением спина s и проекции спина j; индекс τ указывает на принадлежность данного состояния к неприводимому представлению τ. Матрица Γ_4 имеет вид прямой суммы

$$\Gamma_4 = \oplus \sum_s C^s \otimes \mathrm{I}_{2s+1}\ , \tag{47}$$

где C^s – спиновый блок, соответствующий спину s в том смысле, что если у матрицы C^s имеются ненулевые корни (собственные значения), то частица обладает спином s. Спиновый блок $C^s = c^s_{\tau\,\tau'}$ формируется неприводимыми представлениями $\tau \sim (l_1,\,l_2)$, $\tau' \sim (l'_1,\ l'_2)$, удовлетворяющими условию

$$|l_1 - l_2| \le s \le l_1 + l_2\,, \quad \left|l'_1 - l'_2\right| \le s \le l'_1 + l'_2\ . \tag{48}$$

Ненулевые значения массы $m^{(s)}_k$ микрообъекта, описываемого РВУ (45), вычисляются по формуле

$$m^{(s)}_k = \frac{m}{\left|\lambda^{(s)}_k\right|}\ , \tag{49}$$

где $\lambda^{(s)}_k$ – ненулевые корни матрицы Γ_4 (блока C^s), отвечающие единичной клетке в структуре (46) матрицы Γ_0. Корни матрицы Γ_4, которые "обрезаются" проективной матрицей Γ_0, соответствуют нулевой массе (как, например, в десятимерной формулировке уравнений Максвелла).

Введём для упрощения дальнейших обозначений следующую нумерацию представлений, содержащихся в схеме (14):

$$\begin{aligned}
&(0,\,0) \sim 1\ \ (\psi_0)\,; \quad \left(\tfrac12,\,\tfrac12\right)' \sim 2\ \ \left(\psi_{[\mu\,\nu\,\alpha]}\right); \quad \left(\tfrac12,\,\tfrac12\right) \sim 3\ \ (\psi_\mu)\ ;\\
&(0,\,1)\,,\,(1,\,0)\ \sim 4,\,5\ \ \left(\psi_{[\mu\,\nu]}\right)\ ; \quad \left(\tfrac12,\,\tfrac12\right) \sim 6\ \ \left(\varphi_{[\mu\,\nu\,\alpha]}\right)\ ;\\
&\left(\tfrac12,\,\tfrac12\right) \sim 7\ \ (\varphi_\mu)\ ; \quad (0,\,1)\,,\,(1,\,0)\ \sim 8,\,9\ \ \left(\varphi_{[\mu\,\nu]}\right)\ .
\end{aligned} \tag{50}$$

Представления (50) участвуют в формировании только спинов $s = 0,\ 1$. Поэтому матрица Γ_4 РВУ (45), эквивалентного тензорной системе (15)–(21), в базисе Гельфанда–Яглома будет иметь вид

$$\Gamma_4 = C^0 \oplus \left(C^1 \otimes \mathrm{I}_3\right) \ . \tag{51}$$

Для блоков $C^0,\ C^1$ с учётом условия (48), схемы зацеплений (14) и нумерации (50) получаем выражения

$$C^0 = \begin{pmatrix} 0 & c^0_{1\,2} & 0 & 0 & c^0_{1\,7} \\ c^0_{2\,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c^0_{7\,1} & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$C^1 = \begin{pmatrix} 0 & 0 & c^1_{2\,4} & c^1_{2\,5} & 0 & 0 & 0 & 0 \\ 0 & 0 & c^1_{3\,4} & c^1_{3\,5} & 0 & 0 & 0 & 0 \\ c^1_{4\,2} & c^1_{4\,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ c^1_{5\,2} & c^1_{5\,3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c^1_{6\,8} & c^1_{6\,9} \\ 0 & 0 & 0 & 0 & 0 & 0 & c^1_{7\,8} & c^1_{7\,9} \\ 0 & 0 & 0 & 0 & c^1_{8\,6} & c^1_{8\,7} & 0 & 0 \\ & & & & c^1_{9\,6} & c^1_{9\,7} & 0 & 0 \end{pmatrix} , \tag{52}$$

где ненулевые элементы $c^s_{\tau\,\tau'}$ соответствуют зацепляющимся представлениям в (14).

Инвариантность рассматриваемого РВУ относительно преобразований группы Лоренца никаких дополнительных ограничений на числа $c^s_{\tau\,\tau'}$ не накладывает.

При построении лагранжиана

$$L = -\bar{\psi}\ (\Gamma_\mu \partial_\mu + \Gamma_0)\ \psi, \tag{53}$$

из которого может быть получено уравнение (45), используется лоренц-инвариантная вещественная билинейная форма $\bar{\psi}\,\psi = \psi^+ \eta\,\psi$. В базисе Гельфанда–Яглома матрица η имеет структуру, аналогичную (47):

$$\eta = \oplus \sum_S \eta^S \otimes I_{2S+1}. \tag{54}$$

В блоках η^S отличными от нуля являются лишь элементы $\eta^S_{\tau\dot{\tau}} \quad (\tau \sim (l_1,\, l_2)\ ,\ \dot{\tau} \sim (l_2,\, l_1))$, причём

$$\eta^S_{\tau\dot{\tau}} = \eta^S_{\dot{\tau}\tau} = -\eta^{S+1}_{\tau\dot{\tau}}. \tag{55}$$

Не уменьшая общности, матрицу η можно нормировать так, что в блоках η^S будут встречаться только числа ± 1.

Требование возможности лагранжевой формулировки РВУ (45) приводит к условию [3]

$$c^S_{\tau\tau/}\eta^S_{\tau/\dot{\tau}/} = \left(c^S_{\dot{\tau}/\dot{\tau}}\right)^* \eta^S_{\tau\dot{\tau}}. \tag{56}$$

Применяя условие (56) к элементам матриц C^0, C^1 (52), получаем соотношения

$$c^0_{2\,1} = f\left(c^0_{12}\right)^*, \quad c^0_{71} = g\left(c^0_{17}\right)^*, \quad c^1_{42} = h\left(c^1_{25}\right)^*, \quad c^1_{5\,2} = h\left(c^1_{24}\right)^*,$$

$$c^1_{4\,3} = p\left(c^1_{3\,5}\right)^*, \quad c^1_{5\,3} = p\left(c^1_{3\,4}\right)^*, \quad c^1_{8\,6} = g\left(c^1_{6\,9}\right)^*, \quad c^1_{9\,6} = q\left(c^1_{6\,8}\right)^*,$$

$$c^1_{8\,7} = r\left(c^1_{7\,9}\right)^*, \quad c^1_{9\,7} = r\left(c^1_{7\,8}\right)^*, \tag{57}$$

где для краткости записи введены обозначения

$$\begin{aligned} f &= \eta^0_{2\,2}/\eta^0_{1\,1}\,, \qquad g = \eta^0_{7\,7}/\eta^0_{1\,1}\,, \qquad h = \eta^1_{4\,5}/\eta^1_{2\,2}\,, \\ p &= \eta^1_{4\,5}/\eta^1_{3\,3}\,, \qquad q = \eta^1_{8\,9}/\eta^1_{6\,6}\,, \qquad r = \eta^1_{8\,9}/\eta^1_{7\,7}\ . \end{aligned} \tag{58}$$

Чтобы получить матричное РВУ, эквивалентное тензорной системе (15)–(21), можно выбрать, например:

$$\begin{aligned} &c^0_{1\,2} = c^0_{1\,7} = \tfrac{1}{\sqrt{2}}\,; \quad c^1_{2\,4} = c^1_{2\,5} = c^1_{3\,4} = -c^1_{3\,5} = \tfrac{1}{\sqrt{2}}\ ; \\ &c^1_{6\,8} = c^1_{6\,9} = \sqrt{\tfrac{\delta}{2}}\ , \quad c^1_{7\,8} = -c^1_{7\,9} = \sqrt{\tfrac{\rho}{2}}\ ; \end{aligned} \tag{59}$$

$$\eta^0_{1\,1} = \eta^0_{2\,2} = \eta^0_{7\,7} = -\eta^1_{2\,2} = \eta^1_{3\,3} = \eta^1_{6\,6} = -\eta^1_{7\,7} = -\eta^1_{4\,5} = \eta^1_{8\,9} = 1\ . \tag{60}$$

Заметим, что существуют и другие возможности, но все они приводят к унитарно-эквивалентным теориям.

В результате для спиновых блоков $C^0\,,\ C^1$ (52) получаются выражения

$$C^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}\ , \qquad C^1 = \begin{pmatrix} \big(C^1\big)' & \\ & \big(C^1\big)'' \end{pmatrix}\ , \tag{61}$$

$$\big(C^1\big)' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}\ , \qquad \big(C^1\big)'' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \delta & \delta \\ 0 & 0 & \rho & -\rho \\ \delta^* & \rho^* & 0 & 0 \\ \delta^* & -\rho^* & 0 & 0 \end{pmatrix}\ .$$

Вид блоков $\eta^0\,,\ \eta^1$ матрицы билинейной формы η (52) вытекает из (60).

Блок C^0 имеет корни 0, ±1. Ненулевой корень ±1 соответствует массе m скалярной частицы. Блок $\big(C^1\big)'$ имеет двукратно вырожденные корни ±1, ±1, один из которых в силу проективности матрицы Γ_0 описывает безмассовое векторное поле, второй – массивное векторное поле с массой m. Корни $\pm\delta$, $\pm\rho$ блока $\big(C^1\big)''$ соответствуют векторным полям с массами $\frac{m}{|\delta|}$ и $\frac{m}{|\rho|}$.

4 Обсуждение и выводы

Таким образом, схема зацеплений (14) позволяет построить релятивистское волновое уравнение, описывающее векторное поле с четырьмя типами квантов – одним безмассовым и тремя массивными. При этом в теории неизбежно возникает скалярная частица с ненулевой массой, которая собственно и обеспечивает единство компонент указанного векторного поля. Скалярное и векторное поля описываются совместно в рамках одной не распадающейся по группе Лоренца системы уравнений и, следовательно, с точки зрения положений теории РВУ представляют собой единый физический объект – массивно-безмассовое скалярно-векторное поле. Очевидно, что его векторная составляющая может интерпретироваться как “электрослабое” поле, а скалярная – как аналог бозона Хиггса.

Вопрос о происхождении массы не может быть решен в рамках стандартной теории РВУ; наличие или отсутствие массы является в этой теории заданным фактом. В предлагаемой модели масса скалярного бозона должна совпадать с массой одного из промежуточных векторных бозонов (очевидно, Z^0). Массы всех частиц выступают в качестве произвольных параметров теории.

Литература

1. Плетюхов, В.А. Описание массивных и безмассовых полей в теории обобщённых релятивистских волновых уравнений / В.А. Плетюхов, В.И. Стражев // Вучоныя запіскі БрДУ імя А.С. Пушкіна. – 2007. – Т. 3. – Ч. 2. – С. 50–66.

2. Pletyukhov, V.A. Massless and massive gauge-invariant fields in the theory of relativistic wave equations / V.A. Pletyukhov, V.I. Strazhev // arXiv:1002.0720v1 [hep-th]3 Feb 2010.

3. Гельфанд, И.М. Представления группы вращений и группы Лоренца / И.М. Гельфанд, Р.А. Минлос, З.Я. Шапиро // – М. : Физматгиз, 1958. – 368 с.

Relativistic canonical quantum mechanics of arbitrary spin

V.M. Simulik
vsimulik@gmail.com; Institute of Electron Physics of NAS, Uzhgorod, Ukraine.

Abstract

The axiomatic level description of the relativistic canonical quantum mechanics of the arbitrary mass and spin has been considered briefly. The level of von Neumann's consideration of non-relativistic case is maintained. The special attention for the axioms of relativistic invariance of the theory and of the Clifford–Dirac algebra has been given. The 64-dimensional $Cl^{\mathbb{R}}(0,6)$ algebra in terms of Dirac gamma matrices has been suggested.

Keywords: relativistic quantum mechanics, Schrödinger–Foldy equation, Dirac equation, Clifford algebra, arbitrary spin

1 Introduction

Recently in [1] the interesting results in the area of relativistic quantum mechanics have been presented. Here in this article this investigation has been continued by the general forms of the partial cases visualized in [1]. The general forms of quantum-mechanical equations for arbitrary spin are presented. The corresponding relativistic quantum mechanics of arbitrary spin is given as the system of axioms.

Note that in the Dirac model [2], [3] the quantum-mechanical interpretation is not evident. It has been demonstrated in [1], [4], [5] that the quantum-mechanical interpretation is much more clear in the Foldy–Wouthuysen (FW) model [4], [5]. Nevertheless, the complete quantum-mechanical picture is possible only in the framework of relativistic canonical quantum mechanics (RCQM). This assertion is one of the main conclusions proved in [1].

The relativistic quantum mechanics under consideration is called *canonical* due to three main reasons. (i) The model under consideration has direct link with nonrelativistic quantum mechanics based on nonrelativistic Schrödinger equation. The principles of heredity and correspondence with other models of physical reality leads directly to nonrelativistic Schrödinger quantum mechanics. (ii) The FW model is already called by many authors as the canonical representation of the Dirac equation or a canonical field model, see, e. g., the paper [5]. And the difference between the field model given by FW and the RCQM is minimal – in corresponding equations it is only the presence and absence of beta matrix. (iii) The list of relativistic quantum-mechanical models is long. The Dirac model and the FW model are called by the "'old"' physicists as the relativistic quantum mechanics as well (one of my tasks in this paper is to show in visual and demonstrative way that these models have only weak quantum-mechanical interpretation). Further, the fractional relativistic quantum mechanics and the proper-time relativistic quantum mechanics can be listed (recall matrix formulation by W. Heisenberg, Feynman's sum over path's quantum theory, many-worlds interpretation by H. Everett), etc. Therefore, in order to avoid a confusion the model under consideration must have its proper name. Due to the reasons (i)–(iii) the best name for it is RCQM.

The concepts, definitions and notations here are the same as in [1]. For example, in the Minkowski space-time

$$\mathrm{M}(1,3)=\{x\equiv(x^{\mu})=(x^{0}=t,\ \overrightarrow{x}\equiv(x^{j}))\};\quad \mu=\overline{0,3},\ j=1,2,3, \tag{1}$$

x^{μ} denotes the Cartesian (covariant) coordinates of the points of the physical space-time in the arbitrary-fixed inertial reference frame (IRF). We use the system of units with $\hbar=c=1$. The metric tensor is given by

$$g^{\mu\nu}=g_{\mu\nu}=g^{\mu}_{\nu},\ (g^{\mu}_{\nu})=\mathrm{diag}\,(1,-1,-1,-1)\,;\quad x_{\mu}=g_{\mu\nu}x^{\mu}, \tag{2}$$

and summation over the twice repeated indices is implied.

2 Brief review of the relativistic canonical quantum mechanics status quo

Note at first that the square-root operator equation, which is the main equation of RCQM, has been rejected by Dirac. In his consideration in [3] (chapter 11, section 67) of the main steps of [2] Dirac discussed briefly this equation. Today another reason of Dirac's rejection of the square-root operator equation is evident as well. The operations with the Dirac's Hamiltonian are too much easier than the operations with the pseudo-differential Hamiltonian of the equation

$$i\partial_{t}f(x)=\sqrt{m^{2}-\Delta}f(x) \tag{3}$$

for the N-component wave function

$$f\equiv\mathrm{column}(f^{1},f^{2},...,f^{\mathrm{N}}),\quad \mathrm{N}=2s+1,\quad f\in\mathrm{H}^{3,\mathrm{N}}, \tag{4}$$

in the case of particle singlet and for the M-component wave function ($\mathrm{M}=2\mathrm{N}=2(2s+1)$) in the case of particle-antiparticle doublet. Here $\mathrm{H}^{3,\mathrm{N}}$ is the Hilbert space of N-component functions. Note that namely (3) is the equation of motion in the RCQM of arbitrary spin, see [1] for the details.

Nevertheless, today, contrary to the year 1928, the definition of the pseudo-differential (non-local) operator

$$\widehat{\omega}\equiv\sqrt{\widehat{\overrightarrow{p}}^{2}+m^{2}}=\sqrt{-\Delta+m^{2}}\geq m>0,\quad \widehat{\overrightarrow{p}}\equiv(\widehat{p}^{j})=-i\nabla,\quad \nabla\equiv(\partial_{\ell}), \tag{5}$$

is well known. The action of the operator (5) in the coordinate representation (see, e. g. [6]) is given by

$$\widehat{\omega}f(t,\overrightarrow{x})=\int d^{3}yK(\overrightarrow{x}-\overrightarrow{y})f(t,\overrightarrow{y}), \tag{6}$$

where the function $K(\overrightarrow{x}-\overrightarrow{y})$ has the form $K(\overrightarrow{x}-\overrightarrow{y})=-\frac{2m^{2}K_{2}(m|\overrightarrow{x}-\overrightarrow{y}|)}{(2\pi)^{2}|\overrightarrow{x}-\overrightarrow{y}|^{2}}$ and $K_{\nu}(z)$ is the modified Bessel function (Macdonald function), $|\overrightarrow{a}|$ designates the norm of the vector $\overrightarrow{a}$.

Further, the following integral form

$$(\widehat{\omega}f)(t,\overrightarrow{x})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^{3}ke^{i\overrightarrow{k}\,\overrightarrow{x}}\widetilde{\omega}\widetilde{f}(t,\overrightarrow{k}),\quad \widetilde{\omega}\equiv\sqrt{\overrightarrow{k}^{2}+m^{2}},\quad \widetilde{f}\in\widetilde{\mathrm{H}}^{3,\mathrm{N}}, \tag{7}$$

of the operator $\widehat{\omega}$ is used often, see, e. g., [5], [7], where f and $\widetilde{f}$ are linked by the 3-dimensional Fourier transformations

$$f(t,\overrightarrow{x})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\overrightarrow{k}\overrightarrow{x}}\widetilde{f}(t,\overrightarrow{k})\Leftrightarrow\widetilde{f}(t,\overrightarrow{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{-i\overrightarrow{k}\overrightarrow{x}}\widetilde{f}(t,\overrightarrow{x}),\tag{8}$$

(in (8) $\overrightarrow{k}$ belongs to the spectrum $\mathrm{R}^3_{\overrightarrow{k}}$ of the operator $\widehat{\overrightarrow{p}}$, and the parameter $t\in(-\infty,\infty)\subset\mathrm{M}(1,3)$).

Note that the space of states $\mathrm{H}^{3,\mathrm{N}}$ is invariant with respect to the Fourier transformation (8). Therefore, both $\overrightarrow{x}$-realization $\mathrm{H}^{3,\mathrm{N}}$ and $\overrightarrow{k}$-realization $\widetilde{\mathrm{H}}^{3,\mathrm{N}}$ of the space of states are suitable for the purposes of our consideration. In the $\overrightarrow{k}$-realization the Schrödinger–Foldy equation has the algebraic-differential form

$$i\partial_t\widetilde{f}(t,\overrightarrow{k})=\sqrt{\overrightarrow{k}^2+m^2}\widetilde{f}(t,\overrightarrow{k});\quad\overrightarrow{k}\in\mathrm{R}^3_{\overrightarrow{k}},\quad\widetilde{f}\in\widetilde{\mathrm{H}}^{3,\mathrm{N}}.\tag{9}$$

Below in the places, where misunderstanding is impossible, the symbol "tilde" is omitted.

Thus, today on the basis of above given definitions the difficulties, which stopped Dirac in 1928, can be overcome.

The name of the person, whose contribution in the theoretical model based on the equation (3) was decisive, is Leslie Lawrance Foldy (1919–2001). His interesting biography is presented in [8]. The most interesting points are as follows. He was born in Sabinov, Czechoslovakia, on 26 October 1919, into a family with Hungarian roots. His parents named him Laszlo Földi. All his scientific activity was made in USA. His co-author in most cited paper [4] was netherlander Siegfried (Sieg) Wouthuysen (pronounced Vout-high-sen). Taking into account the L. Foldy's contribution in the construction of RCQM and his proof of the principle of correspondence between RCQM and non-relativistic quantum mechanics, I proposed in [1], [9] and [10] *to call the N-component square-root operator equation (3) as the Schrödinger–Foldy equation.* Note here that equation (3), which is a direct sum of N-s one component spinless Salpeter equations [11], has been introduced in the formula (21) of [5].

Contrary to the times of papers [4], [5], [11], the RCQM today is enough approbated and generally accepted theory. There is a big number of articles, results and approaches. In the contemporary RCQM were developed both the construction of mathematical foundations and the solution of concrete quantum-mechanical problems for different potentials. A brief review of the RCQM status quo has been presented in [9], see the refs. [12]–[35] in [9]. My contribution is in formulation [1], [9] and [10] of axiomatic approach for arbitrary spin, in both side links between the RCQM and field theory, in derivation of arbitrary spin field equations on the bases of start from the RCQM.

3 Axioms of the relativistic canonical quantum mechanics of an arbitrary spin

The RCQM of the arbitrary spin given in sections 2 and 18 of [1] can be formulated at the level of von Neumann's consideration [12]. The difference is only in relativistic invariance and in the consideration of multicomponent and multidimensional objects.

The partial case of axiomatic formulation is already given in section 7 of [1] at the example of spin s=1/2 particle-antiparticle doublet. The RCQM of the arbitrary spin particle-antiparticle doublet (or particle singlet) can be formulated similarly as the corresponding generalization of this partial case.

Below the brief presentation of the *list of the axioms* is given. Note that some particular content of these axioms is already given in section 2 of [1], where the RCQM of the arbitrary spin particle singlet has been formulated.

3.1 On the space of states

The space of states of isolated arbitrary spin particle singlet in an arbitrarily-fixed inertial frame of reference (IFR) in its $\overrightarrow{x}$-realization is the Hilbert space

$$\mathrm{H}^{3,\mathrm{N}} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes \mathrm{N}} = \{f = (f^{\mathrm{N}}) : \mathrm{R}^3 \to \mathrm{C}^{\otimes \mathrm{N}}; \quad \int d^3x |f(t, \overrightarrow{x})|^2 < \infty\}, \quad \mathrm{N} = 2s + 1, \quad (10)$$

of complex-valued N-component square-integrable functions of $x \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (similarly, in momentum, $\overrightarrow{p}$-realization). In (10) d^3x is the Lebesgue measure in the space $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ of the eigenvalues of the position operator $\overrightarrow{x}$ of the Cartesian coordinate of the particle in an arbitrary-fixed IFR. Further, $\overrightarrow{x}$ and $\overrightarrow{p}$ are the operators of canonically conjugated dynamical variables of the spin s=(1/2,1/2) particle-antiparticle doublet, and the vectors f, $\tilde{f}$ in $\overrightarrow{x}$- and $\overrightarrow{p}$-realizations are linked by the 3-dimensional Fourier transformation (the variable t is the parameter of time-evolution).

The mathematical correctness of the consideration demands the application of the rigged Hilbert space

$$\mathrm{S}^{3,\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}. \quad (11)$$

where the Schwartz test function space $\mathrm{S}^{3,\mathrm{N}}$ is the core (i. e., it is dense both in $\mathrm{H}^{3,\mathrm{N}}$ and in the space $\mathrm{S}^{3,\mathrm{N}*}$ of the N-component Schwartz generalized functions). The space $\mathrm{S}^{3,\mathrm{N}*}$ is conjugated to that of the Schwartz test functions $\mathrm{S}^{3,\mathrm{N}}$ by the corresponding topology (see, e. g. [13], [14]).

Note that the Schrödinger–Foldy equation (3) has generalized solutions, which do not belong to the space $\mathrm{H}^{3,\mathrm{N}}$ (10). Therefore, the application of the rigged Hilbert space $\mathrm{S}^{3,\mathrm{N}} \subset \mathrm{H}^{3,\mathrm{N}} \subset \mathrm{S}^{3,\mathrm{N}*}$ (11) is necessary.

Note finally that in the case of arbitrary spin particle-antiparticle doublet the dimension of spaces (10), (11) is M=2N=2(2s+1). The complete text of this axiom can be found in [9].

3.2 On the time evolution of the state vectors

The time dependence of the state vectors $f \in \mathrm{H}^{3,\mathrm{N}}$ (time t is the parameter of evolution) is given either in the integral form by the unitary operator

$$u\,(t_0, t) = \exp\left[-i\widehat{\omega}(t - t_0)\right]; \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad (12)$$

(below $t_0 = t$ is put), or in the differential form by the Schrödinger–Foldy equation of motion (3) with the wave function (4). In terms of operator (5)–(7) this equation is given by

$$(i\partial_0 - \widehat{\omega})f(x) = 0. \quad (13)$$

Note that here the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ is the relativistic analog of the energy operator (Hamiltonian) of nonrelativistic quantum mechanics. The Minkowski space-time M(1,3) is pseudo Euclidean with metric $g = \mathrm{diag}(+1, -1, -1, -1)$. The step from the particle singlet of arbitrary spin to the corresponding particle-antiparticle doublet is evident.

Thus, for the arbitrary spin particle-antiparticle doublet the system of two N-component equations $(i\partial_0 - \widehat{\omega})f(x) = 0$ and $(i\partial_0 - \widehat{\omega})f(x) = 0$ is used. Therefore, the corresponding Schrödinger–Foldy equation is given by (13), where the 2N-component wave function is the

direct sum of the particle and antiparticle wave functions, respectively. Due to the historical tradition of the physicists the antiparticle wave function is put in the down part of the 2N-column.

The general solution of the Schrödinger–Foldy equation of motion (13) (in the case of particle-antiparticle arbitrary spin doublet) has the form

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} a^{2\mathrm{N}} \left(\overrightarrow{k}\right) \mathrm{d}_{2\mathrm{N}}, \quad kx \equiv \omega t - \overrightarrow{k}\,\overrightarrow{x}, \quad \omega \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{14}$$

where the orts of the N-dimensional Cartesian basis are given, e. g., in [1] by the formulae (10).

3.3 On the fundamental dynamical variables

The dynamical variable $\overrightarrow{x} \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (as well as the variable $\overrightarrow{k} \in \mathrm{R}^3_{\overrightarrow{k}}$) represents the external degrees of freedom of the arbitrary spin particle-antiparticle doublet. The spin $\overrightarrow{s}$ of the particle-antiparticle doublet is the first in the list of the carriers of the internal degrees of freedom. Taking into account the Pauli principle and the fact that experimentally an antiparticle is observed as the mirror reflection of a particle, the operators of the charge sign and the spin of the arbitrary particle-antiparticle doublet are taken in the form

$$g \equiv -\Gamma^0_{2\mathrm{N}} \equiv -\sigma^3_{2\mathrm{N}} = \left| \begin{array}{cc} -\mathrm{I}_\mathrm{N} & 0 \\ 0 & \mathrm{I}_\mathrm{N} \end{array} \right|, \quad \overrightarrow{s}_{2\mathrm{N}} = \left| \begin{array}{cc} \overrightarrow{s}_\mathrm{N} & 0 \\ 0 & -\hat{C}\overrightarrow{s}_\mathrm{N}\hat{C} \end{array} \right|, \quad \mathrm{N} = 2s+1, \tag{15}$$

where $\Gamma^0_{2\mathrm{N}}$ is the $2\mathrm{N}\times 2\mathrm{N}$ Dirac Γ^0 matrix, $\sigma^3_{2\mathrm{N}}$ is the $2\mathrm{N}\times 2\mathrm{N}$ Pauli σ^3 matrix, $\hat{C}$ is the operator of complex conjugation in the form of $\mathrm{N}\times\mathrm{N}$ diagonal matrix, the operator of involution in $\mathrm{H}^{3,2\mathrm{N}}$, and I_N is $\mathrm{N}\times\mathrm{N}$ unit matrix.

Thus, the spin is given by the generators of SU(2) algebra!

The spin matrices $\overrightarrow{s}_{2\mathrm{N}}$ (15) satisfy the commutation relations

$$\left[s^j_{2\mathrm{N}}, s^\ell_{2\mathrm{N}}\right] = i\varepsilon^{j\ell n} s^n_{2\mathrm{N}}, \quad \varepsilon^{123} = +1, \tag{16}$$

of the algebra of SU(2) group, where $\varepsilon^{j\ell n}$ is the Levi-Civita tensor and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian $2\mathrm{N}\times 2\mathrm{N}$ matrices (15) – the generators of a 2N-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)$\subset$SO(1,3) group).

The Casimir operator for the RCQM representation of SU(2) spin given in (15) has the form

$$\overrightarrow{s}^2_{2\mathrm{N}} = \mathrm{s}(\mathrm{s}+1)\mathrm{I}_{2\mathrm{N}}, \tag{17}$$

where $\mathrm{I}_{2\mathrm{N}}$ is $2\mathrm{N}\times 2\mathrm{N}$ unit matrix.

Above in the text of this axiom the case of arbitrary spin *particle-antiparticle doublet* has been considered. For the case of arbitrary spin *particle singlet* the operator of the charge sign is absent. The spin operator $\overrightarrow{s}_\mathrm{N}$ is given by the Hermitian $\mathrm{N}\times\mathrm{N}$ matrices – the generators of a N-dimensional irreducible representation of the spin group SU(2). Therefore, the SU(2) commutation relations in such notations have the explicit form

$$\left[s^j_\mathrm{N}, s^\ell_\mathrm{N}\right] = i\varepsilon^{j\ell n} s^n_\mathrm{N}, \quad \varepsilon^{123} = +1, \tag{18}$$

and the corresponding Casimir operator is given by

$$\overrightarrow{s}^2_\mathrm{N} = \mathrm{s}(\mathrm{s}+1)\mathrm{I}_\mathrm{N}. \tag{19}$$

The validity of these assertions is proved by numerous partial examples presented in [1].

3.4 On the external and internal degrees of freedom

The coordinate $\overrightarrow{x}$ (as an operator in $\mathrm{H}^{3,2\mathrm{N}}$) is an analog of the discrete index of generalized coordinates $q \equiv (q_1, q_2, ...)$ in non-relativistic quantum mechanics of the finite number degrees of freedom. In other words the coordinate $\overrightarrow{x} \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ is the continuous carrier of the external degrees of freedom of a multiplet (the similar consideration was given in [15]). The coordinate operator together with the operator $\widehat{\overrightarrow{p}}$ determines the operator $m_{ln} = x_l\widehat{p}_n - x_n\widehat{p}_l$ of an orbital angular momentum, which also is connected with the external degrees of freedom.

However, the RCQM doublet has the additional characteristics such as the spin operator $\overrightarrow{s}$ (15), which is the carrier of the internal degrees of freedom of this multiplet. The set of generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ (formulae (22), (23) below) of the main dynamical variables (formulae (73) in [1]) of the doublet are the functions of the following basic set of 9 functionally independent operators

$$\overrightarrow{x} = (x^j),\ \widehat{\overrightarrow{p}} = (\widehat{p}^j),\ \overrightarrow{s}_{2\mathrm{N}} \equiv \left(s^j_{2\mathrm{N}}\right) = (s_{23}, s_{31}, s_{12})\,. \tag{20}$$

Note that spin $\overrightarrow{s}_{2\mathrm{N}}$ (15) commutes both with $(\overrightarrow{x}, \widehat{\overrightarrow{p}})$ and with the operator $i\partial_t - \sqrt{-\Delta + m^2}$ of the Schrödinger–Foldy equation (13). Thus, for the free doublet the external and internal degrees of freedom are independent. Therefore, 9 operators (20) in $\mathrm{H}^{3,2\mathrm{N}}$, which have the univocal physical sense, are the *generating* operators not only for the 10 $\mathcal{P}$ generators $(\widehat{p}_\mu, \widehat{j}_{\mu\nu})$ (22), (23), but also for other operators of any experimentally observable quantities of the doublet.

3.5 On the algebra of observables

Using the operators of canonically conjugated coordinate $\overrightarrow{x}$ and momentum $\overrightarrow{p}$ (where $\left[x^j, \widehat{p}^\ell\right] = i\delta^{j\ell}$, $\left[x^j, x^\ell\right] = \left[\widehat{p}^j, \widehat{p}^\ell\right] = 0$,) in $\mathrm{H}^{3,2\mathrm{N}}$, being completed by the operators $\overrightarrow{s}_{2\mathrm{N}}$ and g (15), we construct the algebra of observables as the Hermitian functions of 10 $(\overrightarrow{x},\ \overrightarrow{p},\ \overrightarrow{s}_{2\mathrm{N}},\ -\Gamma^0_{2\mathrm{N}})$ generating elements of the algebra.

3.6 On the relativistic invariance of the theory

The relativistic invariance of the model under consideration (the relativistic invariance of the Schrödinger–Foldy equation (13)) requires, as a first step, consideration of its invariance with respect to the proper ortochronous Lorentz $\mathrm{L}^\uparrow_+ = \mathrm{SO}(1,3) = \{\Lambda = (\Lambda^\mu_\nu)\}$ and Poincaré $\mathrm{P}^\uparrow_+ = \mathrm{T}(4)\times\mathrm{L}^\uparrow_+ \supset \mathrm{L}^\uparrow_+$ groups. This invariance in an arbitrary relativistic model is the implementation of the Einstein's relativity principle in the special relativity form. Note that the mathematical correctness requires the invariance mentioned above to be considered as the invariance with respect to the *universal coverings* $\mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ and $\mathcal{P} \supset \mathcal{L}$ of the groups $\mathrm{L}^\uparrow_+$ and $\mathrm{P}^\uparrow_+$, respectively.

For the group $\mathcal{P}$ we choose real parameters $a = (a^\mu) \in \mathrm{M}(1,3)$ and $\varpi \equiv (\varpi^{\mu\nu} = -\varpi^{\nu\mu})$ with well-known physical meaning. For the standard $\mathcal{P}$ generators $(p_\mu, j_{\mu\nu})$ we use commutation relations in the manifestly covariant form

$$[p_\mu, p_\nu] = 0,\ [p_\mu, j_{\rho\sigma}] = ig_{\mu\rho}p_\sigma - ig_{\mu\sigma}p_\rho,\ [j_{\mu\nu}, j_{\rho\sigma}] = -i\left(g_{\mu\rho}j_{\nu\sigma} + g_{\rho\nu}j_{\sigma\mu} + g_{\nu\sigma}j_{\mu\rho} + g_{\sigma\mu}j_{\rho\nu}\right). \tag{21}$$

The following assertion should be noted. Not a matter of fact that *non-covariant* objects such as the Lebesgue measure d^3x and *non-covariant (non-Lie)* generators of algebras are explored, the model of RCQM of arbitrary spin is a relativistic invariant in the following sense. The Schrödinger–Foldy equation (13) and the set of its solution $\{f\}$ (14) are invariant with respect

to the irreducible unitary representation of the group $\mathcal{P}$, the N×N matrix-differential generators of which are given by the following nonlocal operators

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \quad \widehat{j}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n}, \tag{22}$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \check{s}_\ell\right), \tag{23}$$

where the orbital parts of the generators are not changed under the transition from one spin to another. Under such transitions only the spin parts (15), (16) of the expressions (22), (23) are changed. Indeed, the direct calculations visualize that generators (22), (23) commute with the operator of equation (13) and satisfy the commutation relations (21) of the Lie algebra of the Poincaré group $\mathcal{P}$. In formulae (22), (23), the SU(2)-spin generators $s^{\ell n}$ have particular specific forms for each representation of the SU(2) group (see the list of examples in [1]). The generators (22), (23) are known from the formulae (B-25) – (B-28) of the paper [5].

Thus, the irreducible unitary representation of the Poincaré group $\mathcal{P}$ in the space (11), with respect to which the Schrödinger-Foldy equation (13) and the set of its solution $\{f\}$ (14) are invariant, is given by a series converges in this space

$$(a, \varpi) \to U(a, \varpi) = \exp(-ia^0\widehat{p}_0 - i\overrightarrow{a}\widehat{\overrightarrow{p}} - \frac{i}{2}\varpi^{\mu\nu}\widehat{j}_{\mu\nu}) \tag{24}$$

where the generators $(\widehat{p}^\mu, \widehat{j}^{\mu\nu})$ are given in (22), (23) with the arbitrary values of the SU(2) spins $\overrightarrow{s} = (s^{\ell n})$ (15), (16).

The validity of this assertion is verified by the following three steps. (i) The calculation that the $\mathcal{P}$-generators (22), (23) commute with the operator $i\partial_0 - \widehat{\omega}$ of the Schrödinger–Foldy equation (13). (ii) The verification that the $\mathcal{P}$-generators (22), (23) satisfy the commutation relations (21) of the Lie algebra of the Poincaré group $\mathcal{P}$. (iii) The proof that generators (22), (23) realize the spin s(s+1) representation of this group. Therefore, the Bargman–Wigner classification on the basis of the corresponding Casimir operators calculation should be given. These three steps can be made by direct and non-cumbersome calculations.

The expression (24) is well known, but rather formal. In fact the transition from a Lie algebra to a finite group transformations in the case of non-Lie generators is a rather non-trivial action. The mathematical justification of (24) can be fulfilled in the framework of Schwartz test function space and will be given in next special publication.

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_\mathrm{N}, \tag{25}$$

$$W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = \mathrm{s}(\mathrm{s}+1) m^2 \mathrm{I}_\mathrm{N}, \tag{26}$$

where I_N is the N × N unit matrix and s =1/2, 1, 3/2, 2, ...

Note that together with the generators (22), (23) another set of 10 operators commutes with the operator of equation (13), satisfies the commutation relations (21) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (22) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (23). Thus, this second set of Poincaré generators is given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \quad \widehat{m}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell, \quad \widehat{m}_{0\ell} = -\widehat{m}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\{x_\ell, \widehat{\omega}\}. \tag{27}$$

Note that in the case s=0 only generators (27) form the Poincaré symmetry.

Note that the modern definition of $\mathcal{P}$ invariance (or $\mathcal{P}$ symmetry) of the equation of motion (13) in $\mathrm{H}^{3,\mathrm{N}}$ is given by the following assertion, see, e. g. [16]. *The set* $\mathrm{F} \equiv \{f\}$ *of all possible solutions of the equation (13) is invariant with respect to the* $\mathcal{P}^{\mathrm{f}}$*-representation of the group* $\mathcal{P}$*, if for arbitrary solution* f *and arbitrarily-fixed parameters* (a, ϖ) *the assertion*

$$(a, \varpi) \to U(a, \varpi)\,\{f\} = \{f\} \equiv \mathrm{F} \tag{28}$$

is valid.

3.7 On the dynamic and kinematic aspects of the relativistic invariance

Consider briefly some detalizations of the relativistic invariance of the Schrödinger–Foldy equation (13). Note that for the free particle-antiparticle doublet of arbitrary spin the equation (13) has one and the same explicit form in arbitrary-fixed IFR (its set of solutions is one and the same in every IFR). Therefore, the algebra of observables and the conservation laws (as the functionals of the free particle-antiparticle doublet states) have one and the same form too. This assertion explains the dynamical sense of the $\mathcal{P}$ invariance (the invariance with respect to the dynamical symmetry group $\mathcal{P}$).

Another, kinematical, aspect of the $\mathcal{P}$ invariance of the RQCM model has the following physical sense. Note at first that any solution of the Schrödinger–Foldy equation (13) is determined by the concrete given set of the amplitudes $\{A\}$. It means that if f with the fixed set of amplitudes $\{A\}$ is the state of the doublet in some arbitrary IFR, then for the observer in the (a, ϖ)-transformed IFR$'$ this state f' is determined by the amplitudes $\{A'\}$. The last ones are received from the given $\{A\}$ by the unitary $\mathcal{P}^{\mathrm{A}}$ -transformation (24).

3.8 On the Clifford–Dirac algebra

This axiom is additional and is not necessary. Nevertheless, such axiom is very useful for the dimensions, where the Γ matrices exist.

Application of the Clifford–Dirac algebra is the useful method of calculations in RCQM. Three different definitions of the Clifford algebra and their equivalence are considered in [17]. In different approaches to the relativistic quantum mechanics the matrix representation of the Clifford algebra in terms of the Dirac gamma matrices is used. This representation is called the Clifford–Dirac algebra.

For our purposes the anticommutation relations of the Clifford–Dirac algebra generators are taken in the general form

$$\Gamma^{\bar{\mu}}_{2\mathrm{N}}\Gamma^{\bar{\nu}}_{2\mathrm{N}} + \Gamma^{\bar{\nu}}_{2\mathrm{N}}\Gamma^{\bar{\mu}}_{2\mathrm{N}} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu}, \bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \tag{29}$$

where $\Gamma^{\bar{\mu}}_{2\mathrm{N}}$ are the $2\mathrm{N} \times 2\mathrm{N}$ Dirac $\Gamma^{\bar{\mu}}$ matrices ($2\mathrm{N} \times 2\mathrm{N}$ generalization of the Dirac 4×4 γ matrices), $\Gamma^{4}_{2\mathrm{N}} \equiv \Gamma^{0}_{2\mathrm{N}}\Gamma^{1}_{2\mathrm{N}}\Gamma^{2}_{2\mathrm{N}}\Gamma^{3}_{2\mathrm{N}}$. Here and in our publications (see, e. g. the last years articles [18], [19]) we use the $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$ matrix instead of the γ^5 matrix of other authors. Our γ^4 is equal to $i\gamma^5_{\mathrm{standard}}$. Notation γ^5 is used in [18], [19] for a completely different matrix $\gamma^5 \equiv \gamma^1\gamma^3\hat{C}$. As well as the element $\Gamma^{4}_{2\mathrm{N}} \equiv \Gamma^{0}_{2\mathrm{N}}\Gamma^{1}_{2\mathrm{N}}\Gamma^{2}_{2\mathrm{N}}\Gamma^{3}_{2\mathrm{N}}$ of (29) is dependent the algebra basis is formed by 4=1+3 independent elements. Therefore, such Clifford algebra over the field of complex numbers is denoted $\mathrm{Cl}^{\mathbb{C}}(1,3)$ and the dimension of the algebra is $2^4 = 16$.

Note that relations (29) are valid only for the dimensions, where the $\Gamma_{2\mathrm{N}}$ are defined (where the matrix representation of the Clifford algebra exists). Therefore, the relations (29) can be useful not for all multiplets of RCQM. Nevertheless, existing relations (29) are very useful in

order to operate with spins, with standard FW transformation and in order to formulate the FW transformation generalizations for the dimensions 2(2s+1)>4.

It has been explained in [1] (and in [18], [19] in details) that the Clifford–Dirac algebra should be introduced into consideration in the FW representation [4] of the spinor field. The reasons are as follows. Part of the Clifford–Dirac algebra operators are directly related to the spin 1/2 doublet operators ($s^1 \equiv \frac{1}{2}\gamma^2\gamma^3$, $s^2 \equiv \frac{1}{2}\gamma^3\gamma^1$, $s^3 \equiv \frac{1}{2}\gamma^1\gamma^2$) (in the anti-Hermitian form). In the FW representation for the spinor field [4] these spin operators commute with the Hamiltonian and with the operator of the FW equation of motion $i\partial_0 - \gamma^0\widehat{\omega}$. In the Pauli–Dirac representation these operators do not commute with the Dirac equation operator. Only the sums of the orbital operators and such spin operators commute with the Diracian. So *if we want to relate the orts γ^μ of the Clifford–Dirac algebra with the actual spin we must introduce this algebra into the FW representation.*

Here in general N-dimensional formalism of RCQM the situation is similar. The anticommutation relations of the Clifford–Dirac algebra generators (29) and corresponding $\Gamma_{2\mathrm{N}}$ matrices must be introduced in the FW representation. Therefore, in order to apply (29) in RCQM one must transform the $\Gamma_{2\mathrm{N}}$ matrices from the FW representation into the RCQM representation. Corresponding operator transformation is given by

$$v_{2\mathrm{N}} = \left| \begin{array}{cc} \mathrm{I_N} & 0 \\ 0 & \hat{C}\mathrm{I_N} \end{array} \right|, \quad v_{2\mathrm{N}}^{-1} = v_{2\mathrm{N}}^{\dagger} = v_{2\mathrm{N}}, \quad v_{2\mathrm{N}} v_{2\mathrm{N}} = \mathrm{I}_{2\mathrm{N}}, \quad \mathrm{N} = 2s+1, \tag{30}$$

where $\hat{C}\mathrm{I_N}$ is the $\mathrm{N}\times\mathrm{N}$ operator of complex conjugation. Indeed, the operator (30) translates any operator from canonical field FW representation into the RCQM representation and vice versa:

$$v_{2\mathrm{N}} \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}} v_{2\mathrm{N}} = \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}, \quad v_{2\mathrm{N}} \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}} v_{2\mathrm{N}} = \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}. \tag{31}$$

Here $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 2N-component particle-antiparticle doublet in the anti-Hermitian form, e. g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion (13), the operator of spin $\overrightarrow{s}_{2\mathrm{N}}$ (15) taken in anti-Hermitian form, etc., $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 2N-component particle-antiparticle doublet in the anti-Hermitian form. Thus, the only warning is that operators here must be taken in anti-Hermitian form, see section 9 in [1] for the details and see [20], [21] for the mathematical correctness of anti-Hermitian operators application.

Further, the operator (30) translates

$$\phi = v_{2\mathrm{N}} f, \quad f = v_{2\mathrm{N}} \phi, \tag{32}$$

the solution (14) of the Schrödinger–Foldy equation (13) into the solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^{\mathrm{N}}(\overrightarrow{k})\mathrm{d_N} + e^{ikx} a^{*\breve{\mathrm{N}}}(\overrightarrow{k})\mathrm{d}_{\breve{\mathrm{N}}} \right], \tag{33}$$

$\mathrm{N} = 1, 2, ..., \mathrm{N}, \quad \breve{\mathrm{N}} = \mathrm{N}+1, \mathrm{N}+2, ..., 2\mathrm{N}$, of the FW equation

$$(i\partial_0 - \Gamma_{2\mathrm{N}}^0 \widehat{\omega})\phi(x) = 0, \quad \Gamma_{2\mathrm{N}}^0 \equiv \sigma_{2\mathrm{N}}^3 = \left| \begin{array}{cc} \mathrm{I_N} & 0 \\ 0 & -\mathrm{I_N} \end{array} \right|, \tag{34}$$

$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \mathrm{N} = 2s+1$, and vice versa. Thus, the transformation (30), (31) translates the matrices $\Gamma_{2\mathrm{N}}^0$ and

$$\Gamma_{2\mathrm{N}}^j = \left| \begin{array}{cc} 0 & \Sigma_{\mathrm{N}}^j \\ -\Sigma_{\mathrm{N}}^j & 0 \end{array} \right|, \quad j = 1, 2, 3, \tag{35}$$

into the RCQM representation

$$\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}} = v_{2\mathrm{N}}\Gamma^{\bar{\mu}}_{2\mathrm{N}}v_{2\mathrm{N}}, \tag{36}$$

where matrices $\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}$ satisfy the anticommutation relations

$$\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}\bar{\Gamma}^{\bar{\nu}}_{2\mathrm{N}} + \bar{\Gamma}^{\bar{\nu}}_{2\mathrm{N}}\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu},\bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \tag{37}$$

of the Clifford–Dirac algebra generators as well. In (35) Σ^j_{N} are the N × N Pauli matrices. The explicit forms of the RCQM representation of the $\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}$ matrices are given by

$$\bar{\Gamma}^0_{2\mathrm{N}} = \Gamma^0_{2\mathrm{N}}, \quad \bar{\Gamma}^1_{2\mathrm{N}} = \Gamma^1_{2\mathrm{N}}\widehat{C}, \quad \bar{\Gamma}^2_{2\mathrm{N}} = \Gamma^0_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}\widehat{C}, \quad \bar{\Gamma}^3_{2\mathrm{N}} = \Gamma^3_{2\mathrm{N}}\widehat{C}, \quad \bar{\Gamma}^4_{2\mathrm{N}} = \Gamma^0_{2\mathrm{N}}\Gamma^4_{2\mathrm{N}}\widehat{C}, \tag{38}$$

where $\widehat{C}$ is the 2N × 2N operator of complex conjugation and matrices $\Gamma^{\bar{\mu}}_{2\mathrm{N}}$ are given in (34), (35).

Note that in the terms of $\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}$ matrices (38) the RCQM spin operator (15) has the form

$$\overrightarrow{s} = \frac{i}{2}(\bar{\Gamma}^2_{2\mathrm{N}}\bar{\Gamma}^3_{2\mathrm{N}},\, \bar{\Gamma}^3_{2\mathrm{N}}\bar{\Gamma}^1_{2\mathrm{N}},\, \bar{\Gamma}^1_{2\mathrm{N}}\bar{\Gamma}^2_{2\mathrm{N}}). \tag{39}$$

Note further that formula (39) is valid for the multiplets of arbitrary dimension but only for the spin s=1/2, whereas the formula (15) is valid for arbitrary spin. Furthermore, the complete analogy between the (39) and the particle-antiparticle spin s=1/2 doublet of arbitrary dimension in the FW representation exists

$$\overrightarrow{s}_{\mathrm{FW}} = \frac{i}{2}(\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}},\, \Gamma^3_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}},\, \Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}). \tag{40}$$

It is very useful to consider a wider then $\mathrm{Cl}^{\mathbb{C}}(1,3)$ Clifford–Dirac algebra. In [18], [19] such additional algebras have been introduced for the purposes of finding links between the fermionic and bosonic states of the spinor field. New algebra can be formed by the generators of $\mathrm{Cl}^{\mathbb{C}}(1,3)$ together with the generators of the Pauli–Gursey–Ibragimov algebra [22].

The main structure elements of such set are given by $(\gamma^0,\gamma^1,\gamma^2,\gamma^3,i,\hat{C}\mathrm{I}_4)$, where γ^μ are 4×4 Dirac matrices in standard representation. It is easy to see that simplest set of the Clifford–Dirac algebra generators can be constructed from these elements in the form $(i\gamma^0,i\gamma^1,\gamma^2,i\gamma^3,\hat{C}\mathrm{I}_4,i\hat{C}\mathrm{I}_4)$. Therefore, the 2N × 2N matrix generators of the corresponding Clifford–Dirac algebra over the field of real numbers can be found by the simple redefinition

$$\tilde{\Gamma}^1_{2\mathrm{N}} \equiv i\Gamma^1_{2\mathrm{N}}, \quad \tilde{\Gamma}^2_{2\mathrm{N}} \equiv i\Gamma^3_{2\mathrm{N}}, \quad \tilde{\Gamma}^3_{2\mathrm{N}} \equiv \widehat{C}\mathrm{I}_{2\mathrm{N}}, \quad \tilde{\Gamma}^4_{2\mathrm{N}} \equiv i\widehat{C}\mathrm{I}_{2\mathrm{N}}, \quad \tilde{\Gamma}^5_{2\mathrm{N}} \equiv i\Gamma^0_{2\mathrm{N}}, \quad \tilde{\Gamma}^6_{2\mathrm{N}} \equiv -\Gamma^2_{2\mathrm{N}}, \tag{41}$$

of the matrices $(i\Gamma^0_{2\mathrm{N}},i\Gamma^1_{2\mathrm{N}},\Gamma^2_{2\mathrm{N}},i\Gamma^3_{2\mathrm{N}},\widehat{C}\mathrm{I}_{2\mathrm{N}},i\widehat{C}\mathrm{I}_{2\mathrm{N}})$, where $\Gamma^\mu_{2\mathrm{N}}$ are given in (34), (35).

Matrices (41) together with the matrix $\tilde{\Gamma}^7_{2\mathrm{N}} \equiv \tilde{\Gamma}^1_{2\mathrm{N}}\tilde{\Gamma}^2_{2\mathrm{N}}\tilde{\Gamma}^3_{2\mathrm{N}}\tilde{\Gamma}^4_{2\mathrm{N}}\tilde{\Gamma}^5_{2\mathrm{N}}\tilde{\Gamma}^6_{2\mathrm{N}} = \Gamma^4_{2\mathrm{N}}$ satisfy the anticommutation relations of the Clifford–Dirac algebra generators in the form

$$\tilde{\Gamma}^{\mathrm{A}}_{2\mathrm{N}}\tilde{\Gamma}^{\mathrm{B}}_{2\mathrm{N}} + \tilde{\Gamma}^{\mathrm{B}}_{2\mathrm{N}}\tilde{\Gamma}^{\mathrm{A}}_{2\mathrm{N}} = 2g^{\mathrm{AB}}; \quad \mathrm{A},\mathrm{B} = \overline{1,7}, \quad (g^{\mathrm{AB}}) = (+++----). \tag{42}$$

As well as in (29) among the generators of (42) only the 4+2=6 matrices (41) are independent and form the basis of the algebra. Therefore, the found above algebra over the field of real numbers is defined as $\mathrm{Cl}^{\mathbb{R}}(4,2)$ and the dimension of this algebra is $2^6 = 64$.

Useful realization of (41), (42) is given in terms of completely anti-Hermitian generators

$$\left\{\Gamma^1_{2\mathrm{N}},\, \Gamma^2_{2\mathrm{N}},\, \Gamma^3_{2\mathrm{N}},\, \Gamma^4_{2\mathrm{N}} = \Gamma^0_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}},\, \Gamma^5_{2\mathrm{N}} = \Gamma^1_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}\widehat{C},\, \Gamma^6_{2\mathrm{N}} = i\Gamma^1_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}}\widehat{C},\, \Gamma^7_{2\mathrm{N}} = i\Gamma^0_{2\mathrm{N}}\right\}, \tag{43}$$

where matrices $\{\Gamma^{\mu}_{2N}\}$ are givem in (34), (35). Matrices (43) obey the anticommutation relations of 64-dimensional $Cl^{R}(0,6)$ algebra in the form

$$\Gamma^{A}_{2N}\Gamma^{B}_{2N} + \Gamma^{B}_{2N}\Gamma^{A}_{2N} = -2\delta^{AB}, \quad A,B = \overline{1,7}. \tag{44}$$

Note that here as well only 6 operators are independent generators: $\Gamma^{4}_{2N} = -i\Gamma^{7}_{2N}\Gamma^{1}_{2N}\Gamma^{2}_{2N}\Gamma^{3}_{2N}$.

Operators (43) generate also the 28 orts $\alpha^{\bar{A}\bar{B}} = 2s^{\bar{A}\bar{B}}$:

$$s^{\bar{A}\bar{B}} = \{s^{AB} = \frac{1}{4}[\Gamma^{A}_{2N}, \Gamma^{B}_{2N}],\ s^{A8} = -s^{8A} = \frac{1}{2}\Gamma^{A}_{2N}\}, \quad \bar{A},\bar{B} = \overline{1,8}, \tag{45}$$

where generators $s^{\bar{A}\bar{B}}$ satisfy the commutation relations of SO(8) algebra

$$[s^{\bar{A}\bar{B}}, s^{\bar{C}\bar{D}}] = \delta^{\bar{A}\bar{C}}s^{\bar{B}\bar{D}} + \delta^{\bar{C}\bar{B}}s^{\bar{D}\bar{A}} + \delta^{\bar{B}\bar{D}}s^{\bar{A}\bar{C}} + \delta^{\bar{D}\bar{A}}s^{\bar{C}\bar{B}}. \tag{46}$$

Of course, the algebra SO(8) is considered over the field of real numbers.

As the consequences of the equalities

$$\Gamma^{4}_{2N} \equiv \prod_{\mu=0}^{3}\Gamma^{\mu}_{2N} \to \prod_{\bar{\mu}=0}^{4}\Gamma^{\bar{\mu}}_{2N} = -I, \tag{47}$$

known from the standard Clifford–Dirac algebra $Cl^{C}(1,3)$, and the anticommutation relations (44), in $Cl^{R}(0,6)$ algebra for the matrices Γ^{A}_{2N} (38) the following extended equalities are valid:

$$\Gamma^{7}_{2N} \equiv -\prod_{\underline{A}=1}^{6}\Gamma^{\underline{A}}_{2N} \to \prod_{A=1}^{7}\Gamma^{A}_{2N} = I, \quad \Gamma^{5}_{2N}\Gamma^{6}_{2N} = i. \tag{48}$$

The relationship (45) between the Clifford–Dirac algebra $Cl^{R}(0,6)$ and algebra SO(8) is similar to the relationship between the standard Clifford–Dirac algebra $Cl^{C}(1,3)$ and algebra SO(3,3) found in [23]; for the relationship between $Cl^{C}(1,3)$ and SO(1,5) see in [18], [19].

Note that subalgebra SO(6)$\subset$SO(8) is the algebra of invariance of the Dirac equation in the FW representation [4], [5]. The 16 elements of this SO(6) algebra are given by

$$\left\{I,\ \alpha^{\underline{A}\,\underline{B}} = 2s^{\underline{A}\,\underline{B}}\right\}, \quad \underline{A},\underline{B} = \overline{1,6}, \tag{49}$$

where

$$\left\{s^{\underline{A}\,\underline{B}}\right\} = \left\{s^{\underline{A}\,\underline{B}} \equiv \frac{1}{4}\left[\Gamma^{\underline{A}}_{2N}, \Gamma^{\underline{B}}_{2N}\right]\right\}. \tag{50}$$

Now only the first 6 matrices

$$\left\{\Gamma^{\underline{A}}_{2N}\right\} = \left\{\Gamma^{1}_{2N}, \Gamma^{2}_{2N}, \Gamma^{3}_{2N}, \Gamma^{4}_{2N}, \Gamma^{5}_{2N}, \Gamma^{6}_{2N}\right\} \tag{51}$$

from the set (43) play the role of generating operators in the constructions (50), (51).

The maximal pure matrix algebra of invariance of the FW equation (34) is the 32-dimensional algebra $SO(6)\oplus i\Gamma^{0}_{2N}\cdot SO(6)\oplus i\Gamma^{0}_{2N}$.

The additional possibilities, which are open by the 29 orts of the algebra SO(8) in comparison with 16 orts of the well-known algebra SO(1,5), are principal in description of the Bose states in the framework of the Dirac theory [18], [19]. The algebra SO(8) includes two independent spin s=1/2 SU(2) subalgebras ($s^1 \equiv \frac{1}{2}\Gamma^{2}_{2N}\Gamma^{3}_{2N}$, $s^2 \equiv \frac{1}{2}\Gamma^{3}_{2N}\Gamma^{1}_{2N}$, $s^3 \equiv \frac{1}{2}\Gamma^{1}_{2N}\Gamma^{2}_{2N}$) and ($\check{s}^1 \equiv \frac{1}{2}\Gamma^{5}_{2N}\Gamma^{6}_{2N}$, $\check{s}^2 \equiv \frac{1}{2}\Gamma^{6}_{2N}\Gamma^{4}_{2N}$, $\check{s}^3 \equiv \frac{1}{2}\Gamma^{4}_{2N}\Gamma^{5}_{2N}$) (in the anti-Hermitian form), whereas

the SO(1,5) algebra includes only one set of SU(2) generators given by the elements ($s^1 \equiv \frac{1}{2}\Gamma^2_{2N}\Gamma^3_{2N}$, $s^2 \equiv \frac{1}{2}\Gamma^3_{2N}\Gamma^1_{2N}$, $s^3 \equiv \frac{1}{2}\Gamma^1_{2N}\Gamma^2_{2N}$). Moreover, the algebra SO(6), which is the algebra of invariance of the FW equation (34), includes these two independent SU(2) subalgebras as well. As long as these two spin s=1/2 SU(2) sets of generators commute between each other, their combination gives the generators of spin s=1 representation of SU(2) algebra. Such SU(2) algebra is the building element in construction of spin s=1 Lorentz and Poincaré algebras, with respect to which the FW equation (34) is invariant.

The transition to the RCQM is given by the transformation (30), (31). Thus, in the quantum-mechanical representation the 7 Γ matrices (43) (in the terms of standard $\Gamma^{\bar{\mu}}_{2N}$ matrices (34), (35)) have the form (38) together with

$$\bar{\Gamma}^5_{2N} \equiv \Gamma^1_{2N}\Gamma^3_{2N}\widehat{C}, \quad \bar{\Gamma}^6_{2N} \equiv -i\Gamma^2_{2N}\Gamma^4_{2N}\widehat{C}, \quad \bar{\Gamma}^7_{2N} \equiv i; \quad \bar{\Gamma}^1_{2N}\bar{\Gamma}^2_{2N}\bar{\Gamma}^3_{2N}\bar{\Gamma}^4_{2N}\bar{\Gamma}^5_{2N}\bar{\Gamma}^6_{2N}\bar{\Gamma}^7_{2N} = \mathrm{I}_{2N}, \quad (52)$$

and satisfy the anticommutation relations of the Clifford–Dirac algebra $\mathrm{Cl}^{\mathbb{R}}(0,6)$ representation in the following form

$$\bar{\Gamma}^{\mathrm{A}}_{2N}\bar{\Gamma}^{\mathrm{B}}_{2N} + \bar{\Gamma}^{\mathrm{B}}_{2N}\bar{\Gamma}^{\mathrm{A}}_{2N} = -2\delta^{\mathrm{AB}}, \quad \mathrm{A},\mathrm{B} = \overline{1,7}. \quad (53)$$

The RCQM representation of the algebra SO(8) is given by

$$\bar{s}^{\bar{\mathrm{A}}\bar{\mathrm{B}}} = \{\bar{s}^{\mathrm{AB}} = \frac{1}{4}[\bar{\Gamma}^{\mathrm{A}}_{2N}, \bar{\Gamma}^{\mathrm{B}}_{2N}],\ \bar{s}^{\mathrm{A8}} = -\bar{s}^{8\mathrm{A}} = \frac{1}{2}\bar{\Gamma}^{\mathrm{A}}_{2N}\}, \quad \bar{\mathrm{A}},\bar{\mathrm{B}} = \overline{1,8}, \quad (54)$$

where the matrices $\bar{\Gamma}^{\mathrm{A}}_{2N}$ are given in (38), (52) and generators $\bar{s}^{\bar{\mathrm{A}}\bar{\mathrm{B}}}$ satisfy the commutation relations

$$[\bar{s}^{\bar{\mathrm{A}}\bar{\mathrm{B}}}, \bar{s}^{\bar{\mathrm{C}}\bar{\mathrm{D}}}] = \delta^{\bar{\mathrm{A}}\bar{\mathrm{C}}}\bar{s}^{\bar{\mathrm{B}}\bar{\mathrm{D}}} + \delta^{\bar{\mathrm{C}}\bar{\mathrm{B}}}\bar{s}^{\bar{\mathrm{D}}\bar{\mathrm{A}}} + \delta^{\bar{\mathrm{B}}\bar{\mathrm{D}}}\bar{s}^{\bar{\mathrm{A}}\bar{\mathrm{C}}} + \delta^{\bar{\mathrm{D}}\bar{\mathrm{A}}}\bar{s}^{\bar{\mathrm{C}}\bar{\mathrm{B}}}. \quad (55)$$

The RCQM representation of the algebra SO(6)$\subset$SO(8) is given by

$$\left\{\mathrm{I},\ \alpha^{\underline{\mathrm{A}}\,\underline{\mathrm{B}}} = 2\bar{s}^{\underline{\mathrm{A}}\,\underline{\mathrm{B}}}\right\}, \quad \underline{\mathrm{A}},\underline{\mathrm{B}} = \overline{1,6}, \quad (56)$$

where

$$\left\{\bar{s}^{\underline{\mathrm{A}}\,\underline{\mathrm{B}}}\right\} = \left\{\bar{s}^{\underline{\mathrm{A}}\,\underline{\mathrm{B}}} \equiv \frac{1}{4}\left[\bar{\Gamma}^{\underline{\mathrm{A}}}_{2N}, \bar{\Gamma}^{\underline{\mathrm{B}}}_{2N}\right]\right\}. \quad (57)$$

The maximal pure matrix algebra of invariance of the Schrödinger–Foldy equation (13) is the 32-dimensional algebra a_{32} =SO(6)$\oplus i\cdot$SO(6)$\oplus i$.

In the RCQM representation two independent sets of generators of SU(2) subalgebra of SO(6) algebra, with respect to which the Schrödinger–Foldy equation (13) is invariant, have the form

$$\bar{s}^1 \equiv \frac{1}{2}\bar{\Gamma}^2_{2N}\bar{\Gamma}^3_{2N},\ \bar{s}^2 \equiv \frac{1}{2}\bar{\Gamma}^3_{2N}\bar{\Gamma}^1_{2N},\ \bar{s}^3 \equiv \frac{1}{2}\bar{\Gamma}^1_{2N}\bar{\Gamma}^2_{2N}, \quad (58)$$

$$\bar{\bar{s}}^1 \equiv \frac{1}{2}\bar{\Gamma}^5_{2N}\bar{\Gamma}^6_{2N},\ \bar{\bar{s}}^2 \equiv \frac{1}{2}\bar{\Gamma}^6_{2N}\bar{\Gamma}^4_{2N},\ \bar{\bar{s}} \equiv \frac{1}{2}\bar{\Gamma}^4_{2N}\bar{\Gamma}^5_{2N}, \quad (59)$$

where the matrices $\bar{\Gamma}^{\mathrm{A}}_{2N}$ are given in (38), (52). The situation here is similar to the FW representation. As long as these two spin s=1/2 SU(2) sets of generators commute between each other, their combination gives the generators of spin s=1 representation of SU(2) algebra. Such SU(2) algebra is the building element in construction of spin s=1 Lorentz and Poincaré algebras, with respect to which the Schrödinger–Foldy equation (13) is invariant. Note that formulae (58),

(59) are valid for the N-dimensional multiplet but only for the spin $s = 1/2$. The arbitrary spin cannot be written in terms of Γ_{2N} matrices as (58), (59).

The partial case of 4 component formalism and 4×4 γ matrices algebra follows from the above given consideration after corresponding substitutions 2N=4, etc. The improved consideration of this axiom in [1] for the spin 1/2 particle-antiparticle doublet should be taken from the text above as the corresponding particular case.

An important fact is that in Pauli–Dirac representation the 32 dimensional algebra $\mathrm{SO}(6) \oplus i\Gamma^0_{2\mathrm{N}} \cdot \mathrm{SO}(6) \oplus i\Gamma^0_{2\mathrm{N}}$ still is the algebra of invariance of the Dirac equation (or 2N component Dirac-like equation). The only difference is that in Pauli–Dirac representation the operators $s^{\underline{\mathrm{A}}\,\underline{\mathrm{B}}}$ and gamma matrices $\Gamma^{\mathrm{A}}_{2\mathrm{N}}$ are not pure matrix. They are the nonlocal pseudo-differential operators, which contain the operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$. For the standard 4 component Dirac equation the corresponding gamma matrices are given by (18)–(22) in [19].

Another interesting fact is that operator of the Dirac equation with Hamiltonian $H \equiv \gamma^0 \overrightarrow{\gamma} \cdot \overrightarrow{p} + \gamma^0 m - e^2/|\overrightarrow{x}|$ commutes with all 32 generators of the $\mathrm{SO}(6) \oplus i\gamma^0 \cdot \mathrm{SO}(6) \oplus i\gamma^0$ algebra, which elements are given in terms of gamma matrices from (18)–(22) in [19]. Therefore, the relativistic hydrogen atom has wide additional symmetries given by the 31 nontrivial generators from $\mathrm{SO}(6) \oplus i\gamma^0 \cdot \mathrm{SO}(6) \oplus i\gamma^0$. Moreover, the relativistic hydrogen atom has additional spin 1 symmetries, which for the free Dirac equation are known from [18], [19] and which are the consequences of the algebra $\mathrm{SO}(6) \oplus i\gamma^0 \cdot \mathrm{SO}(6) \oplus i\gamma^0$. Indeed, much more than SO(4) of V. Fock.

Note that consideration of this axiom above is given as the simple generalization of the correct spin 1/2 particle-antiparticle doublet formalism. In reality the dimensions of Clifford–Dirac (and SO(n)) algebras for the higher spin cases $2(2s+1)\geq8$ can be more higher then considered above 64-dimensional $\mathrm{Cl}^{\mathbb{R}}(4,2)$ and $\mathrm{Cl}^{\mathbb{R}}(0,6)$ algebras (and SO(8) algebra). The reason of this situation is the possibility to construct for $2(2s+1)\geq8$ cases the additional Γ matrices obeying the Clifford–Dirac anticommutation relations. The concrete examples of such additional Γ matrices can be found in [24], [25].

3.9 Briefly on other axioms

Other axioms of arbitrary spin RCQM are given in [9]. The list of these axioms is as follows. *On the main and additional conservation laws, on the stationary complete sets of operators, on the solutions of the Schrödinger-Foldy equation, on the mean value of the operators of observables, on the principles of heredity and the correspondence, on the second quantization (external axiom), on the physical interpretation.* All axioms of this section eventually need to be reconciled with three levels of description used in [1], [9] and [10]: RCQM, canonical FW and Dirac models. Nevertheless, this interesting problem cannot be considered in few pages. The readers of this paper can compare the axioms of RCQM given above with the main principles of the Dirac model given in B. Thaller's monograph [26] on the high mathematical level.

The author is grateful to A.K. Prykarpatski for useful discussion on Clifford algebras representations.

References

[1] V.M. Simulik. Covariant local field theory equations following from the relativistic canonical quantum mechanics of arbitrary spin. arXiv: 1409.2766v2 [quant-ph, hep-th] 19 Sep 2014, 67 p.

[2] P.A.M. Dirac. The quantum theory of the electron. Proc. R. Soc. Lond. A. 1928. Vol. 117. P. 610–624.

[3] P.A.M. Dirac. The principles of quantum mechanics, 4-th edition. – Clarendon Press: Oxford, 1958.

[4] L.L. Foldy, S.A. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit. Phys. Rev. 1950. Vol. 78. P. 29–36.

[5] L.L. Foldy. Synthesis of covariant particle equations. Phys. Rev. 1956. Vol. 102. P. 568–581.

[6] K. Kowalski, J. Rembielinski. Salpeter equation and probability current in the relativistic Hamiltonian quantum mechanics. Phys. Rev. A. 2011. Vol. 84. P. 012108 (1–11).

[7] V. Simulik, I. Krivsky, I. Lamer. Lagrangian for the spinor field in the Foldy–Wouthuysen representation. Int. J. Theor. Math. Phys. 2013. Vol. 3. P. 109–116.

[8] R.W. Brown, L.M. Krauss, P.L. Taylor. Leslie Lawrance Foldy. Physics Today. 2001. Vol. 54. P. 75–76.

[9] V.M. Simulik. General form of the covariant field equations of arbitrary spin and the relativistic canonical quantum mechanics. arXiv: 1509.0463v1 [quant-ph, hep-th] 12 Sep 2015, 42 p.

[10] V.M. Simulik. Derivation of the Dirac and Dirac-like equations of arbitrary spin from the corresponding relativistic canonical quantum mechanics. Ukr. J. Phys. 2015. Vol. 60. P. 985–1006.

[11] E.E. Salpeter. Mass corrections to the fine structure of hydrogen-like atoms. Phys. Rev. 1952. Vol. 87. P. 328–343.

[12] von J. Neumann. Mathematische Grundlagen der Quantenmechanik. – Verlag von Julius Springer: Berlin, 1932.

[13] V.S. Vladimirov. Methods of the theory of generalized functions. – Taylor and Francis: London, 2002.

[14] N.N. Bogoliubov, A.A. Logunov, I.T. Todorov. Introduction to axiomatic quantum field theory. – W.A. Benjamin, Inc.: Florida, 1975.

[15] P. Garbaczewski. Boson-fermion duality in four dimensions: comments on the paper of Luther and Schotte. Int. J. Theor. Phys. 1986. Vol. 25. P. 1193–1208.

[16] W.I. Fushchich, A.G. Nikitin. Symmetries of equations of quantum mechanics. – Allerton Press Inc.: New York, 1994.

[17] P. Lounesto. Clifford algebras and spinors, 2-nd edition. – Cambridge University Press: Cambridge, 2001.

[18] V.M. Simulik, I.Yu. Krivsky. Bosonic symmetries of the Dirac equation. Phys. Lett. A. 2011. Vol. 375. P. 2479–2483.

[19] V.M. Simulik, I.Yu. Krivsky, I.L. Lamer. Bosonic symmetries, solutions and conservation laws for the Dirac equation with nonzero mass. Ukr. J. Phys. 2013. Vol. 58. P. 523–533.

[20] J. Elliott, P. Dawber. Symmetry in Physics, vol.1. – Macmillian Press: London, 1979.

[21] B. Wybourne. Classical groups for Physicists. – John Wiley and sons: New York, 1974.

[22] N. Kh. Ibragimov. Invariant variational problems and conservation laws (remarks on Noether's theorem). Theor. Math. Phys. 1969. Vol. 1. P. 267–274.

[23] M. Petras. The SO(3,3) group as a common basis for Dirac's and Proca's equations. Czech. J. Phys. 1995. Vol. 46. P. 455–464.

[24] I. Yu. Krivsky, V.M. Simulik. The Dirac equation and spin 1 representations, a connection with symmetries of the Maxwell equations. Theor. Math. Phys. 1992. Vol. 90. P. 265–276.

[25] I. Yu. Krivsky, R.R. Lompay, V.M. Simulik. Symmetries of the complex Dirac–Kähler equation. Theor. Math. Phys. 2005. Vol. 143. P. 541–558.

[26] B. Thaller. The Dirac equation. – Springer: Berlin, 1992.

Manifestly covariant field equations for arbitrary spin as a consequences of relativistic canonical quantum mechanics

V.M. Simulik
vsimulik@gmail.com; Institute of Electron Physics of NAS, Uzhgorod, Ukraine.

Abstract

The operator link between the relativistic canonical quantum mechanics of arbitrary spin and the corresponding locally covariant field theory is suggested. The arbitrary spin manifestly covariant field equations that follow are considered. The resulting equations are compared with the known field equations for arbitrary spin. The advantages of the new equations are considered briefly. The extended Foldy–Wouthuysen transwormation between the 2(2s+1)-component field theory and the corresponding relativistic canonical quantum mechanics is put into consideration.

Keywords: relativistic quantum mechanics, Dirac-like equations, equations for arbitrary spin.

1 Introduction

The investigation of different particle equations for an arbitrary spin is the actual problem of contemporary theoretical physics. Here only the first-order particle and the field equations (together with their canonical nonlocal pseudo-differential representations) are considered. The second order equations (like the Klein–Gordon–Fock equation) are not the subject of this investigation.

Different approaches to the description of the field theory of an arbitrary spin can be found in [1]–[18]. As well as in the field theory of spinor s=(1/2,1/2) field Dirac [1] was the first in this activity. The activity of scientists from Belorussia is reviewed briefly in [14], [15]. My contribution is presented in [16]–[18]. Here and in [16]–[18] only the approach started in [4] is the basis for further application. Other results given in [1]–[3], [5]–[15] are not used here.

Note only some general deficiencies of the known equations for arbitrary spin [1]–[3], [5]–[15]. The consideration of the partial cases, when the substitution of the fixed value of spin is fulfilled, is not successful in all cases. For example, for the spin $s > 1$ existing equations have the redundant components and should be complemented by some additional conditions. Indeed, the known equations [19], [20] for the spin s=3/2 (and their confirmation in [21]) should be essentially complemented by the additional conditions. The main difficulty in the models of an arbitrary spin is the interaction between the fields of higher-spin. Even the quantization of higher-spin fields generated the questions. These and other deficiencies of the known equations for higher-spin are considered, e. g., in [22]–[31] (a brief review of deficiencies see in [26]).

Equations suggested in [16]–[18] and here are free of these deficiencies. The start of such consideration is taken from [4], where the main foundations of the relativistic canonical quantum mechanics (RCQM) are formulated. In the text of [16]–[18] and here the results of [4] are generalized and extended. The operator link between the results of [32] and [4] (between the canonical Foldy–Wouthuysen (FW) type field theory and the RCQM) is suggested. Note that

the cases s=3/2 and s=2 are not presented in [4], especially in explicit demonstrative forms. Some results of [16]–[18] are closest to the given in [3], [6]. The results of the section 3 below demonstrate the difference.

Even this brief analysis makes us sure in the prospects of the investigations started in [16]. The successful description of the arbitrary spin field models is not the solved problem today.

The main idea and main new feature of given here approach to the field theory of arbitrary spin is the start from the arbitrary spin RCQM.

The concepts, definitions and notations here are the same as in [16]. The pseudo-Euclidean metric of field theory is used.

2 Main assertions of the relativistic canonical quantum mechanics

The square-root operator equation $i\partial_t f(x) = \sqrt{m^2 - \Delta} f(x)$ for the free arbitrary spin particle-antiparticle doublet is considered in the form

$$(i\partial_0 - \widehat{\omega}) f(x) = 0, \tag{1}$$

where $f(x)$ is the M=2N-component wave function

$$f \equiv \mathrm{column}(f^1, f^2, ..., f^{2\mathrm{N}}), \quad \mathrm{N} = 2s+1, \quad f \in \mathrm{H}^{3,2\mathrm{N}}, \tag{2}$$

and the pseudo-differential (non-local) operator $\widehat{\omega}$ is given by

$$\widehat{\omega} \equiv \sqrt{\widehat{\overrightarrow{p}}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0, \quad \widehat{\overrightarrow{p}} \equiv (\widehat{p}^j) = -i\nabla, \quad \nabla \equiv (\partial_\ell). \tag{3}$$

The definition of action of the operator (3) can be found, e. g., in [4], [17], [18], [33]. Here $\mathrm{H}^{3,2\mathrm{N}}$ is the Hilbert space of 2N-component functions

$$\mathrm{H}^{3,2\mathrm{N}} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes 2\mathrm{N}} = \{f = (f^{2\mathrm{N}}) : \mathrm{R}^3 \to \mathrm{C}^{\otimes 2\mathrm{N}}; \quad \int d^3x |f(t, \overrightarrow{x})|^2 < \infty\}, \quad \mathrm{N} = 2s+1. \tag{4}$$

For the mathematical correctness of the consideration the application of the rigged Hilbert space

$$\mathrm{S}^{3,2\mathrm{N}} \equiv \mathrm{S}(\mathrm{R}^3) \times \mathrm{C}^{2\mathrm{N}} \subset \mathrm{H}^{3,2\mathrm{N}} \subset \mathrm{S}^{3,2\mathrm{N}*} \tag{5}$$

is necessary.

The general solution of the Schrödinger–Foldy equation of motion (1) (in the case of particle-antiparticle arbitrary spin doublet) has the form

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} a^{2\mathrm{N}}\left(\overrightarrow{k}\right) \mathrm{d}_{2\mathrm{N}}, \quad kx \equiv \omega t - \overrightarrow{k}\,\overrightarrow{x}, \quad \omega \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \tag{6}$$

where the orts of the N-dimensional Cartesian basis are given, e. g., in [16] by the formulae (10).

Let us consider briefly the outstanding role of the different *complete sets* of operators from the algebra of observables A_S. If one does not appeal to the complete sets of operators, then the solutions of the the Schrödinger-Foldy equation (1) are linked directly only with the Sturm-Liouville problem for the energy operator (3). In this case one comes to so-called "degeneration" of solutions. Recall that for an arbitrary complete sets of operators the notion of degeneration is absent in the Sturm-Liouville problem (see, e.g., [34]): only one state vector corresponds to any one point of the common spectrum of a complete set of operators. To wit, for a complete set of

operators there is a one to one correspondence between any point of the common spectrum and an eigenvector.

The *stationary complete sets* play the special role among the complete sets of operators. Recall that the stationary complete set is the set of all functionally independent mutually commuting operators, each of which commutes with the operator of energy (in our case with the operator (3)). The examples of the stationary complete sets in $\mathrm{H}^{3,2\mathrm{N}}$ are given by $(\widehat{\overrightarrow{p}},\ s_z \equiv s^3_{2\mathrm{N}},\ g)$, $(\overrightarrow{p},\ \overrightarrow{s}_{2\mathrm{N}} \cdot \overrightarrow{p},\ g)$, ets; g is the charge sign operator. The set $(\overrightarrow{x},\ s_z,\ g)$ is an example of non-stationary complete set. The $\overrightarrow{x}$-realization (4) of the space $\mathrm{H}^{3,2\mathrm{N}}$ and of quantum-mechanical Schrödinger-Foldy equation (1) are related just to this complete set.

For the goals of this paper the stationary complete set $(\widehat{\overrightarrow{p}},\ s_z \equiv s^3_{2\mathrm{N}},\ g)$ is chosen. The series of partial examples of equations on eigenvectors and eigenvalues for this stationary complete set is given in [16]. Consider here as an example the spin s=3/2 particle-antiparticle doublet case. Corresponding equations on eigenvectors and eigenvalues are given by

$$\widehat{\overrightarrow{p}} e^{-ikx} \mathrm{d}_{\bar{\mathrm{A}}} = \overrightarrow{k} e^{-ikx} \mathrm{d}_{\bar{\mathrm{A}}}, \quad \bar{\mathrm{A}} = \overline{1,8}, \tag{7}$$

$$s^3_8 \mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\ s^3_8 \mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2,\ s^3_8 \mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\ s^3_8 \mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4, \tag{8}$$

$$s^3_8 \mathrm{d}_5 = -\frac{3}{2}\mathrm{d}_5,\ s^3_8 \mathrm{d}_6 = -\frac{1}{2}\mathrm{d}_6,\ s^3_8 \mathrm{d}_7 = \frac{1}{2}\mathrm{d}_7,\ s^3_8 \mathrm{d}_8 = \frac{3}{2}\mathrm{d}_8,$$

$$g\mathrm{d}_1 = -\mathrm{d}_1,\ g\mathrm{d}_2 = -\mathrm{d}_2,\ g\mathrm{d}_3 = -\mathrm{d}_3,\ g\mathrm{d}_4 = -\mathrm{d}_4, \tag{9}$$

$$g\mathrm{d}_5 = +\mathrm{d}_5,\ g\mathrm{d}_6 = +\mathrm{d}_6,\ g\mathrm{d}_7 = +\mathrm{d}_7,\ g\mathrm{d}_8 = +\mathrm{d}_8,$$

where the Cartesian orts $\{\mathrm{d}_{\bar{\mathrm{A}}}\}$ are given by $\{\mathrm{d}_{\bar{\mathrm{A}}}\} = \{\delta^{\bar{\mathrm{B}}}_{\bar{\mathrm{A}}}\}$, $\bar{\mathrm{A}}, \bar{\mathrm{B}} = \overline{1,8}$.

Comparison of the given in equations (8) spin s=3/2 case with the spin s=2 case ((194) in [16]) shows that the general form is out of clear visualization. Thus, the appealing to such general form is absent here.

The interpretation of the amplitudes in the general solution (6) follows from equations of (7)–(9) type for the fixed value of arbitrary spin. In general, the functions $a^{2\mathrm{N}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes. Thus, the first half of functions $a^{2\mathrm{N}}(\overrightarrow{k})$ is the momentum-spin amplitudes of the particle with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(-)$ and corresponding spin projections, respectively. Further, the second (bottom) half of functions $a^{2\mathrm{N}}(\overrightarrow{k})$ is the momentum-spin amplitudes of the antiparticle with the momentum $\widehat{\overrightarrow{p}}$, sign of the charge $(+)$ and opposite spin projections, respectively.

Thus, the conclusion about the fermionic (or bosonic) spin features of the solution (6) (i. e. the interpretation of the solution (6)) follows from the equations on eigenvectors and eigenvalues (of (7)–(9) type) and the above given interpretation of the amplitudes.

The spin $\overrightarrow{s}$ of the particle-antiparticle doublet is the first in the list of the carriers of the internal degrees of freedom. Taking into account the Pauli principle and the fact that experimentally an antiparticle is observed as the mirror reflection of a particle, the operators of the charge sign and the spin of the arbitrary particle-antiparticle doublet are taken in the form

$$g \equiv -\Gamma^0_{2\mathrm{N}} \equiv -\sigma^3_{2\mathrm{N}} = \left| \begin{array}{cc} -\mathrm{I}_{\mathrm{N}} & 0 \\ 0 & \mathrm{I}_{\mathrm{N}} \end{array} \right|, \quad \overrightarrow{s}_{2\mathrm{N}} = \left| \begin{array}{cc} \overrightarrow{s}_{\mathrm{N}} & 0 \\ 0 & -\hat{C}\overrightarrow{s}_{\mathrm{N}}\hat{C} \end{array} \right|, \quad \mathrm{N} = 2s+1, \tag{10}$$

where $\Gamma^0_{2\mathrm{N}}$ is the $2\mathrm{N} \times 2\mathrm{N}$ Dirac Γ^0 matrix, $\sigma^3_{2\mathrm{N}}$ is the $2\mathrm{N} \times 2\mathrm{N}$ Pauli σ^3 matrix, $\hat{C}$ is the operator of complex conjugation in the form of $\mathrm{N} \times \mathrm{N}$ diagonal matrix, the operator of involution in $\mathrm{H}^{3,2\mathrm{N}}$, and I_{N} is $\mathrm{N} \times \mathrm{N}$ unit matrix.

Thus, the spin is given by the generators of SU(2) algebra! The spin matrices $\overrightarrow{s}_{2N}$ (10) satisfy the commutation relations

$$\left[s^j_{2N}, s^\ell_{2N}\right] = i\varepsilon^{j\ell n} s^n_{2N}, \quad \varepsilon^{123} = +1, \tag{11}$$

of the algebra of SU(2) group, where $\varepsilon^{j\ell n}$ is the Levi-Civita tensor and $s^j = \varepsilon^{j\ell n} s_{\ell n}$ are the Hermitian $2N \times 2N$ matrices (10) – the generators of a 2N-dimensional reducible representation of the spin group SU(2) (universal covering of the SO(3)⊂SO(1,3) group). The validity of these assertions is proved by numerous partial examples presented in [16].

The Schrödinger–Foldy equation (1) and the set of its solution $\{f\}$ (6) are invariant with respect to the irreducible unitary representation of the group $\mathcal{P}$, the $N \times N$ matrix-differential generators of which are given by the following nonlocal operators

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \quad \widehat{j}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n}, \tag{12}$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\left\{x_\ell, \widehat{\omega}\right\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \breve{s}_\ell\right), \tag{13}$$

where the orbital parts of the generators are not changed under the transition from one spin to another. Under such transitions only the spin parts (10), (11) of the expressions (12), (13) are changed. Indeed, the direct calculations visualize that generators (12), (13) commute with the operator of equation (1) and satisfy the commutation relations

$$[p_\mu, p_\nu] = 0,\ [p_\mu, j_{\rho\sigma}] = ig_{\mu\rho}p_\sigma - ig_{\mu\sigma}p_\rho,\ [j_{\mu\nu}, j_{\rho\sigma}] = -i\left(g_{\mu\rho}j_{\nu\sigma} + g_{\rho\nu}j_{\sigma\mu} + g_{\nu\sigma}j_{\mu\rho} + g_{\sigma\mu}j_{\rho\nu}\right). \tag{14}$$

of the Lie algebra of the Poincaré group $\mathcal{P}$. In formulae (12), (13), the SU(2)-spin generators $s^{\ell n}$ have particular specific forms for each representation of the SU(2) group (see the list of examples in [16]).

The corresponding Casimir operators have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_N, \quad W = w^\mu w_\mu = m^2 \overrightarrow{s}^2 = \mathrm{s}(\mathrm{s}+1)m^2 \mathrm{I}_N, \tag{15}$$

where I_N is the $N \times N$ unit matrix and s =1/2, 1, 3/2, 2, ...

Note that in the case s=0 only generators (16) form the Poincaré symmetry.

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \quad \widehat{m}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell, \quad \widehat{m}_{0\ell} = -\widehat{m}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\left\{x_\ell, \widehat{\omega}\right\}. \tag{16}$$

For our purposes the anticommutation relations of the Clifford–Dirac algebra generators are taken in the general form

$$\Gamma^{\bar{\mu}}_{2N}\Gamma^{\bar{\nu}}_{2N} + \Gamma^{\bar{\nu}}_{2N}\Gamma^{\bar{\mu}}_{2N} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu}, \bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \tag{17}$$

where $\Gamma^{\bar{\mu}}_{2N}$ are the $2N \times 2N$ Dirac $\Gamma^{\bar{\mu}}$ matrices ($2N \times 2N$ generalization of the Dirac 4×4 γ matrices), $\Gamma^4_{2N} \equiv \Gamma^0_{2N}\Gamma^1_{2N}\Gamma^2_{2N}\Gamma^3_{2N}$.

Other details of RCQM are given in this Proceedings in the paper *Relativistic canonical quantum mechanics of arbitrary spin.*

3 General description of the arbitrary spin field theory

The step by step consideration of the different partial examples in [16] (sections 21–27) enabled us to rewrite them in the general form, which is valid for arbitrary spin. Therefore, the generalization of the consideration given in [16] leads to the general formalism of the arbitrary spin fields.

The formalism presented below in this section is valid for an arbitrary particle-antiparticle multiplet in general and for the particle-antiparticle doublet in particular.

3.1 The canonical (Foldy–Wouthuysen type) model of the arbitrary spin particle-antiparticle field

The operator, which transform the RCQM of the arbitrary spin particle-antiparticle multiplet into the corresponding canonical FW type particle-antiparticle field, is given by

$$v_{2\mathrm{N}} = \left| \begin{array}{cc} \mathrm{I}_{\mathrm{N}} & 0 \\ 0 & \hat{C}\mathrm{I}_{\mathrm{N}} \end{array} \right|, \quad v_{2\mathrm{N}}^{-1} = v_{2\mathrm{N}}^{\dagger} = v_{2\mathrm{N}}, \quad v_{2\mathrm{N}} v_{2\mathrm{N}} = \mathrm{I}_{2\mathrm{N}}, \quad \mathrm{N} = 2s+1, \tag{18}$$

where $\hat{C}\mathrm{I}_{\mathrm{N}}$ is the $\mathrm{N} \times \mathrm{N}$ operator of complex conjugation. Indeed, the operator (18) translates any operator from canonical field FW representation into the RCQM representation and vice versa:

$$v_{2\mathrm{N}} \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}} v_{2\mathrm{N}} = \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}, \quad v_{2\mathrm{N}} \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}} v_{2\mathrm{N}} = \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}. \tag{19}$$

Here $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 2N-component particle-antiparticle doublet in the anti-Hermitian form, e. g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion (1), the operator of spin $\overrightarrow{s}_{2\mathrm{N}}$ (10) taken in anti-Hermitian form, etc., $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 2N-component particle-antiparticle doublet in the anti-Hermitian form. Thus, the only warning is that operators here must be taken in anti-Hermitian form, see section 9 in [16] for the details and see [35], [36] for the mathematical correctness of anti-Hermitian operators application.

Further, the operator (18) translates

$$\phi = v_{2\mathrm{N}} f, \quad f = v_{2\mathrm{N}} \phi, \tag{20}$$

the solution (6) of the Schrödinger–Foldy equation (1) into the solution

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} a^{\mathrm{N}}(\overrightarrow{k}) \mathrm{d}_{\mathrm{N}} + e^{ikx} a^{*\breve{\mathrm{N}}}(\overrightarrow{k}) \mathrm{d}_{\breve{\mathrm{N}}} \right], \tag{21}$$

$\mathrm{N} = 1, 2, ..., \mathrm{N}, \quad \breve{\mathrm{N}} = \mathrm{N}+1, \mathrm{N}+2, ..., 2\mathrm{N}$, of the FW equation

$$(i\partial_0 - \Gamma_{2\mathrm{N}}^0 \widehat{\omega})\phi(x) = 0, \quad \Gamma_{2\mathrm{N}}^0 \equiv \sigma_{2\mathrm{N}}^3 = \left| \begin{array}{cc} \mathrm{I}_{\mathrm{N}} & 0 \\ 0 & -\mathrm{I}_{\mathrm{N}} \end{array} \right|, \tag{22}$$

$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \mathrm{N} = 2s+1$, and vice versa. Thus, the transformation (18), (19) translates the matrices $\Gamma_{2\mathrm{N}}^0$ and

$$\Gamma_{2\mathrm{N}}^j = \left| \begin{array}{cc} 0 & \Sigma_{\mathrm{N}}^j \\ -\Sigma_{\mathrm{N}}^j & 0 \end{array} \right|, \quad j = 1, 2, 3, \tag{23}$$

which satisfy the anticommutation relations (17), into the RCQM representation (and vice versa)

$$\bar{\Gamma}_{2\mathrm{N}}^{\bar{\mu}} = v_{2\mathrm{N}} \Gamma_{2\mathrm{N}}^{\bar{\mu}} v_{2\mathrm{N}}, \tag{24}$$

where matrices $\bar{\Gamma}_{2\mathrm{N}}^{\bar{\mu}}$ satisfy the anticommutation relations

$$\bar{\Gamma}_{2\mathrm{N}}^{\bar{\mu}} \bar{\Gamma}_{2\mathrm{N}}^{\bar{\nu}} + \bar{\Gamma}_{2\mathrm{N}}^{\bar{\nu}} \bar{\Gamma}_{2\mathrm{N}}^{\bar{\mu}} = 2g^{\bar{\mu}\bar{\nu}}; \quad \bar{\mu}, \bar{\nu} = \overline{0,4}, \quad (g^{\bar{\mu}\bar{\nu}}) = (+----), \tag{25}$$

of the Clifford–Dirac algebra generators as well. In (23) Σ_{N}^j are the $\mathrm{N} \times \mathrm{N}$ Pauli matrices. The explicit forms of the RCQM representation of the $\bar{\Gamma}_{2\mathrm{N}}^{\bar{\mu}}$ matrices are given by

$$\bar{\Gamma}_{2\mathrm{N}}^0 = \Gamma_{2\mathrm{N}}^0, \quad \bar{\Gamma}_{2\mathrm{N}}^1 = \Gamma_{2\mathrm{N}}^1 \widehat{C}, \quad \bar{\Gamma}_{2\mathrm{N}}^2 = \Gamma_{2\mathrm{N}}^0 \Gamma_{2\mathrm{N}}^2 \widehat{C}, \quad \bar{\Gamma}_{2\mathrm{N}}^3 = \Gamma_{2\mathrm{N}}^3 \widehat{C}, \quad \bar{\Gamma}_{2\mathrm{N}}^4 = \Gamma_{2\mathrm{N}}^0 \Gamma_{2\mathrm{N}}^4 \widehat{C}, \tag{26}$$

where $\widehat{C}$ is the 2N × 2N operator of complex conjugation and matrices $\Gamma^{\bar{\mu}}_{2\mathrm{N}}$ are given in (22), (23).

Note that in the terms of $\bar{\Gamma}^{\bar{\mu}}_{2\mathrm{N}}$ matrices (26) the RCQM spin operator (10) has the form

$$\overrightarrow{s} = \frac{i}{2}(\bar{\Gamma}^2_{2\mathrm{N}}\bar{\Gamma}^3_{2\mathrm{N}},\ \bar{\Gamma}^3_{2\mathrm{N}}\bar{\Gamma}^1_{2\mathrm{N}},\ \bar{\Gamma}^1_{2\mathrm{N}}\bar{\Gamma}^2_{2\mathrm{N}}), \tag{27}$$

which is similar to the FW spin form, see formula (29) below. Note further that formula (27) is valid for the multiplets of arbitrary dimension but only for the spin s=1/2, whereas the formula (10) is valid for arbitrary spin.

The formulas mentioned below are found from the corresponding formulas of RCQM with the help of the operator (18) on the basis of its properties (19), (20). For the general form of arbitrary spin canonical particle-antiparticle field the equation of motion of the FW type is given by (22). The general solution has the form (21), where $a^{\mathrm{N}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the particle and $a^{\check{\mathrm{N}}}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle, {d} is 2N-component Cartesian basis.

It is evident from (21) that the model under consideration is not quantum mechanics. Indeed, contrary to (6) the solution (21) contains positive and negative frequency terms and, as a consequence, equation (22) is dealing with positive and negative energies (contrary to equation (1)).

The spin operator, which follows after the tansformation (18) from the RCQM arbitrary spin (10), has the form

$$\overrightarrow{s}^{\mathrm{FW}}_{2\mathrm{N}} = \begin{vmatrix} \overrightarrow{s}_{\mathrm{N}} & 0 \\ 0 & \overrightarrow{s}_{\mathrm{N}} \end{vmatrix}, \quad \mathrm{N} = 2s+1, \tag{28}$$

where $\overrightarrow{s}_{\mathrm{N}}$ are N × N generators of arbitrary spin irreducible representations of SU(2) algebra, which satisfy the commutation relations (11). Further, the spin operator, which follows after the tansformation (18) from the partial representation (27) is given by

$$\overrightarrow{s}^{\mathrm{FW}}_{2\mathrm{N}} = \frac{i}{2}(\Gamma^2_{2\mathrm{N}}\Gamma^3_{2\mathrm{N}},\ \Gamma^3_{2\mathrm{N}}\Gamma^1_{2\mathrm{N}},\ \Gamma^1_{2\mathrm{N}}\Gamma^2_{2\mathrm{N}}), \quad \mathrm{N} = 2s+1. \tag{29}$$

The representations (27), (29) are valid only for the case s=1/2, hence, not for arbitrary spin. The RCQM and canonical FW type arbitrary spin SU(2) operators are given by (10) and (28), respectively.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (22) and the set $\{\phi\}$ of its solutions (21) are invariant, are given by

$$\widehat{p}^0 = \Gamma^0_{2\mathrm{N}}\widehat{\omega} \equiv \Gamma^0_{2\mathrm{N}}\sqrt{-\Delta + m^2}, \quad \widehat{p}^\ell = -i\partial_\ell, \quad \widehat{j}^{\ell n} = x^\ell \widehat{p}^n - x^n\widehat{p}^\ell + s^{\ell n}_{2\mathrm{N}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{2\mathrm{N}}, \tag{30}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\Gamma^0_{2\mathrm{N}}\left\{x^\ell, \widehat{\omega}\right\} + \Gamma^0_{2\mathrm{N}}\frac{(\overrightarrow{s}_{2\mathrm{N}} \times \overrightarrow{p})^\ell}{\widehat{\omega} + m}, \tag{31}$$

where arbitrary spin SU(2) generators $\overrightarrow{s}_{2\mathrm{N}} = (s^{\ell n}_{2\mathrm{N}})$ have the form (28), $\Gamma^0_{2\mathrm{N}}$ is given in (22).

Note that together with the generators (30), (31) another set of 10 operators commutes with the operator of equation (22), satisfies the commutation relations (14) of the Lie algebra of Poincaré group $\mathcal{P}$, and, therefore, can be chosen as the Poincaré symmetry of the model under consideration. This second set is given by the generators $\widehat{p}^0$, $\widehat{p}^\ell$ from (30) together with the orbital parts of the generators $\widehat{j}^{\ell n}$, $\widehat{j}^{0\ell}$ from (30), (31), respectively. In another way this set follows from the set (16) after the transformation (18), (19).

The calculation of the Casimir operators $p^2 = \widehat{p}^\mu\widehat{p}_\mu$, $W = w^\mu w_\mu$ (w^μ is the Pauli–Lubanski pseudovector) for the fixed value of spin completes the brief description of the model.

3.2 The locally covariant model of the arbitrary spin particle-antiparticle field

The operator, which transform the canonical (FW type) model of the arbitrary spin particle-antiparticle field into the corresponding locally covariant particle-antiparticle field, is the generalized FW operator and is given by

$$V^{\mp}=\frac{\mp\overrightarrow{\Gamma}_{2\mathrm{N}}\cdot\overrightarrow{p}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\quad V^{-}=(V^{+})^{\dagger},\quad V^{-}V^{+}=V^{+}V^{-}=\mathrm{I}_{2\mathrm{N}},\quad \mathrm{N}=2s+1, \tag{32}$$

where $\Gamma^{j}_{2\mathrm{N}}$ are known from (23) and Σ^{j}_{N} are the $\mathrm{N}\times\mathrm{N}$ Pauli matrices.

Of course, for the matrices $\Gamma^{\mu}_{2\mathrm{N}}$ (22), (23) (together with the matrix $\Gamma^{4}_{2\mathrm{N}}\equiv\Gamma^{0}_{2\mathrm{N}}\Gamma^{1}_{2\mathrm{N}}\Gamma^{2}_{2\mathrm{N}}\Gamma^{3}_{2\mathrm{N}}$) the relations (17) are valid.

Note that in formulas (32) and in all formulas before the end of the subsection the values of N are only even. Therefore, the canonical field equation (22) describes the larger number of multiplets then the generalized Dirac equation (33) given below.

The formulas (33)–(38) below are found from the corresponding formulas (21), (22), (28), (30), (31) of canonical field model on the basis of the operator (32).

For the general form of arbitrary spin locally covariant particle-antiparticle field the Dirac-like equation of motion follows from the equation (22) after the transformation (32) and is given by

$$\left[i\partial_{0}-\Gamma^{0}_{2\mathrm{N}}(\overrightarrow{\Gamma}_{2\mathrm{N}}\cdot\overrightarrow{p}+m)\right]\psi(x)=0. \tag{33}$$

The general solution has the form

$$\psi(x)=V^{-}\phi(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^{3}k\left[e^{-ikx}a^{\mathrm{N}}(\overrightarrow{k})\mathrm{v}^{-}_{\mathrm{N}}(\overrightarrow{k})+e^{ikx}a^{*\check{\mathrm{N}}}(\overrightarrow{k})\mathrm{v}^{+}_{\check{\mathrm{N}}}(\overrightarrow{k})\right], \tag{34}$$

where amplitudes and notation $\check{\mathrm{N}}$ are the same as in (21); $\left\{\mathrm{v}^{-}_{\mathrm{N}}(\overrightarrow{k}),\,\mathrm{v}^{+}_{\check{\mathrm{N}}}(\overrightarrow{k})\right\}$ are 2N-component Dirac basis spinors with properties of orthonormalisation and completeness similar to 4-component Dirac spinors from [37].

The spin operator is given by

$$\overrightarrow{s}_{\mathrm{D}}=V^{-}\overrightarrow{s}_{2\mathrm{N}}V^{+}, \tag{35}$$

where operator $\overrightarrow{s}_{2\mathrm{N}}$ is known from (28). The explicit forms of few partial cases of spin operators (35) are given in formulae (259)–(261), (284)–(286), (359) of [16] for the particle-antiparticle multiplets s=(1,0,1,0), s=(3/2,3/2), s=(2,1,2,1), respectively.

The generators of the reducible unitary representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant field equation (33) and the set $\{\psi\}$ of its solutions (34) are invariant, have the form

$$\widehat{p}^{0}=\Gamma^{0}_{2\mathrm{N}}(\overrightarrow{\Gamma}_{2\mathrm{N}}\cdot\overrightarrow{p}+m),\quad \widehat{p}^{\ell}=-i\partial_{\ell},\quad \widehat{j}^{\ell n}=x^{\ell}_{\mathrm{D}}\widehat{p}^{n}-x^{n}_{\mathrm{D}}\widehat{p}^{\ell}+s^{\ell n}_{\mathrm{D}}\equiv\widehat{m}^{\ell n}+s^{\ell n}_{\mathrm{D}}, \tag{36}$$

$$\widehat{j}^{0\ell}=-\widehat{j}^{\ell 0}=x^{0}\widehat{p}^{\ell}-\frac{1}{2}\left\{x^{\ell}_{\mathrm{D}},\widehat{p}^{0}\right\}+\frac{\widehat{p}^{0}(\overrightarrow{s}_{\mathrm{D}}\times\overrightarrow{p})^{\ell}}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{37}$$

where the spin matrices $\overrightarrow{s}_{\mathrm{D}}=(s^{\ell n}_{\mathrm{D}})$ are given in (35) and the operator $\overrightarrow{x}_{\mathrm{D}}$ has the form

$$\overrightarrow{x}_{\mathrm{D}}=\overrightarrow{x}+\frac{i\overrightarrow{\Gamma}_{2\mathrm{N}}}{2\widehat{\omega}}-\frac{\overrightarrow{s}^{\Gamma}_{2\mathrm{N}}\times\overrightarrow{p}}{\widehat{\omega}(\widehat{\omega}+m)}-\frac{i\overrightarrow{p}(\overrightarrow{\Gamma}_{2\mathrm{N}}\cdot\overrightarrow{p})}{2\widehat{\omega}^{2}(\widehat{\omega}+m)}, \tag{38}$$

where specific spin matrices $\overrightarrow{s}^{\Gamma}_{2N}$ are given by

$$\overrightarrow{s}^{\Gamma}_{2N} \equiv \overrightarrow{s}_{FW} = \left(s^1_{2N}, s^2_{2N}, s^3_{2N}\right) = \frac{i}{2}(\Gamma^2_{2N}\Gamma^3_{2N},\ \Gamma^3_{2N}\Gamma^1_{2N},\ \Gamma^1_{2N}\Gamma^2_{2N}). \tag{39}$$

Note that for corresponding partial cases of $\overrightarrow{x}_D$ in [16] ((267) for s=(1,0,1,0), once more (267) but for s=(3/2,3/2), (333) for s=(2,0,2,0), (363) for s=(2,1,2,1)) the corresponding $\overrightarrow{s}^{\Gamma}_{2N}$ are given by

$$\overrightarrow{s}^{\Gamma}_{8} \equiv \left(s^1_8, s^2_8, s^3_8\right) = \frac{i}{2}(\Gamma^2_8\Gamma^3_8,\ \Gamma^3_8\Gamma^1_8,\ \Gamma^1_8\Gamma^2_8),\ \overrightarrow{s}^{\Gamma}_{12} \equiv \left(s^1_{12}, s^2_{12}, s^3_{12}\right) = \frac{i}{2}(\Gamma^2_{12}\Gamma^3_{12},$$

$$\Gamma^3_{12}\Gamma^1_{12},\ \Gamma^1_{12}\Gamma^2_{12}),\ \overrightarrow{s}^{\Gamma}_{16} \equiv \left(s^1_{16}, s^2_{16}, s^3_{16}\right) = \frac{i}{2}(\Gamma^2_{16}\Gamma^3_{16},\ \Gamma^3_{16}\Gamma^1_{16},\ \Gamma^1_{16}\Gamma^2_{16}).$$

It is easy to verify that the generators (36), (37) for any N commute with the operator of equation (33), and satisfy the commutation relations (14) of the Lie algebra of the Poincaré group. The last step in the brief description of the model is the calculation of the Casimir operators $p^2 = \widehat{p}^{\mu}\widehat{p}_{\mu}$, $W = w^{\mu}w_{\mu}$ (w^{μ} is the Pauli–Lubanski pseudovector) for the fixed value of spin.

3.3 The example of spin s=(0,0) particle-antiparticle doublet

The partial cases of the above given general arbitrary spin formalism are presented in [16]. The completeness of simplest spin multiplets and doublets consideration of [16] is achieved by the supplementation of this example. The formalism follows from the general formalism of arbitrary spin after the substitution s=0.

The Schrödinger–Foldy equation of RCQM is given by (1) for N=1, i. e. it is 2-component equation. The solution is given by (6) for N=1. The Poincaré group $\mathcal{P}$ generators, with respect to which the equation (1) for s=(0,0) is invariant, are given by (12), (13) taken in the form of 2×2 matrices with spin terms equal to zero, i. e. the corresponding generators are given by (16).

The corresponding FW type equation of canonical field theory is given by

$$(i\partial_0 - \sigma^3\widehat{\omega})\phi(x) = 0, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \widehat{\omega} \equiv \sqrt{-\Delta + m^2}. \tag{40}$$

The general solution is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx}a^1(\overrightarrow{k})\mathrm{d}_1 + e^{ikx}a^{*2}(\overrightarrow{k})\mathrm{d}_2\right]. \tag{41}$$

The Poincaré group $\mathcal{P}$ generators, with respect to which the equation (40) and the set $\{\phi\}$ of its solutions (41) are invariant, have the form

$$\widehat{p}^0 = \sigma^3\widehat{\omega} \equiv \sigma^3\sqrt{-\Delta + m^2}, \quad \widehat{p}^{\ell} = -i\partial_{\ell}, \quad \widehat{j}^{\ell n} = x^{\ell}\widehat{p}^{n} - x^{n}\widehat{p}^{\ell}, \tag{42}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^{\ell} - \frac{1}{2}\sigma^3\left\{x^{\ell}, \widehat{\omega}\right\}. \tag{43}$$

Generators (42), (43) are the partial 2×2 matrix form of operators (30), (31) taken with the spin terms equal to zero.

4 Brief conclusions

The field (particle) equations for an arbitrary spin given above are the direct consequences of the corresponding equations of RCQM. Such method of derivation of the covariant field equations is well-defined and new. Therefore, these equations are valid and new as well. Moreover, the comparison with the known arbitrary spin field equations demonstrates the difference and advantages of the found here equations. For example, in the partial case s=3/2 the equation (33) has 8 component and does not have redundand components. The Rarita-Schwinger equation has 16 components and essential edditional condition. The details are explained in [17], [18].

References

[1] P.A.M. Dirac. Relativistic wave equations. Proc. R. Soc. Lond. A. 1936. Vol. 155. P. 447–459.

[2] H.J. Bhabha. Relativistic wave equations for the elementary particles. Rev. Mod. Phys. 1945. Vol. 17. P. 200–216.

[3] V. Bargman, E.P. Wigner. Group theoretical discussion of relativistic wave equations. Proc. Nat. Acad. Sci. USA. 1948. Vol. 34. P. 211–223.

[4] L.L. Foldy. Synthesis of covariant particle equations. Phys. Rev. 1956. Vol. 102. P. 568–581.

[5] D.L. Weaver, C.L. Hammer and R.H. Good Jr. Description of a particle with arbitrary mass and spin. Phys. Rev. 1964. Vol. 135. P. B241–B248.

[6] D.L. Pursey. A Foldy-Wouthuysen transformation for particles of arbitrary spin. Nucl. Phys. 1964. Vol. 53. P. 174–176.

[7] D.L. Pursey. General theory of covariant particle equations. Ann. Phys. (N.Y.) 1965. Vol. 32. P. 157–191.

[8] P.M. Mathews. Relativistic Schrödinger equations for particles of arbitrary spin. Phys. Rev. 1966. Vol. 143. P. 978–985.

[9] T.J. Nelson, R.H. Good Jr. Second-quantization process for particles with any spin and with internal symmetry. Rev. Mod. Phys. 1968. Vol. 40. P. 508–522.

[10] R.F. Guertin. Relativistic Hamiltonian equations for any spin. Ann. Phys. (N.Y.) 1974. Vol. 88. P. 504–553.

[11] R.A. Kraicik, M.M. Nieto. Bhabha first-order wave equations. VI. Exact, closed-form, Foldy-Wouthuysen transformations and solutions. Phys. Rev. D. 1977. Vol. 15. P. 433–444.

[12] R-K. Loide, I. Ots, R. Saar. Bhabha relativistic wave equations. J. Phys. A. 1997. Vol. 30. P. 4005–4017.

[13] D.M. Gitman, A.L. Shelepin. Fields on the Poincaré group: arbitrary spin description and relativistic wave equations. Int. J. Theor, Phys. 2001. Vol. 40. P. 603–684.

[14] V.M. Red'kov. Fields in Riemannian space and the Lorentz group (in Russian). – Publishing House Belarusian Science: Minsk, 2009.

[15] V.M. Red'kov. Particle with spin s = 3/2 in Riemannian space-time. arXiv: 1109.3871v1 [math-ph] 18 Sep 2011, 12 p.

[16] V.M. Simulik. Covariant local field theory equations following from the relativistic canonical quantum mechanics of arbitrary spin. arXiv: 1409.2766v2 [quant-ph, hep-th] 19 Sep 2014, 67 p.

[17] V.M. Simulik. General form of the covariant field equations of arbitrary spin and the relativistic canonical quantum mechanics. arXiv: 1509.0463v1 [quant-ph, hep-th] 12 Sep 2015, 42 p.

[18] V.M. Simulik. Derivation of the Dirac and Dirac-like equations of arbitrary spin from the corresponding relativistic canonical quantum mechanics. Ukr. J. Phys. 2015. Vol. 60. P. 985–1006.

[19] M. Fierz, W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Proc. R. Soc. Lond. A. 1939. Vol. 173. P. 211–232.

[20] W. Rarita, J. Schwinger. On a theory of particles with half-integral spin. Phys. Rev. 1941. Vol. 60. P. 61–64.

[21] A.S. Davydov. Wave equations of a particle having spin 3/2 in the absence of a field (in Russian). J. Exper. Theor. Phys. 1943. Vol. 13. P. 313–319.

[22] K. Johnson, E.C.G. Sudarshan. Inconsistency of the local field theory of charged spin 3/2 particles. Ann. Phys. (N.Y.) 1961. Vol. 13. P. 126–145.

[23] G. Velo and D. Zwanziger. Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential. Phys. Rev. 1969. Vol. 186. P. 1337–1341.

[24] G. Velo and D. Zwanziger. Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. Phys. Rev. 1969. Vol. 188. P. 2218–2222.

[25] C. R. Hagen. New inconsistencies in the quantization of spin-$\frac{3}{2}$ fields. Phys. Rev. D. 1971. Vol. 4. P. 2204–2208.

[26] A.Z. Capri, R.L. Kobes. Further problems in spin-3/2 field theories. Phys. Rev. D. 1980. Vol. 22. P. 1967–1978.

[27] T. Darkhosh. Is there a solution to the Rarita-Schwinger wave equation in the presence of an external electromagnetic field? Phys. Rev. D. 1985. Vol. 32. P. 3251–3256.

[28] W. Cox. On the Lagrangian and Hamiltonian constraint algorithms for the Rarita-Schwinger field coupled to an external electromagnetic field. J. Phys. A. 1989. Vol. 22. P. 1599–1608.

[29] T. Pilling. Symmetry of massive Rarita-Schwinger fields. Int. J. Mod. Phys. A. 2005. Vol. 20. P. 2715–2742.

[30] M. Napsuciale, M. Kirchbach, S. Rodriguez. Spin 3/2 beyond the Rarita-Schwinger framework. Eur. Phys. J. A. 2006. Vol. 29. P. 289–306.

[31] A.E. Kaloshin, V.P. Lomov. Rarita-Schwinger field and multi-component wave equation. Particles and Nuclei, Letters. 2011. Vol. 8. P. 517–520.

[32] L.L. Foldy, S.A. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit. Phys. Rev. 1950. Vol. 78. P. 29–36.

[33] K. Kowalski, J. Rembielinski. Salpeter equation and probability current in the relativistic Hamiltonian quantum mechanics. Phys. Rev. A. 2011. Vol. 84. P. 012108 (1–11).

[34] N.N. Bogoliubov, A.A. Logunov, I.T. Todorov. Introduction to axiomatic quantum field theory. – W.A. Benjamin, Inc.: Florida, 1975.

[35] J. Elliott, P. Dawber. Symmetry in Physics, vol.1. – Macmillian Press: London, 1979.

[36] B. Wybourne. Classical groups for Physicists. – John Wiley and sons: New York, 1974.

[37] N.N. Bogoliubov, D.V. Shirkov. Introduction to the theory of quantized fields. – John Wiley and sons: New York, 1974.

Field equations for spin s=3/2 and s=2

V.M. Simulik
vsimulik@gmail.com; Institute of Electron Physics of NAS, Uzhgorod, Ukraine.

Abstract

New manifestly covariant relativistic equations for the particles with spin s=3/2 and s=2 are considered. The equations under consideration are the corresponding partial examples of the general equation for an arbitrary spin introduced in arXiv: 1509.0463v1 [quant-ph, hep-th] 12 Sep 2015 and in Ukr. J. Phys. 2015, 60, 985. The property of the Fermi–Bose triality of the field equation for spin s=(3/2,3/2) particle-antiparticle doublet is discussed briefly. The hypothesis that the particle with spin s=5/2 exists in reality is suggested.

Keywords: relativistic quantum mechanics, Dirac-like equations, equations for arbitrary spin, equations for spin s=3/2 and s=2.

1 Introduction

The partial cases of the general approach suggested in [1]–[3] are under consideration. Here in Proceedings of the conference this general approach is presented in two articles: *Relativistic canonical quantum mechanics of arbitrary spin* and *Manifestly covariant field equations for arbitrary spin as a consequences of relativistic canonical quantum mechanics*. Below in the text the case s=3/2 is considered in details and the case s=2 is considered briefly.

2 Spin s=(3/2,3/2) particle-antiparticle doublet in quantum mechanics

This model is constructed in complete analogy with the relativistic canonical quantum mechanics (RCQM) of the 4-component spin s=(1/2,1/2) particle-antiparticle doublet [1]. The difference is only in the dimensions of the corresponding spaces and matrices. The model below is useful for the Σ-hyperon description.

The 8-component fermionic spin s=(3/2,3/2) particle-antiparticle doublet is constructed as the direct sum of the two spin s=3/2 singlets. The spin s=3/2 singlet description can be found in the section 5 of [1]. The most important fact is that here the link with the Dirac-like equation is similar to that between the spin s=(1/2,1/2) particle-antiparticle doublet and the standard 4-component Dirac equation. Only the dimension of the matrices is changed. Therefore, the spin s=(3/2,3/2) particle-antiparticle doublet is of special interest.

Thus, the Schrödinger–Foldy equation has the form

$$(i\partial_0 - \widehat{\omega})f(x) = 0, \quad f = \text{column}\left(f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8\right), \tag{1}$$

where the operator $\widehat{\omega}$ is given by

$$\widehat{\omega} \equiv \sqrt{\widehat{\overrightarrow{p}}^2 + m^2} = \sqrt{-\Delta + m^2} \geq m > 0, \quad \widehat{\overrightarrow{p}} \equiv (\widehat{p}^j) = -i\nabla, \quad \nabla \equiv (\partial_\ell).$$

The space of the states is as follows

$$\mathrm{S}^{3,8} \subset \mathrm{H}^{3,8} \subset \mathrm{S}^{3,8*}. \tag{2}$$

The general solution of the equation (1) for the spin s=(3/2,3/2) particle-antiparticle doublet is given by

$$f(x) = \left| \begin{array}{c} f_{\mathrm{part}} \\ f_{\mathrm{antipart}} \end{array} \right| = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{-ikx} b^{\mathrm{A}}(\overrightarrow{k}) \mathrm{d}_{\mathrm{A}}, \quad \mathrm{A} = \overline{1,8}, \tag{3}$$

where the orts of the 8-component Cartesian basis have the form

$$\mathrm{d}_1 = \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_2 = \left| \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_3 = \left| \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_4 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_5 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_6 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array} \right|, \mathrm{d}_7 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array} \right|, \mathrm{d}_8 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right|. \tag{4}$$

The generators of the corresponding SU(2)-spin that satisfy the commutation relations of the SU(2) algebra are as follows

$$\overrightarrow{s}_8 = \left| \begin{array}{cc} \overrightarrow{s} & 0 \\ 0 & -C\overrightarrow{s}C \end{array} \right|, \quad \left[s_8^j, s_8^\ell \right] = i\varepsilon^{j\ell n} s_8^n, \tag{5}$$

where $C\mathrm{I}_4$ is the diagonal 4×4 operator of the complex conjugation and the matrices $\overrightarrow{s}$ for the single spin s=3/2 particle are given in [1], see the formulas (39) therein. In the explicit form the SU(2) spin operators (5) are given by

$$s_8^1 = \frac{1}{2} \left| \begin{array}{cccccccc} 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \end{array} \right|, \tag{6}$$

$$s_8^2 = \frac{i}{2} \left| \begin{array}{cccccccc} 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{array} \right|, s_8^3 = \frac{1}{2} \left| \begin{array}{cccccccc} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array} \right|.$$

The Casimir operator has the form of the following 8×8 diagonal matrix

$$\overrightarrow{s}^2 = \frac{15}{4} \mathrm{I}_8 = \frac{3}{2} \left(\frac{3}{2} + 1 \right) \mathrm{I}_8, \tag{7}$$

where I_8 is the 8×8 unit matrix.

The stationary complete set of operators is given by

$$g = \left| \begin{array}{cc} -\mathrm{I}_4 & 0 \\ 0 & \mathrm{I}_4 \end{array} \right|, \quad \widehat{\overrightarrow{p}} \equiv (\widehat{p}^j) = -i\partial_j, \quad s^3_8 \Rightarrow (6)\,, \tag{8}$$

where g is the charge sign operator, $\widehat{\overrightarrow{p}}$ is the momentum operator and $s^3_8 = s_z$ is the operator of the spin (5) projection on the axe z.

The equations on the eigenvalues of the operators $\widehat{\overrightarrow{p}}$, g, $s^3_8 = s_z$ have the form

$$\widehat{\overrightarrow{p}} e^{-ikx} \mathrm{d}_{\bar{\mathrm{A}}} = \overrightarrow{k} e^{-ikx} \mathrm{d}_{\bar{\mathrm{A}}}, \quad \bar{\mathrm{A}} = \overline{1,8}, \tag{9}$$

$$\begin{array}{c} g\mathrm{d}_1 = -\mathrm{d}_1,\, g\mathrm{d}_2 = -\mathrm{d}_2,\, g\mathrm{d}_3 = -\mathrm{d}_3,\, g\mathrm{d}_4 = -\mathrm{d}_4, \\ g\mathrm{d}_5 = +\mathrm{d}_5, g\mathrm{d}_6 = +\mathrm{d}_6,\, g\mathrm{d}_7 = +\mathrm{d}_7,\, g\mathrm{d}_8 = +\mathrm{d}_8, \end{array} \tag{10}$$

$$\begin{array}{c} s^3_8\mathrm{d}_1 = \frac{3}{2}\mathrm{d}_1,\, s^3_8\mathrm{d}_2 = \frac{1}{2}\mathrm{d}_2,\, s^3_8\mathrm{d}_3 = -\frac{1}{2}\mathrm{d}_3,\, s^3_8\mathrm{d}_4 = -\frac{3}{2}\mathrm{d}_4, \\ s^3_8\mathrm{d}_5 = -\frac{3}{2}\mathrm{d}_5, s^3_8\mathrm{d}_6 = -\frac{1}{2}\mathrm{d}_6,\, s^3_8\mathrm{d}_7 = \frac{1}{2}\mathrm{d}_7, s^3_8\mathrm{d}_8 = \frac{3}{2}\mathrm{d}_8 \end{array} \tag{11}$$

Therefore, the functions $b^1(\overrightarrow{k})$, $b^2(\overrightarrow{k})$, $b^3(\overrightarrow{k})$, $b^4(\overrightarrow{k})$ in solution (3) are the quantum-mechanical momentum-spin amplitudes of the massive fermion with the spin s=3/2 and the spin projection $(3/2, 1/2, -1/2, -3/2)$, respectively; $b^5(\overrightarrow{k})$, $b^6(\overrightarrow{k})$, $b^7(\overrightarrow{k})$, $b^8(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle (antifermion) with the spin s=3/2 and the spin projection $(-3/2, -1/2, 1/2, 3/2)$, respectively.

The Schrödinger–Foldy equation (1) (and the set $\{\mathrm{f}\}$ of its solutions (3)) is invariant with respect to the reducible unitary fermionic representation of the Poincaré group $\mathcal{P}$, whose Hermitian 8×8 matrix-differential generators are given by

$$\widehat{p}_0 = \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \widehat{p}_\ell = i\partial_\ell, \quad \widehat{j}_{\ell n} = x_\ell \widehat{p}_n - x_n \widehat{p}_\ell + s_{ln} \equiv \widehat{m}_{\ell n} + s_{\ell n}, \tag{12}$$

$$\widehat{j}_{0\ell} = -\widehat{j}_{\ell 0} = t\widehat{p}_\ell - \frac{1}{2}\left\{x_\ell, \widehat{\omega}\right\} - \left(\frac{s_{\ell n}\widehat{p}_n}{\widehat{\omega} + m} \equiv \breve{s}_\ell\right), \tag{13}$$

where the spin s=(3/2,3/2) SU(2) generators $s^{\ell n} \equiv s^{\ell n}_8$ are given in (6).

The Casimir operators of this reducible fermionic spin s=(3/2,3/2) representation of the group $\mathcal{P}$ have the form

$$p^2 = \widehat{p}^\mu \widehat{p}_\mu = m^2 \mathrm{I}_8, \quad W = w^\mu w_\mu = m^2 \overrightarrow{s}^2_8 = \frac{3}{2}\left(\frac{3}{2} + 1\right) m^2 \mathrm{I}_8, \tag{14}$$

where I_8 is the 8×8 unit matrix.

Thus, above the foundations of the RCQM of the 8-component spin s=(3/2,3/2) particle-antiparticle doublet are considered briefly. It is the basis for the transition to the covariant local field theory of the spin s=(3/2,3/2) particle-antiparticle doublet given below.

3 Equation for spin s=(3/2,3/2) canonical Foldy–Wouthuysen type particle-antiparticle field

The second step is the transition from the Schrödinger–Foldy equation (1) to the corresponding canonical Foldy–Wouthuysen (FW) type particle-antiparticle field equation. The similar

equation for the spin s=(1/2,1/2) particle-antiparticle field is well known as the FW equation [4]. This step is possible only for the anti-Hermitian form of the operators. The mathematical correctness is demonstrated in [1] and the appealing to the anti-Hermitian form of the operators is explained in [5], [6]. Nevertheless, the resulting operators can be turned back to the standard Hermitian form and do not contain the operator C of complex conjugation.

Thus, the canonical field equation for the 8-component spin s=(3/2,3/2) fermionic particle-antiparticle doublet (8 component analogy of the FW equation) and its solution are found from the the Schrödinger–Foldy equation (1) and solution (3) on the basis of the transformation

$$v_8 = \left| \begin{array}{cc} \mathrm{I}_4 & 0 \\ 0 & C\mathrm{I}_4 \end{array} \right|, \quad v_8^{-1} = v_8^{\dagger} = v_8, \quad v_8 v_8 = \mathrm{I}_8, \tag{15}$$

$$\phi = v_8 f, \quad f = v_8 \phi, \quad v_8 \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}} v_8 = \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}, \quad v_8 \hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}} v_8 = \hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}. \tag{16}$$

Here $\hat{q}_{\mathrm{qm}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the RCQM of the 8-component particle-antiparticle doublet in the anti-Hermitian form, e. g., the operator $(\partial_0 + i\widehat{\omega})$ of equation of motion, the operator of spin (6), etc., $\hat{q}_{\mathrm{cf}}^{\mathrm{anti-Herm}}$ is an arbitrary operator from the canonical field theory of the 8-component particle-antiparticle doublet in the anti-Hermitian form, $C\mathrm{I}_4$ is the 4×4 operator of complex conjugation.

Hence, the corresponding canonical field equation (consequence of the equation (1) and transformation (15)) is found in the form

$$(i\partial_0 - \Gamma_8^0 \widehat{\omega})\phi(x) = 0, \quad f = \mathrm{column}\left(\phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8\right), \tag{17}$$

where

$$\widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \quad \Gamma_8^0 = \left| \begin{array}{cc} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{array} \right|, \quad \mathrm{I}_4 = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right|. \tag{18}$$

The general solution of this equation in the case of spin s=(3/2,3/2) fermionic particle-antiparticle doublet is found with the help of the transformation v_8 (15), (16) from the general solution (3) of the Schrödinger–Foldy equation (1) and is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx} b^{\mathrm{A}}(\overrightarrow{k}) \mathrm{d}_{\mathrm{A}} + e^{ikx} b^{*\mathrm{B}}(\overrightarrow{k}) \mathrm{d}_{\mathrm{B}} \right], \tag{19}$$

where $\mathrm{A} = \overline{1,4}$, $\mathrm{B} = \overline{5,8}$, the orts of the 8-component Cartesian basis are given in (4) and the quantum-mechanical interpretation of the amplitudes ($b^{\mathrm{A}}(\overrightarrow{k})$, $b^{*\mathrm{B}}(\overrightarrow{k})$) is given according to (9)–(11).

The SU(2) spin operators, which satisfy the commutation relations from (5) and commute with the operator $(i\partial_0 - \Gamma_8^0 \widehat{\omega})$ of the equation of motion (17), are derived from the corresponding RCQM operators (5), (6) on the basis of transformations (15), (16). These canonical field spin operators are given by

$$\overrightarrow{s}_8 = \left| \begin{array}{cc} \overrightarrow{s} & 0 \\ 0 & \overrightarrow{s} \end{array} \right|, \quad \overrightarrow{s}_8^2 = \frac{3}{2}\left(\frac{3}{2} + 1\right) \mathrm{I}_8, \tag{20}$$

where the 4×4 operators $\overrightarrow{s}$ have the form

$$s^1=\frac{1}{2}\begin{vmatrix} 0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix},\ s^2=\frac{i}{2}\begin{vmatrix} 0 & -\sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & -2 & 0\\ 0 & 2 & 0 & -\sqrt{3}\\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix},\ s^3=\frac{1}{2}\begin{vmatrix} 3 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -3 \end{vmatrix}. \quad (21)$$

The stationary complete set of operators is given by the operators g, $\overrightarrow{p}$, $s^3_8=s_z$ of the charge sign, momentum and spin projection, respectively (see section 2 for details). The equations on the eigenvectors and eigenvalues of the spin projection operator $s^3_8=s_z$ from (20) have the form

$$s^3_8\mathrm{d}_1=\frac{3}{2}\mathrm{d}_1,\ s^3_8\mathrm{d}_2=\frac{1}{2}\mathrm{d}_2,\ s^3_8\mathrm{d}_3=-\frac{1}{2}\mathrm{d}_3,\ s^3_8\mathrm{d}_4=-\frac{3}{2}\mathrm{d}_4,$$
$$s^3_8\mathrm{d}_5=\frac{3}{2}\mathrm{d}_5,\ s^3_8\mathrm{d}_6=\frac{1}{2}\mathrm{d}_6,\ s^3_8\mathrm{d}_7=-\frac{1}{2}\mathrm{d}_7,\ s^3_8\mathrm{d}_8=-\frac{3}{2}\mathrm{d}_8. \quad (22)$$

Therefore, the functions $b^1(\overrightarrow{k})$, $b^2(\overrightarrow{k})$, $b^3(\overrightarrow{k})$, $b^4(\overrightarrow{k})$ in solution (19) are the momentum-spin amplitudes of the massive fermion with s=3/2 and the spin projection $(3/2,1/2,-1/2,-3/2)$, respectively; $b^5(\overrightarrow{k})$, $b^6(\overrightarrow{k})$, $b^7(\overrightarrow{k})$, $b^8(\overrightarrow{k})$ are the momentum-spin amplitudes of the antiparticle (antifermion) with the spin s=3/2 and the spin projection $(3/2,1/2,-1/2,-3/2)$, respectively.

Note that direct quantum-mechanical interpretation of the amplitudes in solution (19) should be given in the framework of the RCQM. Such interpretation is already given in section 2 in paragraph after equations (11).

The generators of the reducible unitary fermionic spin s=(3/2,3/2) doublet representation of the Poincaré group $\mathcal{P}$, with respect to which the canonical field equation (17) and the set $\{\phi\}$ of its solutions (19) are invariant, are derived from the RCQM set of generators (12), (13) with the spin (5), (6) on the basis of the transformations (15), (16). These Hermitian 8×8 matrix-differential generators are given by

$$\widehat{p}^0=\Gamma^0_8\widehat{\omega}\equiv\Gamma^0_8\sqrt{-\Delta+m^2},\quad \widehat{p}^\ell=-i\partial_\ell,\quad \widehat{j}^{\ell n}=x^\ell\widehat{p}^n-x^n\widehat{p}^\ell+s^{\ell n}_8\equiv\widehat{m}^{\ell n}+s^{\ell n}_8, \quad (23)$$

$$\widehat{j}^{0\ell}=-\widehat{j}^{\ell 0}=x^0\widehat{p}^\ell-\frac{1}{2}\Gamma^0_8\left\{x^\ell,\widehat{\omega}\right\}+\Gamma^0_8\frac{(\overrightarrow{s}_8\times\overrightarrow{p})^\ell}{\widehat{\omega}+m}, \quad (24)$$

where the explicit form of SU(2) spin s=(3/2,3/2) is given in (20).

It is easy to prove by the direct verification that the generators (23), (24) commute with the operator $(i\partial_0-\Gamma^0_8\widehat{\omega})$ of the canonical field equation (17) and satisfy the commutation relations of the Lie algebra of the Poincaré group $\mathcal{P}$. The Casimir operators for the representation (23), (24) with SU(2) spin (20) are given by

$$p^2=\widehat{p}^\mu\widehat{p}_\mu=m^2\mathrm{I}_8,\quad W=w^\mu w_\mu=m^2\overrightarrow{s}^2_8=\frac{3}{2}\left(\frac{3}{2}+1\right)\mathrm{I}_8. \quad (25)$$

Thus, due to the eigenvalues in equations (22), positive and negative frequencies form of solution (19) and the Bargman–Wigner analysis of the Casimir operators (25) one can come to a conclusion that equation (17) describes the 8-component canonical field (the fermionic particle-antiparticle doublet) with the spins s=(3/2,3/2) and $m>0$.

4 Covariant equation for the spin s=(3/2,3/2) field

The operator of transition to the covariant local field theory representation (the 8×8 analogy of the 4×4 FW transformation operator [4]) is given by

$$V^{\mp}=\frac{\mp\overrightarrow{\Gamma}_8\cdot\overrightarrow{p}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\quad V^-V^+=V^+V^-=\mathrm{I}_8,\quad V^-=(V^+)^\dagger, \quad (26)$$

$$\psi = V^{-}\phi,\ \phi = V^{+}\psi, \quad \hat{q}_{\rm D} = V^{-}\hat{q}_{\rm CF}V^{+},\ \hat{q}_{\rm CF} = V^{+}\hat{q}_{\rm D}V^{-}, \tag{27}$$

where $\hat{q}_{\rm D}$ is an arbitrary operator (both in the Hermitian and anti-Hermitian form) in the covariant local field theory representation and $\hat{q}_{\rm CF}$ is an arbitrary operator in canonical FW type representation. The inverse transformation is valid as well.

Thus, on the basis of transformation (26), (27) the 8-component Dirac-like equation is found from the canonical field equation (17) in the form

$$\left[i\partial_0 - \Gamma^0_8(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m)\right]\psi(x) = 0. \tag{28}$$

In formula (26) and in equation (28) the Γ^{μ}_8 (and their Σ^j) matrices are given by

$$\Gamma^0_8 = \left|\begin{array}{cc} {\rm I}_4 & 0 \\ 0 & -{\rm I}_4 \end{array}\right|, \quad \Gamma^j_8 = \left|\begin{array}{cc} 0 & \Sigma^j \\ -\Sigma^j & 0 \end{array}\right|, \quad \Sigma^j = \left|\begin{array}{cc} \sigma^j & 0 \\ 0 & \sigma^j \end{array}\right|, \tag{29}$$

where σ^j are the standard 2×2 Pauli matrices. The matrices Σ^j satisfy the similar commutation relations as the standard 2×2 Pauli matrices and have other similar properties. The matrices Γ^{μ}_8 (29) satisfy the anticommutation relations of the Clifford–Dirac algebra in the form

$$\Gamma^{\mu}_8\Gamma^{\nu}_8 + \Gamma^{\nu}_8\Gamma^{\mu}_8 = 2g^{\mu\nu}. \tag{30}$$

Note that equation (28) is not the ordinary direct sum of the two Dirac equations. Therefore, it is not the complex Dirac–Kähler equation [7]. Moreover, it is not the standard 16-component Ivanenko–Landau–Dirac–Kähler equation [8], [9]. Furthermore, for the same reasons it is not the spin 3/2 case of the Bargman–Wigner equation from [10], [11] and not the 16-component Rarita–Schwinger equation [12], [13], the last fact is evident.

The solution of equation (28) is derived on the basis of transformation (26), (27) from the solution (19) of the FW type equation (17) and is given by

$$\psi(x) = V^{-}_8\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\left[e^{-ikx}b^{\rm A}(\overrightarrow{k}){\rm v}^{-}_{\rm A}(\overrightarrow{k}) + e^{ikx}b^{*\rm B}(\overrightarrow{k}){\rm v}^{+}_{\rm B}(\overrightarrow{k})\right], \tag{31}$$

where ${\rm A} = \overline{1,4}$, ${\rm B} = \overline{5,8}$ and the 8-component spinors $({\rm v}^{-}_{\rm A}(\overrightarrow{k}),\ {\rm v}^{+}_{\rm B}(\overrightarrow{k}))$ are given by (257) in [1]. The spinors $({\rm v}^{-}_{\rm A}(\overrightarrow{k}),\ {\rm v}^{+}_{\rm B}(\overrightarrow{k}))$ satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors, see, e. g., [14].

In the covariant local field theory, the operators of the SU(2) spin, which satisfy the corresponding commutation relations $\left[s^j_{\rm 8D}, s^{\ell}_{\rm 8D}\right] = i\varepsilon^{j\ell n}s^n_{\rm 8D}$ and commute with the operator of equation (28), are derived from the pure matrix operators (20) with the help of transition operator (26), (27). The explicit form of these s=(3/2,3/2) SU(2) generators was given already by formulae (284)–(287) in [1].

The equations on eigenvectors and eigenvalues of the operator $s^3_{\rm 8D}$ (286) in [1] follow from the equations (22) and the transformation $V^{\mp}_8$ (26). In addition to it, the action of the operator $s^3_{\rm 8D}$ (286) of [1] for the spinors $({\rm v}^{-}_{\rm A}(\overrightarrow{k}),\ {\rm v}^{+}_{\rm B}(\overrightarrow{k}))$ (257) of [1] also leads to the result

$$s^3_{\rm 8D}{\rm v}^{-}_1(\overrightarrow{k}) = \frac{3}{2}{\rm v}^{-}_1(\overrightarrow{k}),\ s^3_{\rm 8D}{\rm v}^{-}_2(\overrightarrow{k}) = \frac{1}{2}{\rm v}^{-}_2(\overrightarrow{k}),\ s^3_{\rm 8D}{\rm v}^{-}_3(\overrightarrow{k}) =$$

$$= -\frac{1}{2}{\rm v}^{-}_3(\overrightarrow{k}),\ s^3_{\rm 8D}{\rm v}^{-}_4(\overrightarrow{k}) = -\frac{3}{2}{\rm v}^{-}_4(\overrightarrow{k}),$$

$$s^3_{8\mathrm{D}}\mathrm{v}^+_5(\overrightarrow{k}) = \frac{3}{2}\mathrm{v}^+_5(\overrightarrow{k}), s^3_{8\mathrm{D}}\mathrm{v}^+_6(\overrightarrow{k}) = \frac{1}{2}\mathrm{v}^+_6(\overrightarrow{k}),$$
$$s^3_{8\mathrm{D}}\mathrm{v}^+_7(\overrightarrow{k}) = -\frac{1}{2}\mathrm{v}^+_7(\overrightarrow{k}),\ s^3_{8\mathrm{D}}\mathrm{v}^+_8(\overrightarrow{k}) = -\frac{3}{2}\mathrm{v}^+_8(\overrightarrow{k}). \tag{32}$$

In order to verify equations (32) the identity $(\tilde{\omega}+m)^2+(\overrightarrow{k})^2 = 2\tilde{\omega}(\tilde{\omega}+m)$ is used. In the case $\mathrm{v}^+_{\mathrm{B}}(\overrightarrow{k})$ in the expression $s^3_{8\mathrm{D}}(\overrightarrow{k})$ (286) of [1] the substitution $\overrightarrow{k} \to -\overrightarrow{k}$ is made.

The equations (32) determine the interpretation of the amplitudes in solution (31). Nevertheless, the direct quantum-mechanical interpretation of the amplitudes should be made in the framework of the RCQM and is already given in section 2 in paragraph after equations (11).

The explicit form of the $\mathcal{P}$-generators of the fermionic representation of the Poincaré group $\mathcal{P}$, with respect to which the covariant equation (28) and the set $\{\psi\}$ of its solutions (31) are invariant, is derived from the generators (23), (24) in canonical representation with the help of the transformation (26), (27). The corresponding generators are given by

$$\widehat{p}^0 = \Gamma^0_8(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m), \quad \widehat{p}^\ell = -i\partial_\ell, \quad \widehat{j}^{\ell n} = x^\ell_{\mathrm{D}}\widehat{p}^n - x^n_{\mathrm{D}}\widehat{p}^\ell + s^{\ell n}_{8\mathrm{D}} \equiv \widehat{m}^{\ell n} + s^{\ell n}_{8\mathrm{D}}, \tag{33}$$

$$\widehat{j}^{0\ell} = -\widehat{j}^{\ell 0} = x^0\widehat{p}^\ell - \frac{1}{2}\left\{x^\ell_{\mathrm{D}}, \widehat{p}^0\right\} + \frac{\widehat{p}^0(\overrightarrow{s}_{8\mathrm{D}} \times \overrightarrow{p})^\ell}{\widehat{\omega}(\widehat{\omega}+m)}, \tag{34}$$

where the spin matrices $\overrightarrow{s}_{8\mathrm{D}} = (s^{\ell n}_{8\mathrm{D}})$ are given by (284)–(286) in [1] and the operator $\overrightarrow{x}_{\mathrm{D}}$ has the form

$$\overrightarrow{x}_{\mathrm{D}} = \overrightarrow{x} + \frac{i\overrightarrow{\Gamma}_8}{2\widehat{\omega}} - \frac{\overrightarrow{s}^\Gamma_8 \times \overrightarrow{p}}{\widehat{\omega}(\widehat{\omega}+m)} - \frac{i\overrightarrow{p}(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p})}{2\widehat{\omega}^2(\widehat{\omega}+m)}, \tag{35}$$

with specific spin matrices $\overrightarrow{s}^\Gamma_8$ given in

$$\overrightarrow{s}^\Gamma_8 \equiv \left(s^1_8, s^2_8, s^3_8\right)^\Gamma = \frac{i}{2}(\Gamma^2_8\Gamma^3_8,\ \Gamma^3_8\Gamma^1_8,\ \Gamma^1_8\Gamma^2_8). \tag{36}$$

It is not easy but it has been verified by the direct calculations that the generators (33), (34) with SU(2) spin (284)–(286) from [1] commute with the operator $\left[i\partial_0 - \Gamma^0_8(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m)\right]$ of equation (28), satisfy the commutation relations of the Lie algebra of the Poincaré group and the corresponding Casimir operators are given by $p^2 = \widehat{p}^\mu\widehat{p}_\mu = m^2\mathrm{I}_8$, $W = w^\mu w_\mu = m^2\overrightarrow{s}^2_{8\mathrm{D}} = \frac{3}{2}\left(\frac{3}{2}+1\right)m^2\mathrm{I}_8$.

The conclusion that equation (28) describes the local field of fermionic particle-antiparticle doublet of the spin s=(3/2,3/2) and mass $m > 0$ (and its solution (31) is the local fermionic field of the above mentioned spin and nonzero mass) follows from the analysis of equations (32) and the above given calculation of the Casimir operators p^2, $W = w^\mu w_\mu$.

Hence, the equation (28) describes the spin s=(3/2,3/2) particle-antiparticle doublet on the same level, on which the standard 4-component Dirac equation describes the spin s=(1/2,1/2) particle-antiparticle doublet. Moreover, the external argument in the validity of such interpretation is the link with the corresponding RCQM of spin s=(3/2,3/2) particle-antiparticle doublet, where the quantum-mechanical interpretation is direct and evident. Therefore, the fermionic spin s=(3/2,3/2) properties of equation (28) are proved.

Today after [2] and [3] equation (28) can be derived much more easier then here above. Now, contrary to [1], equation (28) for spin s=3/2 can be found as the simple partial case of the general equation for arbitrary spin from [2] and [3]. This equation is given by

$$\left[i\partial_0 - \Gamma^0_{2\mathrm{N}}(\overrightarrow{\Gamma}_{2\mathrm{N}} \cdot \overrightarrow{p} + m)\right]\psi(x) = 0, \quad \mathrm{N} = 2s+1, \tag{37}$$

where Γ^{μ}_{2N} have the form

$$\Gamma^{0}_{2N} \equiv \sigma^{3}_{2N} = \left| \begin{array}{cc} I_N & 0 \\ 0 & -I_N \end{array} \right|, \quad \Gamma^{j}_{2N} = \left| \begin{array}{cc} 0 & \Sigma^{j}_{N} \\ -\Sigma^{j}_{N} & 0 \end{array} \right|, \quad j = 1, 2, 3. \tag{38}$$

Contrary to the bosonic spin s=(1,0,1,0) properties of the equation (28) found in [1] (section 22), the fermionic spin s=(1/2,1/2,1/2,1/2) properties of this equation are evident. The fact that equation (28) describes the multiplet of two fermions with the spin s=1/2 and two antifermions with that spin can be proved much more easier then the above given consideration. The proof is similar to that given in the standard 4-component Dirac model. The detailed consideration can be found in sections 7, 9, 10 of [1]. Therefore, equation (28) has more extended property of the Fermi–Bose duality then the standard Dirac equation [15], [16]. This equation has the property of the Fermi–Bose triality. The property of the Fermi–Bose triality of the manifestly covariant equation (28) means that this equation describes on equal level (i) the spin s=(1/2,1/2,1/2,1/2) multiplet of two spin s=(1/2,1/2) fermions and two spin s=(1/2,1/2) antifermions, (ii) the spin s=(1,0,1,0) multiplet of the vector and scalar bosons together with their antiparticles, (iii) the spin s=(3/2,3/2) particle-antiparticle doublet.

It is evident that equation (28) is new in comparison with the Pauli–Fierz [12] and Rarita–Schwinger [13] equations for the spin s=3/2 particle. Contrary to 16-component equations from [12], [13] equation (28) is 8-component and does not need any additional condition. Formally equation (28) looks like to have some similar features with the Bargman–Wigner equation [10] for arbitrary spin, when the spin value is taken 3/2. The transformation $V^{\mp}_{8} = \frac{\mp \overrightarrow{\Gamma}_{8} \cdot \overrightarrow{p} + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}$ looks like the transformation of [11] in the case of s=3/2. Nevertheless, the difference is clear. Equation (28) for the case s=3/2 has only 8 components and is not the ordinary direct sum of the two Dirac equations. Furthermore, the given here model is derived from the first principles of RCQM (not from the FW type representation of the canonical field theory). Our consideration is original and new. The link with corresponding RCQM, the proof of the symmetry properties and relativistic invariance, the well defined spin operator (284)–(286) in [1], the features of the Fermi–Bose duality (triality) of the equation (28), the interaction with electromagnetic field and many other characteristics are suggested firstly.

Interaction, quantization and Lagrange approach in the above given spin s=(3/2,3/2) model are completely similar to the Dirac 4-component theory and standard quantum electrodynamics. For example, the Lagrange function of the system of interacting 8-component spinor and electromagnetic fields (in the terms of 4-vector potential $A^{\mu}(x)$) is given by

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{i}{2} \left(\overline{\psi}(x) \Gamma^{\mu}_{8} \frac{\partial \psi(x)}{\partial x^{\mu}} - \frac{\partial \overline{\psi}(x)}{\partial x^{\mu}} \Gamma^{\mu}_{8} \psi(x) \right) - \\ - m \overline{\psi}(x) \psi(x) + q \overline{\psi}(x) \Gamma^{\mu}_{8} \psi(x) A_{\mu}(x), \tag{39}$$

where $\overline{\psi}(x)$ is the independent Lagrange variable and $\overline{\psi} = \psi^{\dagger} \Gamma^{0}_{8}$ in the space of solutions $\{\psi\}$. In Lagrangian (101) $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the electromagnetic field tensor in the terms of potentials, which play the role of variational variables in this Lagrange approach.

Therefore, the covariant local quantum field theory model for the interacting particles with spin s=3/2 and photons can be constructed in complete analogy to the construction of the modern quantum electrodynamics. This model can be useful for the investigations of processes with interacting hyperons and photons.

Hence, the given above model for spin s=(3/2,3/2) fields is free from the difficulties of other equations for the spin s=(3/2), which brief review can be found in [17]–[21].

5 Field equations for spin s=2

Field equations for spin s=2 can be derived from corresponding RCQM in complete analogy with above given derivation of spin s=3/2 field equation. In [1] the following new equations have been suggested.

Canonical field theory equation for the spin s=(2,2) particle-antiparticle doublet:

$$(i\partial_0 - \Gamma^0_{10}\widehat{\omega})\phi(x) = 0, \quad \Gamma^0_{10} = \left| \begin{array}{cc} \mathrm{I}_5 & 0 \\ 0 & -\mathrm{I}_5 \end{array} \right|. \tag{40}$$

Canonical and covariant field equation for the spin s=(2,0,2,0) particle-antiparticle multiplet, see, e. g., the covariant version:

$$\left[i\partial_0 - \Gamma^0_{12}(\overrightarrow{\Gamma}_{12} \cdot \overrightarrow{p} + m)\right]\psi(x) = 0, \quad \Gamma^0_{12} = \left| \begin{array}{cc} \mathrm{I}_6 & 0 \\ 0 & -\mathrm{I}_6 \end{array} \right|, \quad \Gamma^j_{12} = \left| \begin{array}{cc} 0 & \Sigma^j_6 \\ -\Sigma^j_6 & 0 \end{array} \right|, \tag{41}$$

where Σ^j_6 are the 6×6 Pauli matrices

$$\Sigma^1_6 = \left| \begin{array}{cc} 0 & \mathrm{I}_3 \\ \mathrm{I}_3 & 0 \end{array} \right|, \Sigma^2_6 = \left| \begin{array}{cc} 0 & -i\mathrm{I}_3 \\ i\mathrm{I}_3 & 0 \end{array} \right|, \Sigma^3_6 = \left| \begin{array}{cc} \mathrm{I}_3 & 0 \\ 0 & -\mathrm{I}_3 \end{array} \right|.$$

Canonical and covariant field equation for the spin s=(2,1,2,1) particle-antiparticle multiplet, see, e. g., the covariant version:

$$\left[i\partial_0 - \Gamma^0_{16}(\overrightarrow{\Gamma}_{16} \cdot \overrightarrow{p} + m)\right]\psi(x) = 0, \quad \Gamma^0_{16} = \left| \begin{array}{cc} \mathrm{I}_8 & 0 \\ 0 & -\mathrm{I}_8 \end{array} \right|, \quad \Gamma^j_{16} = \left| \begin{array}{cc} 0 & \Sigma^j_8 \\ -\Sigma^j_8 & 0 \end{array} \right|, \tag{42}$$

where Σ^j_8 are the 8×8 Pauli matrices $\Sigma^j_8 = \left| \begin{array}{cccc} \sigma^j & 0 & 0 & 0 \\ 0 & \sigma^j & 0 & 0 \\ 0 & 0 & \sigma^j & 0 \\ 0 & 0 & 0 & \sigma^j \end{array} \right|$.

Covariant field theory equation for the spin s=(2,2) particle-antiparticle doublet can not be presented in terms of Γ matrices. Nevertheless, *two ways* of derivation of such equations are evident. (i) Putting equal to zero the scalar components in (41). The similar way was demonstrated in [1] (section 23) in order to find field equations for the spin s=(1,1) particle-antiparticle doublet from the known Dirac-like equation (252) for the spin s=(1,0,1,0) particle-antiparticle multiplet. Similarly to equation (275) in [1] the covariant field equation for the spin s=(2,2) particle-antiparticle doublet can be presented in terms of **curl**, **div** and **grad** operators as well. (ii) Using the FW transformations for arbitrary spin [22]–[27], which do not appeal to Γ matrices.

The equation found by the method (i) has the form

$$i\partial_0\psi^1 - p^3\psi^6 - m\psi^1 = 0,$$

$$i\partial_0\psi^2 - p^3\psi^7 + ip^2\psi^{10} - p^1\psi^{10} - m\psi^2 = 0,$$
$$i\partial_0\psi^3 - p^3\psi^8 + ip^2\psi^9 - p^1\psi^9 - m\psi^3 = 0,$$
$$i\partial_0\psi^4 + p^3\psi^9 - ip^2\psi^8 - p^1\psi^8 - m\psi^4 = 0,$$
$$i\partial_0\psi^5 + p^3\psi^{10} - ip^2\psi^7 - p^1\psi^7 - m\psi^5 = 0,$$
$$i\partial_0\psi^6 - p^3\psi^1 + m\psi^6 = 0,$$

$$i\partial_0\psi^7 - p^3\psi^2 + ip^2\psi^5 - p^1\psi^5 + m\psi^7 = 0, \tag{43}$$

$$i\partial_0\psi^8 - p^3\psi^3 + ip^2\psi^4 - p^1\psi^4 + m\psi^8 = 0,$$

$$i\partial_0\psi^9 + p^3\psi^4 - ip^2\psi^3 - p^1\psi^3 + m\psi^9 = 0,$$

$$i\partial_0\psi^{10} + p^3\psi^5 - ip^2\psi^2 - p^1\psi^2 + m\psi^{10} = 0,$$

$$-ip^2\psi^1 - p^1\psi^1 = 0, \quad -ip^2\psi^6 - p^1\psi^6 = 0,$$

or the form

$$\left[i\partial_0 - \Gamma^0_8(\overrightarrow{\Gamma}_8 \cdot \overrightarrow{p} + m)\right]\psi(x) = 0, \quad (i\partial_0 - \sigma^1 p^3 - \sigma^3 m)\chi = 0, \quad (p^1 + ip^2)\chi = 0, \tag{44}$$

where $\psi = \mathrm{column}(\psi^1, \psi^2, ..., \psi^8)$

$$\chi = \left| \begin{array}{c} \psi^9 \\ \psi^{10} \end{array} \right|, \Gamma^0_8 = \left| \begin{array}{cc} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{array} \right|, \Gamma^j_8 = \left| \begin{array}{cc} 0 & \Sigma^j_4 \\ -\Sigma^j_4 & 0 \end{array} \right|,$$

and Σ^j_4 are the 4×4 Pauli matrices

$$\Sigma^1_4 = \left| \begin{array}{cc} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{array} \right|, \Sigma^2_4 = \left| \begin{array}{cc} 0 & -i\mathrm{I}_2 \\ i\mathrm{I}_2 & 0 \end{array} \right|, \Sigma^3_4 = \left| \begin{array}{cc} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{array} \right|.$$

The form similar to (275) of [1] is found by the method similar to one presented in section 23 of [1].

Main **conclusion** is in approbation of the effective method of deriving of arbitrary spin field equations. The three level model (RCQM, canonical field, covariant field), e. g., of the spin s=(5/2,5/2) particle-antiparticle doublet is formulated similarly to the above given three level spin s=(3/2,3/2) model. In the approach under consideration the particle with spin s=5/2 is the same theoretical reality as the particle with spin s=1/2 (the particle with spin s=2 is the same theoretical reality as the particle with spin s=1). Therefore, I can suggest the hypothesis that the particle with spin s=5/2 exists in reality as well. The time of life of such new particle is smaller then the time of life of spin s=3/2 hyperons.

References

[1] V.M. Simulik. Covariant local field theory equations following from the relativistic canonical quantum mechanics of arbitrary spin. arXiv: 1409.2766v2 [quant-ph, hep-th] 19 Sep 2014, 67 p.

[2] V.M. Simulik. General form of the covariant field equations of arbitrary spin and the relativistic canonical quantum mechanics. arXiv: 1509.0463v1 [quant-ph, hep-th] 12 Sep 2015, 42 p.

[3] V.M. Simulik. Derivation of the Dirac and Dirac-like equations of arbitrary spin from the corresponding relativistic canonical quantum mechanics. Ukr. J. Phys. 2015. Vol. 60. P. 985–1006.

[4] L.L. Foldy, S.A. Wouthuysen. On the Dirac theory of spin 1/2 particles and its non-relativistic limit. Phys. Rev. 1950. Vol. 78. P. 29–36.

[5] J. Elliott, P. Dawber. Symmetry in Physics, vol.1. – Macmillian Press: London, 1979.

[6] B. Wybourne. Classical groups for Physicists. – John Wiley and sons: New York, 1974.

[7] I. Yu. Krivsky, R.R. Lompay, V.M. Simulik. Symmetries of the complex Dirac–Kähler equation. Theor. Math. Phys. 2005. Vol. 143. P. 541–558.

[8] E. Kähler. Der innere Differentialkalkul. Rendiconti di Matematica. Roma. Ser. 5. 1962. Vol. 21. P. 425–523.

[9] S.I. Kruglov. Dirac–Kähler equation. Int. J. Theor. Phys. 2002. Vol. 41. P. 653–687.

[10] V. Bargman, E.P. Wigner. Group theoretical discussion of relativistic wave equations. Proc. Nat. Acad. Sci. USA. 1948. Vol. 34. P. 211–223.

[11] D.L. Pursey. A Foldy-Wouthuysen transformation for particles of arbitrary spin. Nucl. Phys. 1964. Vol. 53. P. 174–176.

[12] M. Fierz, W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Proc. R. Soc. Lond. A. 1939. Vol. 173. P. 211–232.

[13] W. Rarita, J. Schwinger. On a theory of particles with half-integral spin. Phys. Rev. 1941. Vol. 60. P. 61–64.

[14] N.N. Bogoliubov, D.V. Shirkov. Introduction to the theory of quantized fields. – John Wiley and sons: New York, 1974.

[15] V.M. Simulik, I.Yu. Krivsky. Bosonic symmetries of the Dirac equation. Phys. Lett. A. 2011. Vol. 375. P. 2479–2483.

[16] V.M. Simulik, I.Yu. Krivsky, I.L. Lamer. Bosonic symmetries, solutions and conservation laws for the Dirac equation with nonzero mass. Ukr. J. Phys. 2013. Vol. 58. P. 523–533.

[17] K. Johnson, E.C.G. Sudarshan. Inconsistency of the local field theory of charged spin 3/2 particles. Ann. Phys. (N.Y.) 1961. Vol. 13. P. 126–145.

[18] G. Velo and D. Zwanziger. Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. Phys. Rev. 1969. Vol. 188. P. 2218–2222.

[19] A.Z. Capri, R.L. Kobes. Further problems in spin-3/2 field theories. Phys. Rev. D. 1980. Vol. 22. P. 1967–1978.

[20] T. Darkhosh. Is there a solution to the Rarita-Schwinger wave equation in the presence of an external electromagnetic field? Phys. Rev. D. 1985. Vol. 32. P. 3251–3256.

[21] W. Cox. On the Lagrangian and Hamiltonian constraint algorithms for the Rarita-Schwinger field coupled to an external electromagnetic field. J. Phys. A. 1989. Vol. 22. P. 1599–1608.

[22] L.M. Garrido, L. Oliver. On the Foldy-Wouthuysen transformation for particles in an electromagnetic field. Nuov. Cim. A. 1967. Vol. LII. P. 588–605.

[23] G.A. Ramanujam. On the unitary Foldy-Wouthuysen operator for particles of arbitrary spin. Nuov. Cim. A. 1974. Vol. 20. P. 27–34.

[24] A.K. Nagpal, R.S. Kaushal. A note on the unitary Foldy-Wouthuysen operator for arbitrary spin. Lett. Nuov. Cim. 1974. Vol. 9. P. 391–394.

[25] A.R. Tekumalla, T.S. Santhanam. Unitary Foldy-Wouthuysen transformations for particles of arbitrary spin. Lett. Nuov. Cim. 1974. Vol. 10. P. 737–740.

[26] J. Jayaraman. A note on the recent unitary Foldy-Wouthuysen transformations for particles of arbitrary spin. J. Phys. A. 1975. Vol. 8. P. L1–L4.

[27] J. Leon, M. Quiros, J.R. Mittelbrunn. Group content of the Foldy-Wouthuysen transformation and the nonrelativistic limit for arbitrary spin. J. Math. Phys. 1979. Vol. 20. P. 1068–1076.

Частица Дирака с аномальным магнитным моментом во внешнем магнитном поле, построение решений методом квадрирования

В.В. Кисель
Белорусский государственный университет радиоэлектроники и информатики
Войнова Я.А.
voinyuschka@mail.ru; Средняя школа, Ельский район, Беларусь
В.М. Редьков
redkov@dragon.bas-net.by; Институт физики им. Б.И. Степанова

Аннотация

Метод квадрирования положен в основу анализа поведения частицы со спином $1/2$ и аномальным магнитным моментом во внешнем однородном магнитном поле. В волновых функциях выделены компоненты, отвечающие фиксированным значениям третьей проекции спина частицы. Эти компоненты подчиняются независимым уравнениям второго порядка; после разделения переменных каждая сводится к двум обыкновенным дифференциальным уравнениям второго, которые с использованием специальных линейных преобразований удается разделить на независимые. Найденные уравнений решены точно в гипергеометрических функциях, найдены обобщенные формулы для состояний электроны во внешнем поле. Имеются два класса решений, отвечающие им уровни энергии зависят от параметра $\Gamma = \lambda \frac{2eB}{Mc^2}$, соответственно с противоположными знаками; случай $\lambda = 0$ отвечает нулевому аномальному магнитному моменту.

1 Уравнение Дирака, учет аномального магнитного момента

В работе на основе применения метода квадрирования будет решена задача о построении точных решений для частицы со спином $1/2$ и аномальным магнитным полем во внешнем однородном магнитном поле. Используем известное описание однородного магнитного поля: $\mathbf{A} = \frac{1}{2}\, c\mathbf{B} \times \mathbf{r},\ \mathbf{B} = (0, 0, B)$. После преобразования к цилиндрическим координатам имеем

$$A_t = 0, \qquad A_r = 0;, \quad A_z = 0, \quad A_\phi = -Br^2/2, \quad F_{\phi r} = Br. \tag{1}$$

Для описания дираковской частицы в магнитном поле будем использовать тетрадный формализм [1] для цилиндрических координат и тетрады $x^\alpha = (t, r, \phi, z)$:

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2\ , \qquad e^{\beta}_{(a)}(x) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{2}$$

Уравнение Дирака в тетрадном формализме [1] записывается так:

$$\left\{ \gamma^c [\ i\hbar\ (\ e^{\beta}_{(c)} \partial_\beta + \frac{1}{2} \sigma^{ab} \gamma_{abc}\ )\ -\ \frac{e}{c} A_c\ ]\ -\ mc \right\} \Psi = 0\ , \tag{3}$$

где γ_{abc} – коэффициенты вращения Риччи: $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha}e^{\beta}_{(a)}e^{\alpha}_{(c)}$, $A_a = e^{\beta}_{(a)}A_\beta$ – тетрадные компоненты 4-вектора электромагнитного поля A_β; $\sigma^{ab} = 1/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$. Ниже будем использовать сокращенные обозначения: $e/c\hbar \Longrightarrow e$, $mc/\hbar \Longrightarrow M$. Уравнение Дирака принимает вид (пусть $\Psi = \psi/\sqrt{r}$):

$$\left[i\gamma^0\frac{\partial}{\partial t} + i\gamma^1\frac{\partial}{\partial r} + \gamma^2\left(\frac{i\partial_\phi}{r} + \frac{eBr}{2}\right) + i\gamma^3\frac{\partial}{\partial z} - M\right]\psi = 0\,. \tag{4}$$

Решения ищем в виде

$$\psi = e^{-i\epsilon t}e^{im\phi}e^{ikz}\left|\begin{array}{c} f_1(r)\\ f_2(r)\\ f_3(r)\\ f_4(r)\end{array}\right|, \quad \left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \gamma^2\,\mu(r) - k\gamma^3 - M\right]\left|\begin{array}{c} f_1\\ f_2\\ f_3\\ f_4\end{array}\right| = 0\,,$$

где $\mu(r) = m/r - eBr/2$; квантовое число третьей проекции полного момента $m = \pm 1/2, \pm 3/2, ...$

Уравнение Дирака с учетом описания аномального магнитного момента частицы представимо в подходе Петраша так [2, 3]:

$$\left\{\gamma^c[\; i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2}\sigma^{ab}\gamma_{abc}\;) \;-\; \frac{e}{\hbar c}A_c\;] - i\lambda\frac{2e}{Mc^2}\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) - \frac{Mc}{\hbar}\right\}\Psi = 0\,. \tag{5}$$

Размерности величин задаются соотношениями

$$[\frac{Mc}{\hbar}] = l^{-1}, \quad [\frac{e}{\hbar c}A] = l^{-1}, \quad [\frac{e}{\hbar c}F] = l^{-2}, \quad [\frac{eF}{Mc^2}] = l^{-1}\,;$$

т. е. свободный параметр λ является безразмерным.

2 Выделение собственных состояний оператора Σ_3

Будем рассматривать это обобщенное уравнение Дирака во внешнем однородном магнитном поле. Ввиду соотношения $\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) = 2\sigma^{\phi r}F_{\phi r} = -i\gamma^1\gamma^2 B = \Sigma_3 B$ вместо (4) будем иметь

$$\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \gamma^2\mu(r) - k\gamma^3 + \Gamma\Sigma_3 - M\right]\psi = 0, \quad \Sigma_3 = -i\gamma^1\gamma^2,\; \Gamma = \lambda\frac{2eB}{Mc^2}. \tag{6}$$

Ниже для сокращения записи используем следующие обозначения: $eB/2 \Longrightarrow B$. Используем спинорный базис матриц Дирака:

$$\gamma^0 = \left|\begin{array}{cccc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right|, \quad \gamma^1 = \left|\begin{array}{cccc} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right|,$$

$$\gamma^2 = \left|\begin{array}{cccc} 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{array}\right|, \quad \gamma^3 = \left|\begin{array}{cccc} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{array}\right|,$$

$$\Sigma_3 = -i\gamma^1\gamma^2 = \left|\begin{array}{cccc} -1 & 0 & 0 & 0\\ 0 & +1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & +1\end{array}\right|, \quad (\Sigma_3 - 1)\,(\Sigma_3 + 1) = \Sigma_3^2 - 1 = 0.$$

Вводим проекционные операторы

$$P_+ = \frac{1}{2}\left(\Sigma_3 + 1\right) = \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right|, P_+^2 = P_+, \;\; P_- = \frac{1}{2}\left(1 - \Sigma_3\right) = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|, P_-^2 = P_-.$$

Эти операторы выделяют компоненты с фиксированным значением третьей проекции оператора Σ_3: $\psi_+ = P_+\psi$, $\psi_- = P_-\psi$. Операторами P_+ и P_- действуем на уравнение (6), в результате получим

$$\begin{aligned} \left(+\Gamma + \epsilon\gamma^0 - k\gamma^3 - M\right)\psi_+ + P_+\left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi = 0, \\ \left(-\Gamma + \epsilon\gamma^0 - k\gamma^3 - M\right)\psi_- + P_-\left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi = 0. \end{aligned} \tag{7}$$

Учтем равенства

$$\gamma^1 P_+ = P_-\gamma^1, \quad \gamma^2 P_+ = P_-\gamma^2, \quad \gamma^1 P_- = P_+\gamma^1, \quad \gamma^2 P_- = P_+\gamma^2,$$

тогда из (7) приходим к следующему:

$$\begin{aligned} \left(\epsilon\gamma^0 - k\gamma^3 - M + \Gamma\right)\psi_+ + \left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi_- = 0, \\ \left(\epsilon\gamma^0 - k\gamma^3 - M - \Gamma\right)\psi_- + \left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi_+ = 0. \end{aligned} \tag{8}$$

Так как справедлива цепочка равенств

$$\left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi_+ = \left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)P_+\psi_+ =$$

$$= \left(i\gamma^1 P_+\frac{\partial}{\partial r} - \gamma^2 P_+\mu\right)\psi_+ = \left(\frac{i}{2}\left(\gamma^1 + i\gamma^2\right)\frac{d}{dr} - \frac{1}{2}\left(\gamma^2 - i\gamma^1\right)\mu\right)\psi_+ =$$

$$= i\left[\frac{1}{\sqrt{2}}\left(\gamma^1 + i\gamma^2\right)\right]\left[\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m - Br^2}{r}\right)\right]\psi_+,$$

то получаем

$$\left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi_+ = i\gamma_+\,\hat{a}_m\psi_+, \tag{9}$$

где использованы обозначения

$$\gamma_+ = \frac{1}{\sqrt{2}}\left(\gamma^1 + i\gamma^2\right), \quad \hat{a}_m = \frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m - Br^2}{r}\right). \tag{10}$$

Аналогично, находим

$$\left(i\gamma^1\frac{d}{dr} - \gamma^2\mu\right)\psi_- = -i\gamma_-\,\hat{b}_m\psi_- \tag{11}$$

где

$$\gamma_- = \frac{1}{\sqrt{2}}\left(\gamma^1 - i\gamma^2\right), \quad \hat{b}_m = \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m - Br^2}{r}\right). \tag{12}$$

Таким образом, получены равенства

$$\left(i\gamma^1\frac{d}{dr}-\gamma^2\mu\right)\psi_+ = i\gamma_+\hat{a}_m\psi_+, \quad \left(i\gamma^1\frac{d}{dr}-\gamma^2\mu\right)\psi_- = -i\gamma_-\hat{b}_m\psi_-. \tag{13}$$

Следовательно, уравнения (8) можно записать так:

$$\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ - i\gamma_-\,\hat{b}_m\psi_- = 0, \tag{14}$$

$$\left(\epsilon\gamma^0 - k\gamma^3 - D\right)\psi_- + i\gamma_+\,\hat{a}_m\psi_+ = 0, \tag{15}$$

где использованы обозначения: $M-\Gamma = A,\ M+\Gamma = D$.

Умножим уравнение (14) на выражение $(\epsilon^2 - k^2 - D^2)$:

$$\left(\epsilon^2-k^2-D^2\right)\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ - i\gamma_-\hat{b}_m\left(\epsilon^2-k^2-D^2\right)\psi_- = 0; \tag{16}$$

уравнение уравнение (15) – на матрицу $(\epsilon\gamma^0 - k\gamma^3 + D)$:

$$\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\left(\epsilon\gamma^0 - k\gamma^3 - D\right)\psi_- + i\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\gamma_+\hat{a}_m\psi_+ = 0.$$

последнее может быть переписано так:

$$\left(\epsilon^2-k^2-D^2\right)\psi_- = -i\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\gamma_+\hat{a}_m\psi_+.$$

Соответственно, уравнение (16) представить иначе:

$$\left(\epsilon^2-k^2-D^2\right)\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ - \gamma_-\hat{b}_m\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\gamma_+\hat{a}_m\psi_+ = 0;$$

его можно преобразовать в следующее:

$$\left(\epsilon^2-k^2-D^2\right)\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ - \hat{b}_m\hat{a}_m\left(-\epsilon\gamma^0 + k\gamma^3 + D\right)\gamma_-\gamma_+\psi_+ = 0. \tag{17}$$

Убеждаемся в справедливости равенства $\gamma_-\gamma_+ = -2P_+$; следовательно, уравнение (17) принимает вид

$$\left(\epsilon^2-k^2-D^2\right)\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ + 2\hat{b}_m\hat{a}_m\left(-\epsilon\gamma^0 + k\gamma^3 + D\right)\psi_+ = 0.$$

Подействуем на полученное уравнение оператором $(\epsilon\gamma^0 - k\gamma^3 + D)$:

$$\left(\epsilon^2-k^2-D^2\right)\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\left(\epsilon\gamma^0 - k\gamma^3 - A\right)\psi_+ +$$

$$+2\hat{b}_m\hat{a}_m\left(\epsilon\gamma^0 - k\gamma^3 + D\right)\left(-\epsilon\gamma^0 + k\gamma^3 + D\right)\psi_+ = 0$$

или

$$\left(\epsilon^2-k^2-D^2\right)\times$$

$$\left[\epsilon^2\gamma^0\gamma^0 - \epsilon k\gamma^0\gamma^3 - A\epsilon\gamma^0 - \epsilon k\gamma^3\gamma^0 + k^2\gamma^3\gamma^3 + Ak\gamma^3 + D\epsilon\gamma^0 - Dk\gamma^3 - AD\right]\psi_+ +$$

$$+2\hat{b}_m\hat{a}_m\times$$

$$\left[-\epsilon^2\gamma^0\gamma^0 + \epsilon k\gamma^0\gamma^3 + \epsilon D\gamma^0 + k\epsilon\gamma^3\gamma^0 - k^2\gamma^3\gamma^3 - Dk\gamma^3 - D\epsilon\gamma^0 + Dk\gamma^3 + D^2\right]\psi_+ = 0$$

или

$$\left(\epsilon^2-k^2-D^2\right)\left[\epsilon^2 - k^2 - AD - \epsilon\left(A-D\right)\gamma^0 + k\left(A-D\right)\gamma^3\right]\psi_+$$

$$+2\hat{b}_m\hat{a}_m\left[-\epsilon^2+k^2+D^2\right]\psi_+=0,$$

Таким образом, приходим к уравнению для компоненты ψ_+:

$$\left[-2\hat{b}_m\hat{a}_m+\epsilon^2-k^2-AD-(A-D)\left(\epsilon\gamma^0-k\gamma^3\right)\right]\psi_+=0. \tag{18}$$

Аналогично получим уравнение для функции ψ_-:

$$\left[-2\hat{a}_m\hat{b}_m+\epsilon^2-k^2-AD+(A-D)\left(\epsilon\gamma^0-k\gamma^3\right)\right]\psi_-=0. \tag{19}$$

При анализе этих уравнений будем использовать явный вид матриц Дирака в спинорном базисе. Уравнение (18) с учетом равенств

$$\gamma^0\psi_+=\left|\begin{array}{c}0\\ \psi_4\\ 0\\ \psi_2\end{array}\right|,\quad \gamma^3\psi_+=\left|\begin{array}{c}0\\ \psi_4\\ 0\\ -\psi_2\end{array}\right|$$

дает

$$-2\hat{b}_m\hat{a}_m\left|\begin{array}{c}0\\ \psi_2\\ 0\\ \psi_4\end{array}\right|+(\epsilon-k^2-AD)\left|\begin{array}{c}0\\ \psi_2\\ 0\\ \psi_4\end{array}\right|-(A-D)\left\{\epsilon\left|\begin{array}{c}0\\ \psi_4\\ 0\\ \psi_2\end{array}\right|-k\left|\begin{array}{c}0\\ \psi_4\\ 0\\ -\psi_2\end{array}\right|\right\}=0.$$

Это эквивалентно двум уравнениям:

$$\begin{aligned}-2\hat{b}_m\hat{a}_m\psi_2+\left(\epsilon^2-k^2-AD\right)\psi_2-(A-D)(\epsilon-k)\psi_4=0,\\ -2\hat{b}_m\hat{a}_m\psi_4+\left(\epsilon^2-k^2-AD\right)\psi_4-(A-D)(\epsilon+k)\psi_2=0.\end{aligned} \tag{20}$$

Введем новые функции R_1 и R_2:

$$R_1=(\epsilon+k)\ \psi_2+\sqrt{\epsilon^2-k^2}\,\psi_4,\quad R_2=-(\epsilon+k)\ \psi_2+\sqrt{\epsilon^2-k^2}\,\psi_4. \tag{21}$$

Система (20) примет вид

$$\begin{aligned}-2\hat{b}_m\hat{a}_mR_1+\left(\epsilon^2-k^2-AD\right)R_1-\\ -(A-D)\left\{\left(\epsilon^2-k^2\right)\psi_4+(\epsilon+k)\sqrt{\epsilon^2-k^2}\psi_2\right\}=0\quad\Longrightarrow\end{aligned}$$

$$\left\{-2\hat{b}_m\hat{a}_m+\epsilon^2-k^2-AD-(A-D)\sqrt{\epsilon^2-k^2}\right\}R_1=0 \tag{22}$$

и

$$\begin{aligned}-2\hat{b}_m\hat{a}_m\psi_2+\left(\epsilon^2-k^2-AD\right)\psi_2-\\ (A-D)\left[-\left(\epsilon^2-k^2\right)\psi_4+(\epsilon+k)\sqrt{\epsilon^2-k^2}\psi_2\right]=0\quad\Longrightarrow\end{aligned}$$

$$\left[-2\hat{b}_m\hat{a}_m+\epsilon^2-k^2-AD+(A-D)\sqrt{\epsilon^2-k^2}\right]R_2=0. \tag{23}$$

Аналогично с учетом

$$\gamma^0\psi_- = \left|\begin{array}{c}\psi_3\\0\\\psi_1\\0\end{array}\right|, \quad \gamma^3\psi_- = \left|\begin{array}{c}-\psi_3\\0\\\psi_1\\0\end{array}\right|$$

устанавливаем покомпонентную запись уравнения (19)

$$-2\hat{a}_m\hat{b}_m\left|\begin{array}{c}\psi_1\\0\\\psi_3\\0\end{array}\right| + (\epsilon^2 - k^2 - AD)\left|\begin{array}{c}\psi_1\\0\\\psi_3\\0\end{array}\right| + (A-D)\left\{\epsilon\left|\begin{array}{c}\psi_3\\0\\\psi_1\\0\end{array}\right| - k\left|\begin{array}{c}-\psi_3\\0\\\psi_1\\0\end{array}\right|\right\} = 0.$$

Это эквивалентно двум уравнениям:

$$-2\hat{a}_m\hat{b}_m\psi_1 + (\epsilon^2 - k^2 - AD)\psi_1 + (A-D)(\epsilon + k\ )\psi_3 = 0,$$

$$-2\hat{a}_m\hat{b}_m\psi_3 + (\epsilon^2 - k^2 - AD)\psi_3 + (A-D)(\epsilon - k)\psi_1 = 0. \tag{24}$$

Вводим две новые функции:

$$\Phi_1 = (\epsilon - k)\,\psi_1 + \sqrt{\epsilon^2 - k^2}\psi_3, \quad \Phi_2 = -(\epsilon - k)\,\psi_1 + \sqrt{\epsilon^2 - k^2}\psi_3. \tag{25}$$

Систему (24) преобразуем в следующую:

$$-2\widehat{a_m}\hat{b}_m\Phi_1 + \left(\epsilon^2 - k^2 - AD\right)\Phi_1 +$$

$$+ (A-D)\left\{\left(\epsilon^2 - k^2\right)\Phi_3 + (\epsilon - k)\sqrt{\epsilon^2 - k^2}\psi_1\right\} = 0 \quad \Longrightarrow$$

$$\left\{-2\hat{a}_m\hat{b}_m + \epsilon^2 - k^2 - AD + (A-D)\sqrt{\epsilon^2 - k^2}\right\}\Phi_1 = 0 \tag{26}$$

и

$$-2\widehat{a_m}\hat{b}_m\Phi_2 + \left(\epsilon^2 - k^2 - AD\right)\Phi_2 +$$

$$+ (A-D)\left\{-\left(\epsilon^2 - k^2\right)\Phi_3 + (\epsilon - k)\sqrt{\epsilon^2 - k^2}\psi_1\right\} = 0 \quad \Longrightarrow$$

$$\left\{-2\hat{a}_m\widehat{b_m} + \epsilon^2 - k^2 - AB - (A-D)\sqrt{\epsilon^2 - k^2}\right\}\Phi_2 = 0. \tag{27}$$

Учтем равенства:

$$A - D = -2\Gamma, \quad AD = M^2 - \Gamma^2, \quad A = M - \Gamma, \quad D = M + \Gamma,$$

тогда найденные четыре уравнения запишутся так:

$$\left[-2\hat{b}_m\hat{a}_m + \epsilon^2 - k^2 - M^2 + \Gamma^2 + 2\Gamma\sqrt{\epsilon^2 - k^2}\right]R_1 = 0,$$

$$\left[-2\hat{b}_m\hat{a}_m + \epsilon^2 - k^2 - M^2 + \Gamma^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}\right]R_2 = 0,$$

$$\left[-2\hat{a}_m\hat{b}_m + \epsilon^2 - k^2 - M^2 + \Gamma^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}\right]\Phi_1 = 0,$$

$$\left[-2\hat{a}_m\hat{b}_m + \epsilon^2 - k^2 - M^2 + \Gamma^2 + 2\Gamma\sqrt{\epsilon^2 - k^2}\right]\Phi_2 = 0. \tag{28}$$

Учтем выражения для дифференциальных операторов второго порядка:

$$-2\hat{a}_m\hat{b}_m = \frac{d^2}{dr^2} - \frac{m\,(m-1)}{r^2} + (2m+1)\,B - B^2r^2, \tag{29}$$

$$-2\hat{b}_m\hat{a}_m = \frac{d^2}{dr^2} - \frac{m\,(m+1)}{r^2} + (2m-1)\,B - Br^2. \tag{30}$$

Следовательно, четыре уравнения имеют следующий явный вид:

$$\left[\frac{d^2}{dr^2} - \frac{m\,(m+1)}{r^2} + (2m-1)\,B - Br^2 + C_1\right] R_1 = 0, \tag{31}$$

$$\left[\frac{d^2}{dr^2} - \frac{m\,(m+1)}{r^2} + (2m-1)\,B - Br^2 + C_2\right] R_2 = 0, \tag{32}$$

$$\left[\frac{d^2}{dr^2} - \frac{m\,(m-1)}{r^2} + (2m+1)\,B - Br^2 + C_2\right] \Phi_1 = 0, \tag{33}$$

$$\left[\frac{d^2}{dr^2} - \frac{m\,(m-1)}{r^2} + (2m+1)\,B - Br^2 + C_1\right] \Phi_2 = 0, \tag{34}$$

где использованы обозначения

$$C_1 = \epsilon^2 - k^2 - M^2 + \Gamma^2 + 2\Gamma\sqrt{\epsilon^2 - k^2},$$

$$C_2 = \epsilon^2 - k^2 - M^2 + \Gamma^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}.$$

Отмечаем, что уравнения для $R_1 - R_2$, равно как и уравнения для $\Phi_1 - \Phi_2$, различаются между собой только знаком при параметре Γ. Это означает, что в пределах каждой пары достаточно исследовать только один случай.

3 Решение радиальных уравнений

Для анализа возникающих дифференциальных уравнений вводим новую переменную

$$x = |B|r^2, \quad r^2 = \frac{x}{|B|},\ \frac{d}{dr} = 2\sqrt{|B|x}\frac{d}{dx},\ \frac{d^2}{dr^2} = 4|B|\left(\frac{1}{2}\frac{d}{dx} + x\frac{d^2}{dx^2}\right).$$

Тогда уравнение для R_1 запишется в виде

$$\left[x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} - \frac{m\,(m+1)}{4x} + \frac{2m-1}{4}\frac{B}{|B|} - \frac{1}{4}x + \frac{C_1}{4|B|}\right] R_1 = 0. \tag{35}$$

Для определенности вектор магнитного поля $\vec{B}$ направим в положительном направлении оси z, учитываем также отрицательность заряда электрона: $e < 0$. Тогда $B = -|B|$, и уравнение для R_1 запишется так:

$$\left[x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} - \frac{m\,(m+1)}{4x} - \frac{2m-1}{4} - \frac{1}{4}x + \frac{C_1}{4|B|}\right] R_1 = 0. \tag{36}$$

Используем подстановку $R_1 = x^L \epsilon^{-Cx} Q_1(x)$, из уравнения (36) получаем

$$x\frac{d^2Q_1}{dx^2} + \left(\frac{1}{2} + 2L - 2Cx\right)\frac{dQ_1}{dx} + \frac{1}{x}\left[L\left(L-1\right) - \frac{m\left(m+1\right)}{4} + \frac{L}{2}\right]Q_1 +$$

$$+\left(C^2 - \frac{1}{4}\right)xQ_1 - \left[2LC + \frac{1}{2}C + \frac{2m-1}{4} - \frac{C_1}{4|B|}\right]Q_1 = 0.$$

Требуем выполнения равенств:

$$C^2 - \frac{1}{4} = 0, \quad L\left(L-1\right) - \frac{m\left(m+1\right)}{4} + \frac{L}{2} = 0\,;$$

отсюда получаем ограничения: $C = \pm 1/2$ (понятно, что необходимо использовать $C = +1/2$, поскольку только такой выбор может обеспечить затухание решений при$x \to$) и

$$L_1 = \frac{m+1}{2} \Longrightarrow m \geq +\frac{1}{2}, \quad L_2 = -\frac{m}{2} \Longrightarrow m \leq -\frac{1}{2}$$

предполагаем, что волновые функции должны обращаться в ноль также и в начале координат).

Рассматриваем вариант $L_1, m \geq +1/2$:

$$x\frac{d^2Q_1}{dx^2} + \left(m + \frac{3}{2} - x\right)\frac{dQ_1}{dx} - \left(\frac{1}{2} + m - \frac{C_1}{4|B|}\right)Q_1 = 0$$

это уравнение является вырожденным гипергеометрическим уравнением

$$xF'' + \left(\gamma - x\right)F' - \alpha F = 0,$$

с параметрами

$$\gamma = m + \frac{3}{2}, \quad \alpha = m + \frac{1}{2} - \frac{C_1}{4|B|}.$$

Известно, что условием обращения гипергеометрического ряда в полином степени n является следующее ограничение на параметр α:

$$\alpha = -n \quad \Longrightarrow \quad \frac{1}{2} + m - C_1\frac{1}{4|B|} = -n, \quad \Longrightarrow \quad C_1 = 4|B|\left(n + m + \frac{1}{2}\right).$$

Отсюда, вспоминая смысл величины C_1, получаем правило квантования

$$m \geq \frac{1}{2}, \quad \epsilon^2 - k^2 - M^2 + \Gamma^2 + 2\Gamma\sqrt{\epsilon^2 - k^2} = 4|B|\left(m + n + \frac{1}{2}\right);$$

его можно представить в более коротком виде

$$\epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(m + n + \frac{1}{2})} - \Gamma\right)^2, \quad m \geq +\frac{1}{2}. \tag{37}$$

Теперь рассматриваем случай $L_2, m \leq -1/2$:

$$x\frac{d^2Q_1}{dx^2} + \left(\frac{1}{2} - m - x\right)\frac{dQ_1}{dx} - \left(-\frac{C_1}{4|B|}\right)Q_1 = 0.$$

Требование квантования $\gamma = -n$ принимает вид

$$m \leq -\frac{1}{2}, \quad C_1 = 4|B|n,$$

$$\epsilon^2 - k^2 - M^2 + \Gamma^2 + 2\Gamma\sqrt{\epsilon^2 - k^2} - \left\{M^2 - \Gamma^2\right\} = 4n|B|,$$

или

$$m \leq -\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|n} - \Gamma\right)^2. \tag{38}$$

Теперь обращаемся к уравнению для R_2. Поскольку формально оно отличается от уравненияR_1 только заменой $\Gamma \Longrightarrow -\Gamma$, то никаких дополнительных вычислений проводить не нужно. Выражение для спектра можно написать сразу

$$m \geq \frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(m + n + \frac{1}{2})} + \Gamma\right)^2; \tag{39}$$

$$m \leq -\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n + \frac{1}{2})} + \Gamma\right)^2. \tag{40}$$

Рассмотрим уравнение для Φ_2

$$\left[\frac{d^2}{dr^2} - \frac{m\,(m-1)}{r^2} + (2m+1)\,B - Br^2 + C_1\right]\Phi_2 = 0. \tag{41}$$

Для анализа уравнения используем ту же самую переменную x; тогда получаем

$$\left[x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} - \frac{m\,(m-1)}{4x} - \frac{2m+1}{4} - \frac{1}{4} + \frac{C_1}{4|B|}\right]\Phi_2 = 0. \tag{42}$$

Используем подстановку $\Phi_2 = x^L\epsilon^{-Cx}P_2(x)$;

$$x\frac{d^2P_2}{dx^2} + \left(\frac{1}{2} + 2L - 2Cx\right)\frac{dP_2}{dx} + \frac{1}{x}\left[L\,(L-1) - \frac{m\,(m-1)}{4} + \frac{L}{2}\right]P_2 +$$

$$+\left(C^2 - \frac{1}{4}\right)xP_2 - \left[2LC + \frac{1}{2}C + \frac{2m+1}{4} - \frac{C_1}{4|B|}\right]P_2 = 0.$$

Требуем выполнения равенств:

$$C^2 - \frac{1}{4} = 0, \quad L\,(L-1) - \frac{m\,(m-1)}{4} + \frac{L}{2} = 0\,;$$

отсюда получаем ограничения: $C = \pm 1/2$ (используем $C = +1/2$;

$$L_1 = \frac{1-m}{2} \Longrightarrow m \leq -\frac{1}{2}, \quad L_2 = \frac{m}{2} \Longrightarrow m \geq \frac{1}{2}.$$

Рассматриваем вариант $L_1, m \leq -1/2$:

$$x\frac{d^2P_2}{dx^2} + \left(-m + \frac{3}{2} - x\right)\frac{dP_2}{dx} - \left(1 - \frac{C_1}{4|B|}\right)P_2 = 0$$

это уравнение является вырожденным гипергеометрическим уравнением с параметрами

$$\gamma = -m + \frac{3}{2}, \quad \alpha = 1 - \frac{C_1}{4|B|}\,.$$

Условие квантования $\alpha = -n$ дает

$$m \leq -\frac{1}{2}, \quad C_1 = 4|B|(n+1).$$

Учитывая смысл параметра C_1, получаем правило квантования

$$m \leq -\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n+1)} - \Gamma\right)^2, \quad m \leq -\frac{1}{2}. \tag{43}$$

Теперь рассматриваем случай $L_2, m \geq +1/2$:

$$x\frac{d^2P_2}{dx^2} + \left(\frac{1}{2} + m - x\right)\frac{dP_2}{dx} - \left(m + \frac{1}{2} - \frac{C_1}{4|B|}\right)P_2 = 0.$$

Требование квантования $\alpha = -n$ дает

$$m \geq +\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n+m+\frac{1}{2})} - \Gamma\right)^2, \quad m < 0. \tag{44}$$

Результаты для функции Φ_1 получаются простой заменой параметра: $\Gamma \Longrightarrow -\Gamma$.

4 Заключение

Соберем результаты по найденным спектрам энергии вместе:
$R_1, R_2 \longrightarrow (\mp)$,

$$m \geq +\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n+m+\frac{1}{2})} \mp \Gamma\right)^2,$$

$$m \leq -\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|n} \mp \Gamma\right)^2.; \tag{45}$$

$\Phi_2, \Phi_1 \longrightarrow (\mp)$,

$$m \geq +\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n+m+\frac{1}{2})} \mp \Gamma\right)^2, \quad m < 0.$$

$$m \leq -\frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 4|B|(n+1)} \mp \Gamma\right)^2, \quad m \leq -\frac{1}{2}. \tag{46}$$

Можно сделать вывод, что существенно разные спектры энергии имеем для двух классов состояний:

$$\{R_1, R_2 = 0, \Phi_1 = 0, \Phi_2\}, \qquad \{R_1 = 0, R_2, \Phi_1, \Phi_2 = 0\};$$

связь между ними достигается с помощью формальной замены $\Gamma \implies -\Gamma$.

Список литературы

[1] Редьков В.М. Поля частиц в римановом пространстве и группа Лоренца. Белорусская наука, Минск, 2009. 486 стр.

[2] Кисель В.В., Токаревская Н.Г., Редьков В.М. Теория Петраша для частицы со спином 1/2 в искривленном пространстве-времени. / Препринт №̲ 737, Институт физики НАНБ, Минск, 2002, 25 с.

[3] Богуш А.А., Кисель В.В., Токаревская Н.Г., Редьков В.М. Теория Петраша для частицы со спином 1/2 в искривленном пространстве-времени. // Весці НАНБ. Сер. фіз.-мат. навук. 2002. №̲ 1. С. 63 – 68.

О теории релятивистских волновых уравнений для частиц с внутренней структурой: уравнения Кокса, Шамали–Капри, Плетюхова–Федорова

О.В. Веко
vekoolga@mail.ru; Гимназия, г. Калинковичи, Беларусь
Я.А. Войнова
voinyuschka@mail.ru; Средняя школа, Ельский район, Беларусь
В.В. Кисель
Белорусский Государственный Университет Информатики и Радиоэлектроники

Аннотация

Отказ от требования минимальности используемых наборов представлений группы Лоренца существенно расширяет возможности метода релятивистских волновых уравнений с точки зрения пространственно-временного описания внутренней структуры частиц. Получение новых уравнений с более богатой структурой для частицы с заданным спином s возможно либо за счет включения представлений с более высокими спинами, либо за счет использования повторяющихся (кратных) представлений группы Лоренца.

Данная работа посвящена изложенного того, как в рамках общего подхода к теории релятивистских волновых уравнений первого порядка с расширенными наборами представлений группы Лоренца возникают уравнения, известные в литературе как уравнения для частиц со спином ноль и единица, обладающих дополнительной электромагнитной структурой; речь идет об уравнении Кокса для частицы со спином 0, уравнении Шамали–Капри для частицы со спином 1, и уравнениях Плетюхова–Федорова для частиц спинов ноль и единица с дополнительной электромагнитной характеристикой (часто называемой поляризуемостью).

1 Введение

Теория релятивистских волновых уравнений первого порядка, записанных в матричной форме, предполагает в своей исходной формулировке возможность описания только одной, спиновой, внутренней степени свободы элементарных частиц. Математическим отражением этого обстоятельства является использование минимального набора неприводимых представлений группы Лоренца, который необходим для описания частицы с заданным значением спина (уравнение Дирака для спина 1/2, уравнения Даффина–Кеммера для спинов 0 и 1, уравнение Фирца–Паули для спина 3/2). Однако в 1955–1957 гг. Петрашем и Улеглой было построено уравнение для частицы со спином 1/2 и аномальным магнитным моментом, которое возникает за счет привлечения дополнительных по отношению к биспинору неприводимых компонент в пространстве представлений волновой функции частицы. Еще ранее, в 1928 году, английским физиком Дарвином было предложено уравнение (впоследствии получившее название уравнения Дирака–Кэлера), которое содержит двукратные скалярную и векторную компоненты. Данное уравнение допускает трактовку как РВУ для спина 1/2 и дополнительной внутренней степенью свободы. Уравнения Дирака–Кэлера и Петраша–Улеглы можно считать первыми РВУ, которые выходят за рамки стандартных

положений теории поля. Они продемонстрировали, что в подходе теории РВУ при отказе от условия минимальности используемого набора представлений группы Лоренца можно описывать внутреннюю структуру частиц, включая дополнительные степени свободы. С конца 1960-х – начала 1970-х годов данное направление начинает активно развиваться в нашей республике с работ академика Ф. И. Федорова и его учеников.

За прошедшие десятилетия был накоплен богатый материал по развитию теории релятивистских волновых уравнений с расширенными наборами неприводимых представлений группы Лоренца. В частности убедительно доказано, что отказ от требования минимальности используемых наборов представлений группы Лоренца существенно расширяет возможности метода релятивистских волновых уравнений с точки зрения пространственно-временного описания как внутренней структуры, так и изоспиновых степеней свободы частиц. Получение новых уравнений с более богатой структурой для частицы с заданным спином s возможно либо за счет включения представлений с более высокими спинами, либо за счет использования повторяющихся (кратных) представлений группы Лоренца. Определенный итог работе, проделанной в этом направлении белорусской школой теоретической физики был подведен в недавней книге [1].

Данная заметка посвящена изложенного того, как в рамках общего подхода к теории релятивистских волновых уравнений первого порядка с расширенными наборами представлений группы Лоренца возникают уравнения, известные в литературе как уравнения частиц спина ноль и единица, обладающих дополнительной электромагнитной структурой; имеются в виду уравнение Кокса для частицы со спином 0 и уравнение Шамали–Капри для частицы со спином 1, уравнения Плетюхова–Федорова для частиц спинов ноль и единица с дополнительной электромагнитной характеристикой (часто называемой поляризуемостью).

2 Обобщенное уравнение для скалярной частицы частицы

Исходим из системы уравнений для набора тензорных полей: вектора, скаляра, симметричного и антисимметричного тензоров. Она отвечает частице со спином ноль. Учитываем присутствие внешнего электромагнитного поля. Фактически эта схема эквивалентна подходу Кокса [2]:

$$\lambda_1 D_\mu \psi_\mu^{(1)} + M\psi_0^{(1)} = 0, \tag{1a}$$

$$\lambda_1^* D_\mu \psi_0^{(1)} - \lambda_2 D_\rho \psi_{[\rho\mu]}^{(1)} + \lambda_3 D_\rho \psi_{(\rho\mu)}^{(1)} + M\psi_\mu^{(1)} = 0, \tag{1b}$$

$$-\lambda_2^* (D_\mu \psi_\nu^{(1)} - D_\nu \psi_\mu^{(1)}) + M\psi_{[\mu\nu]}^{(1)} = 0, \tag{1c}$$

$$-\lambda_3^* (D_\mu \psi_\nu^{(1)} + D_\nu \psi_\mu^{(1)} - \frac{1}{2}\delta_{\mu\nu} D_\rho \psi_\rho^{(1)}) + M\psi_{(\mu\nu)}^{(1)} = 0. \tag{1d}$$

Здесь λ_k – числовые параметры, удовлетворяющие условиям

$$\lambda_1\lambda_1^* - \frac{3}{2}\lambda_3\lambda_3^* = 1, \qquad \lambda_2\lambda_2^* - \lambda_3\lambda_3^* = 0\ . \tag{2}$$

Эти условия обеспечивают выполнимость требований: спин частицы равен нулю; частица может находиться лишь в одном спиновом состоянии ($s = 0$).

Выразим из четвертого и третьего уравнений системы (1) тензоры $\psi_{(\mu\nu)}^{(1)}$ и $\psi_{[\mu\nu]}^{(1)}$ и подставим их во второе уравнение системы. В полученном уравнении учтем (1a) и (2). В результате приходим к уравнению

$$\frac{1}{\lambda_1} D_\mu \psi_0^{(1)} + 2ie\frac{\lambda_3\lambda_3^*}{M} F_{[\mu\nu]}\psi_\nu^{(1)} + M\psi_\mu^{(1)} = 0.$$

Полагая

$$\psi_0^{(1)} = \Phi_0, \quad \lambda_1 \psi_\mu^{(1)} = \Phi_\mu,$$

получаем систему из двух тензорных уравнений для 4-вектора и скаляра:

$$D_\mu \Phi_\mu + M\Phi_0 = 0, \tag{3a}$$

$$D_\mu \Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M} F_{[\mu\nu]}\Phi_\nu + M\Phi_\mu = 0. \tag{3b}$$

Введенная система уравнений по сравнению со стандартной системой уравнений для скалярной частицы [3] содержит дополнительное слагаемое

$$2ie\frac{\lambda_3\lambda_3^*}{M} F_{[\mu\nu]}\Phi_\nu \ ;$$

оно отвечает некоторой дополнительной электромагнитной характеристике скалярной частицы.

Приведем матричную форму записи системы (3):

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} J^{(0)}_{[\mu\nu]} + M\right)\Phi = 0,$$

где

$$\Phi = \left| \begin{array}{c} \Phi_0 \\ \Phi_\mu \end{array} \right|,$$

β_μ – 5 x 5-матрицы Петьо–Даффина–Кеммера, $J^{(0)}_{[\mu\nu]} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu$.

Получим нерелятивистский аналог системы уравнений (3). Для этого выделим уравнения относительно функций Φ_0, Φ_k, Φ_4, а также произведем (3+1)-расщепление исходных уравнений. Получаем

$$D_4\Phi_4 + D_k\Phi_k + M\Phi_0 = 0,$$

$$D_k\Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[ka]}\Phi_a + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[k4]}\Phi_4 + M\Phi_k = 0,$$

$$D_4\Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[4c]}\Phi_c + M\Phi_4 = 0.$$

В предположении малости параметра $e\frac{\lambda_3\lambda_3^*}{M}$, слабости внешнего поля, малости функции Φ_k имеем

$$\Phi_k = -\frac{1}{M}\left(D_k\Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[k4]}\Phi_4\right).$$

Следовательно

$$D_4\Phi_4 + M\Phi_0 - \frac{1}{M}D_k\left(D_k\Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[k4]}\Phi_4\right) = 0,$$

$$D_4\Phi_0 + M\Phi_4 + 2ie\frac{\lambda_3\lambda_3^*}{M^2}F_{[k4]}\left(D_k\Phi_0 + 2ie\frac{\lambda_3\lambda_3^*}{M}F_{[k4]}\Phi_4\right) = 0.$$

Вводим обозначения ($B(S)$ – большая (малая) составляющие функций Φ_0, Φ_4)

$$\Phi_0 = B + S, \quad \Phi_4 = B - S \ ;$$

приходим к уравнениям

$$D_4(B-S)+M(B+S)-\frac{1}{M}D_kD_k(B+S)-2ie\frac{\lambda_3\lambda_3^*}{M}D_kF_{[k4]}(B-S)=0,$$

$$D_4(B+S)+M(B_S)+2ie\frac{\lambda_3\lambda_3^*}{M^2}F_{[k4]}D_k(B-S)-4e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}F_{[k4]}F_{[k4]}(B-S)=0.$$

Отсюда следует

$$2D_4B+2MB-\frac{1}{M}D_kD_kB-2ie\frac{\lambda_3\lambda_3^*}{M^2}(D_kF_{[k4]}-F_{[k4]}D_k)B+$$
$$2ie\frac{\lambda_3\lambda_3^*}{M^2}(D_kF_{[k4]}+F_{[k4]}D_k)S-4e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}F_{[k4]}F_{[k4]}(B-S)=0,$$

$$-2D_4S+2MS-\frac{1}{M}D_kD_kB-2ie\frac{\lambda_3\lambda_3^*}{M^2}(D_kF_{[k4]}+F_{[k4]}D_k)B+$$
$$2ie\frac{\lambda_3\lambda_3^*}{M^2}(D_kF_{[k4]}-F_{[k4]}D_k)S-4e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}F_{[k4]}F_{[k4]}(B-S)=0.$$

В полученных уравнениях совершаем формальную замену $D_4 \Longrightarrow (D_4-M)$, отвечающую выделению энергии покоя; одновременно пренебрегаем малой компонентой S в сравнении с большой B; кроме того, пренебрегаем членом D_4S по сравнению с MS. В результате приходим к более простым уравнениям (пусть $B=\Psi, S=\psi$):

$$D_4\Psi=\left(\frac{1}{2M}D_kD_k+ie\frac{\lambda_3\lambda_3^*}{M^2}(\partial_kF_{[k4]})+2e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}F_{[k4]}F_{[k4]}\right)\Psi, \tag{4a}$$

$$\psi=\frac{1}{4M}\left(\frac{1}{M}D_kD_k+2ie\frac{\lambda_3\lambda_3^*}{M^2}(D_kF_{[k4]}+F_{[k4]}D_k)+4e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}F_{[k4]}F_{[k4]}\right)\Psi. \tag{4b}$$

Первое уравнение содержит только большую компоненту B, оно представляет собой нерелятивистское обобщенное уравнение Шредингера для скалярной частицы. Согласно анализу Швебера [4], такого типа уравнение можно интерпретировать как уравнение для частицы, заряд которой распределен в объеме радиуса порядка $\sqrt{\lambda_3\lambda_3^*}/M$, обладающей вследствие этого электрической поляризуемостью. Соответствующий параметр поляризуемости определяется соотношением

$$2e^2\frac{(\lambda_3\lambda_3^*)^2}{M^3}. \tag{5}$$

3 Обобщенное уравнение для векторной частицы

Рассмотрим еще один вариант расширенных релятивистских уравнений первого порядка для частицы со спином 1, взаимодействующей с внешним электромагнитным полем (отмечаем, что используемый при этом набор тензорных полей тот же самый, что и в случае скалярной частицы):

$$\lambda_1D_\mu\psi_\mu^{(1)}+M\psi_0^{(1)}=0, \tag{6a}$$

$$\lambda_1^*D_\mu\psi_0^{(1)}-\lambda_2D_\rho\psi_{[\rho\mu]}^{(1)}+\lambda_3D_\rho\psi_{(\rho\mu)}^{(1)}+M\psi_\mu^{(1)}=0, \tag{6b}$$

$$-\lambda_2^*(D_\mu\psi_\nu^{(1)}-D_\nu\psi_\mu^{(1)})+M\psi_{[\mu\nu]}^{(1)}=0, \tag{6c}$$

$$-\lambda_3^*(D_\mu\psi_\nu^{(1)}+D_\nu\psi_\mu^{(1)}-\frac{1}{2}\delta_{\mu\nu}D_\rho\psi_\rho^{(1)})+M\psi_{(\mu\nu)}^{(1)}=0; \tag{6d}$$

причем параметры λ_i подчинены условиям

$$\lambda_1\lambda_1^* - \frac{3}{2}\lambda_3\lambda_3^* = 0, \qquad \lambda_2\lambda_2^* - \lambda_3\lambda_3^* = 1. \tag{7}$$

Выразим из первого и четвертого уравнений системы (6) величины $\psi_0^{(1)}$ и $\psi_{(\mu\nu)}^{(1)}$ через 4-вектор $\psi_\mu^{(1)}$ и подставим их во второе уравнение системы. В результате приходим к уравнению

$$-\lambda_2 D_\nu \psi_{[\nu\mu]}^{(1)} + \frac{\lambda_3\lambda_3^*}{\lambda_2^*} D_\nu \psi_{[\nu\mu]}^{(1)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\rho]}\psi_\rho^{(1)} + M\psi_\mu^{(1)} = 0$$

или (см. (7))

$$-\frac{1}{\lambda_2^*} D_\nu \psi_{[\nu\mu]}^{(1)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\rho]}\psi_\rho^{(1)} + M\psi_\mu^{(1)} = 0. \tag{8}$$

Вводя 4-вектор $\psi_\mu^{(0)}$ и антисимметричный тензор $\psi_{[\mu\nu]}^{(0)}$ согласно соотношениям

$$\psi_\mu^{(0)} = \lambda_2^*\psi_\mu^{(1)}, \qquad \psi_{[\mu\nu]}^{(0)} = \psi_{[\mu\nu]}^{(1)},$$

из (6c), (8) получаем

$$-(D_\mu\psi_\nu^{(0)} - D_\nu\psi_\mu^{(0)}) + M\psi_{[\mu\nu]}^{(0)} = 0, \tag{9a}$$

$$D_\nu\psi_{[\mu\nu]}^{(0)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\rho]}\psi_\rho^{(0)} + M\psi_\mu^{(0)} = 0. \tag{9b}$$

Таким образом, как и в предыдущем случае исходная расширенная система тензорных уравнений (6) (здесь для частицы со спином 1) сводится к системе тензорных уравнений в подходе Петьо–Даффина–Кеммера (9a), (9b) [3], но содержащих дополнительное слагаемое

$$2\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\rho]}\psi_\rho^{(0)}. \tag{10}$$

Отметим также, что при переходе к матричной форме записи системы (9) получаем

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} P J_{[\mu\nu]}^{(0)} + M\right)\psi^{(0)} = 0,$$

где

$$\psi^{(0)} = \left|\begin{array}{c} \psi_\mu^{(0)} \\ \psi_{[\mu\nu]}^{(0)} \end{array}\right|,$$

β_μ – 10x10-матрицы Петьо–Даффина–Кеммера, P – проективный оператор, выделяющий из $\psi^{(0)}$ составляющую $\psi_\mu^{(0)}$, $J_{[\mu\nu]}^{(0)} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu$.

Определим нерелятивистский аналог системы уравнений (9). Проведя (3+1)-расщепление, получаем

$$D_4\psi_{[a4]}^{(0)} + D_k\psi_{[ak]}^{(0)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[ak]}\psi_k^{(0)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[a4]}\psi_4^{(0)} + M\psi_a^{(0)} = 0,$$

$$-D_k\psi_{[k4]}^{(0)} - 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[k4]}\psi_k^{(0)} + M\psi_4^{(0)} = 0,$$

$$D_b\psi_a^{(0)} - D_a\psi_b^{(0)} + M\psi_{[ab]}^{(0)} = 0,$$

$$D_4\psi_a^{(0)} - D_a\psi_4^{(0)} + M\psi_{[a4]}^{(0)} = 0.$$

Отсюда следует

$$\psi_4^{(0)} = \frac{1}{M}D_k\psi_{[k4]}^{(0)} - 2\frac{ie}{M^2}\lambda_3\lambda_3^* F_{[k4]}\psi_k^{(0)},$$

$$\psi_{[ab]}^{(0)} = \frac{1}{M}(D_b\psi_a^{(0)} - D_a\psi_b^{(0)});$$

значит,

$$D_4\psi_a^{(0)} + M\psi_{[a4]}^{(0)} - \frac{1}{M}D_aD_k\psi_{[k4]}^{(0)} - 2\frac{ie}{M^2}\lambda_3\lambda_3^* D_aF_{[k4]}\psi_k^{(0)} = 0,$$

$$D_4\psi_{[a4]}^{(0)} + M\psi_a^{(0)} + \frac{1}{M}D_kD_a\psi_k^{(0)} - \frac{1}{M}D_kD_k\psi_a^{(0)} + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[ak]}\psi_k^{(0)} -$$

$$2\frac{ie}{M^2}\lambda_3\lambda_3^* F_{[a4]}D_k\psi_{[k4]}^{(0)} - 4\frac{e^2}{M^3}(\lambda_3\lambda_3^*)^2 F_{[a4]}F_{[k4]}\psi_k^{(0)} = 0.$$

Полагаем

$$\psi_a^{(0)} = B_a + M_a, \qquad \psi_{[a4]}^{(0)} = B_a - M_a,$$

где B_a, M_a – большая и малая составляющие функций $\psi_a^{(0)}$, $\psi_{[a4]}^{(0)}$. Тогда

$$D_4(B_a + M_a) + M(B_a - M_a) - \frac{1}{M}D_aD_k(B_k - M_k) -$$

$$2\frac{ie}{M^2}\lambda_3\lambda_3^* D_aF_{[k4]}(B_k + M_k) = 0,$$

$$D_4(B_a - M_a) + M(B_a + M_a) +$$

$$\frac{1}{M}D_kD_a(B_k + M_k) - \frac{1}{M}D_kD_k(B_a + M_a) + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[ak]}(B_k + M_k) -$$

$$2\frac{ie}{M^2}\lambda_3\lambda_3^* F_{[a4]}D_k(B_k - M_k) - 4\frac{e^2}{M^3}(\lambda_3\lambda_3^*)^2 F_{[a4]}F_{[k4]}(B_k + M_k) = 0.$$

Отсюда следует

$$2D_4B_a + 2MB_a - \frac{1}{M}(D_aD_k - D_kD_{a)}B_k + \frac{1}{M}(D_aD_k + D_kD_{a)}M_k -$$

$$\frac{1}{M}D_kD_k(B_a + M_a) - 2\frac{ie}{M^2}\lambda_3\lambda_3^* D_aF_{[k4]}(B_k + M_k) + 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[ak]}(B_k + M_k) -$$

$$2\frac{ie}{M^2}\lambda_3\lambda_3^* F_{[a4]}D_k(B_k - M_k) - 4\frac{e^2}{M^3}(\lambda_3\lambda_3^*)^2 F_{[a4]}F_{[k4]}(B_k + M_k) = 0,$$

$$2D_4M_a - 2MM_a - \frac{1}{M}(D_aD_k + D_kD_{a)}B_k + \frac{1}{M}(D_aD_k - D_kD_{a)}M_k +$$

$$\frac{1}{M}D_kD_k(B_a + M_a) - 2\frac{ie}{M^2}\lambda_3\lambda_3^* D_aF_{[k4]}(B_k + M_k) - 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[ak]}(B_k + M_k) +$$

$$2\frac{ie}{M^2}\lambda_3\lambda_3^* F_{[a4]}D_k(B_k - M_k) + 4\frac{e^2}{M^3}(\lambda_3\lambda_3^*)^2 F_{[a4]}F_{[k4]}(B_k + M_k) = 0.$$

Из первого уравнения полученной системы при осуществлении замены $D_4 \Longrightarrow (D_4 - M)$ (выделении энергии покоя) и пренебрежении малой компонентой M_a в сравнении с большой B_a находим (пусть $B_k = \Psi_k$)

$$D_4\Psi_l = \frac{1}{2M} D_k D_k \Psi_l - \frac{ie}{2M}\left(1 + 2\lambda_3\lambda_3^*\right) F_{[lk]}\Psi_k +$$

$$+\frac{ie}{M^2}\lambda_3\lambda_3^*\left(D_l F_{[k4]} + F_{[l4]} D_k\right)\Psi_k + 2\frac{e^2}{M^3}(\lambda_3\lambda_3^*)^2 F_{[l4]}F_{[k4]}\Psi_k = 0. \tag{11}$$

Отмечаем присутствие дополнительных членов взаимодействия частицы с внешними магнитными и электрическими полями. В отсутствие электрического поля уравнение упрощается

$$D_4\Psi_l = \frac{1}{2M} D_k D_k B_l - \frac{ie}{2M}(1 + 2\lambda_3\lambda_3^*)F_{[lk]}\Psi_k \ ; \tag{12}$$

здесь дополнительное слагаемое можно связать с наличием у векторной частицы аномального магнитного момента.

Если исходить из второго уравнения последней системы, выполнить приведенные требования, а также пренебречь на заключительном этапе членом $D_4 M_a$ по сравнению с MM_a, то приходим к уравнению по определению малых функций M_a через большие $B_k = \Psi_k$.

4 Дальнейшие обобщения: частицы с поляризуемостью

Аналогичные обобщенные системы уравнений могут быть развиты на основе зацеплении скаляра, двух векторов, антисимметричного тензора. В случае скалярной частицы исходная система имеет вид [6], [7]

$$\lambda_1 D_\mu \psi_\mu^{(1)} + \lambda_2 D_\mu \varphi_\mu^{(1)} + M\psi_0^{(1)} = 0, \tag{13a}$$

$$\lambda_1^* D_\mu \psi_0^{(1)} + \lambda_3 D_\rho \psi_{[\rho\mu]}^{(1)} + M\psi_\mu^{(1)} = 0, \tag{13b}$$

$$-\lambda_2^* D_\mu \psi_0^{(1)} + \lambda_4 D_\rho \psi_{[\rho\mu]}^{(1)} + M\varphi_\mu^{(1)} = 0, \tag{13c}$$

$$\pm\lambda_3^*(D_\rho\psi_\sigma^{(1)} - D_\sigma\psi_\rho^{(1)}) \mp \lambda_4^*(D_\rho\varphi_\sigma^{(1)} - D_\sigma\varphi_\rho^{(1)}) + M\psi_{[\rho\sigma]}^{(1)} = 0, \tag{13d}$$

где постоянные λ_i, удовлетворяют условиям

$$\lambda_1\lambda_1^* - \lambda_2\lambda_2^* = 1, \qquad \lambda_3\lambda_3^* - \lambda_4\lambda_4^* = 0 \ ; \tag{14}$$

В случае векторной частицы исходная система уравнений (для того же самого набора тензорных полей, но с другой схемой зацепления) следующая [8], [9]:

$$\lambda_1 D_\mu \psi_\mu^{(1)} + \lambda_2 D_\mu \varphi_\mu^{(1)} + M\psi_0^{(1)} = 0, \tag{15a}$$

$$\mp\lambda_1^* D_\mu \psi_0^{(1)} + \lambda_3 D_\rho \psi_{[\rho\mu]}^{(1)} + M\psi_\mu^{(1)} = 0, \tag{15b}$$

$$\pm\lambda_2^* D_\mu \psi_0^{(1)} + \lambda_4 D_\rho \psi_{[\rho\mu]}^{(1)} + M\varphi_\mu^{(1)} = 0, \tag{15c}$$

$$\lambda_3^*(D_\rho\psi_\sigma^{(1)} - D_\sigma\psi_\rho^{(1)}) - \lambda_4^*(D_\rho\varphi_\sigma^{(1)} - D_\sigma\varphi_\rho^{(1)}) + M\psi_{[\rho\sigma]}^{(1)} = 0, \tag{15d}$$

где для постоянных λ_i имеют место ограничения

$$\lambda_1\lambda_1^* - \lambda_2\lambda_2^* = 0, \quad \lambda_3\lambda_3^* - \lambda_4\lambda_4^* = 1. \tag{16}$$

Соответствующая система уравнений для основных компонент в случае скалярной частицы имеет вид (см. обозначения в[10])

$$D_\mu \Phi_\mu \mp \frac{e^2 dd^*}{2M^3} F_{[\rho\mu]} F_{[\rho\mu]} \psi_0^{(0)} + M\psi_0^{(0)} = 0,$$

$$D_\mu \psi_0^{(0)} + M\Phi_\mu = 0. \tag{17}$$

в случае векторной частицы уравнения для основных компонент следующие (см. обозначения в [10]):

$$(D_\sigma \Phi_\rho - D_\rho \Phi_\sigma) \pm \frac{e^2}{2M^3} dd^* F_{[\rho\sigma]} F_{[\mu\nu]} \psi_{[\mu\nu]}^{(0)} + M\psi_{[\rho\sigma]}^{(0)} = 0,$$

$$D_\rho \psi_{[\mu\rho]}^{(0)} + M\Phi_\mu = 0. \tag{18}$$

Описываемые этими уравнения частицы спинов ноль и единица обладают дополнительной электромагнитной характеристикой – поляризуемостью [7].

Рассмотренные примеры указывают на то, что базирующаяся на группе Лоренца и лагранжевом формализме теория релятивистских волновых уравнений первого порядка [1] содержит в себе множество еще неизученных и неиспользованных возможностей для описания дополнительных внутренних структур элементарных частиц.

Список литературы

[1] В.А. Плетюхов, В.М. Редьков, В.И. Стражев. Релятивистские волновые уравнения и внутренние степени свободы. – Беларуская навука: Минск, 2015. – 328 с.

[2] W.J. Cox Higher-rank representations for zero-spin filds theories Phys.: Math. and Gen. 1982. Vol.15, No 2. P. 627–635.

[3] А.А. Богуш. Введение в полевую теорию элементарных частиц. – Мн.: Наука и техника, 1981, – 390 с.

[4] С. Швебер. Введение в релятивистскую квантовую теорию поля. – М.: ИЛ, 1963, – 842с.

[5] A. Shamaly, A.Z. Capri. Unified theories for massive spin 1 fields. Can. J. Phys. 1973. Vol. 51, no 14. P. 1467 – 1470.

[6] В.А. Плетюхов, Ф.И. Федоров Волновое уравнение с кратными представлениями для частицы со спином 0. Весці АН БССР. Сер. фіз.-мат. навук. 1970. no 2. С. 79–85.

[7] А.А. Богуш, В.В. Кисель, М.И. Левчук, Л.Г. Мороз. Ковариантные методы в теоретической физике. Физика элементарных частиц и теории относительности. - ИФ АН БССР: – Минск, 1981. С. 81–90.

[8] В.А. Плетюхов, Ф.И. Федоров. Волновое уравнение с кратными представлениями для частицы со спином 1. Весці АН БССР. Сер. фіз.-мат. навук. 1970. no 3. С. 84–92.

[9] В.В. Кисель. Электрическая поляризуемость частиц со спином 1 втеории релятивистских волновых уравнений. Весці АН БССР. Сер. фіз.-мат. навук. 1982. no 3. С. 73–78.

[10] Редьков В.М. Поля частиц в римановом пространстве и группа Лоренца. Белорусская наука, Минск, 2009. 486 стр.

Три типа дуальности преметрических уравнений электродинамики: геометрический аспект

И. Г. Дудко
igordudko89@gmail.com; Институт физики им. Б.И. Степанова, НАН Беларуси
Е. А. Толкачёв
tea@dragon.bas-net.by; Институт физики им. Б.И. Степанова, НАН Беларуси

Аннотация

Показано, что полевые уравнения макроскопической электродинамики инвариантны относительно преобразований из группы $GL(2,R)$, подгруппами которой являются дуальные преобразования напряжённостей и индукций в псевдоримановых и римановых пространствах. Впервые предложена универсальная "преметрическая" запись полевых уравнений с использованием трёх систем гиперкомплексных чисел, охватывающая также два типа уравнений макроскопической электродинамики, инвариантных относительно групп Галилея и соответствующих дуальных преобразований.

Как известно, в работах И. Е. Тамма [1] был развит геометрический подход к описанию распространения света в анизотропных средах. Его суть в представлении полевых уравнений Максвелла в виде, независящем от геометрии 4-мерного дифференцируемого многообразия, которая определяется выбором уравнений связи. При этом оказалось, что для описания явления двойного лучепреломления следует определять 4-интервал через биквадратичную форму. В этом случае уравнение связи индукций и полей определяются через тензор 4-го ранга, распадающегося на произведение двух метрических тензоров второго ранга только при описании изотропных сред. Конечно, возникающая геометрия является эффективной. Во время написания работ [1] можно было думать, что: "Геометризовать" макроскопическую теорию материальной среды можно только "оптическим" способом. Однако открытие эффекта Черенкова и его последующее объяснение И. Е. Таммом и И. М. Франком, увенчанное Нобелевской премией 1958 года, позволило визуализировать тот факт, что эффективная метрика, определяемая биквадратичной формой в пределах среды, существует на фоне метрики пространства Минковского. Особый интерес проблема двух метрик в электродинамике приобретает в связи с тем, что из эффективного нелинейного лагранжиана квантовой электродинамики следует эффект двойного лучепреломления в сильных полях.

Подход Тамма получил дальнейшее развитие в рамках т.н. преметрической формулировки электродинамики, идеологи которой полагают [2], что: "The Maxwell equation in their premetric form are valid in any 4D differential manifold, provided the latter can be split locally into 1 + 3. Accordingly, they are not only beyond special relativity, but also beyond general relativity". Эта идеологема тривиальна с математической точки зрения, поскольку частная подсистема из двух несвязанных пар четырёх уравнений, относительно шести неизвестных каждая, безусловно, менее ограничена в своих симметрийных и прочих свойствах, нежели полная система с уравнениями связи.

В настоящем докладе продемонстрируем последнее утверждение на примере дуальной симметрии полевых уравнений. Тем более, что недавно было выяснено [3], что дуальная инвариантность свободных уравнений Максвелла в вакууме и дуальная инвариантность

полной системы уравнений микроскопической электродинамики, несмотря на совпадение трансформационных свойств напряжённостей, принципиально различаются на уровне соответствующих лагранжианов и наблюдаемых. Мы покажем, что полевые уравнения макроскопической электродинамики инвариантны относительно линейных преобразований из группы $GL(2.R)$,подгруппами которой являются дуальные преобразования напряжённостей и индукций в псевдоримановых и римановых пространствах. Будет предложена также "преметрическая" запись двух типов уравнений макроскопической электродинамики, инвариантных относительно дуальных преобразований галилеевского вида, вперве найденнных в [4].

Напомним, что в преметрическом подходе фарадеевская пара уравнений для полей B, E записывается через внешнюю производную производную от обычной 2-формы, а максвелловская – относительно H и D как внешняя производная относительно твистированной 2-формы. Законы преобразования тензорных коэффициентов твистированных форм при замене координат включают дополнительное умножение на згак детерминанта Якобиана, что позволяет сохранить инвариантность теории, например, относительно отражений пространственных координат. В этом формализме полевые уравнения выглядят следующим обраом [2]:

$$\begin{gathered} dF = d\left(E \wedge dt + B\right) = 0, \qquad \left(\partial_{[k} F_{lm]} = 0\right), \\ dH = d\left(-H \wedge dt + D\right) = J, \qquad \left(\partial_{[k} H_{lm]} = J_{klm}\right), \\ H = k\left[F\right], \qquad \left(H_{ij} = k_{ij}^{lm} F_{lm}\right), \end{gathered} \tag{1}$$

где $F = F_{lm} dx^l \wedge dx^m/2$ – 2-форма напряжённостей, $H = H_{lm} dx^l \wedge dx^m/2$ – твистированная 2-форма индукции, $J = J_{klm} dx^k \wedge dx^l \wedge dx^m/6$ – твистированная 3-форма тока. Очевидно, что предполагается $(1+3)$ расщепление 4-мерного дифференцируемого многообразия, и, соответственно, E и H – пространственные 1-формы, а B и D – пространственные 2-формы, сопоставляемые полям и индукциям. Тензорная запись уравнений дана в скобках.

Наличие знака минус перед H в (1), образно говоря, является "воспоминанием о будущем" переходе к уравнениям электродинамики в 4-мерном локально псевдоримановом многообразии. Прежде чем убедиться в этом, заметим, что первые два уравнения системы (1)в отсутствии источников не изменяют своего вида при умножении на произвольную $2x2$-матрицу с постоянными действительными коэффициентами, реализующую представление группы $GL(2.R)$. Действительно, переписывая их в виде одного матричного уравнения имеем:

$$\begin{gathered} 0 = d\left[\begin{pmatrix} E \\ -H \end{pmatrix} \wedge dt + \begin{pmatrix} B \\ D \end{pmatrix}\right] \Longrightarrow \\ d\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} E \\ -H \end{pmatrix} \wedge dt + \begin{pmatrix} B \\ D \end{pmatrix}\right] = d\left[\begin{pmatrix} E' \\ -H' \end{pmatrix} \wedge dt + \begin{pmatrix} B' \\ D' \end{pmatrix}\right] \end{gathered} \tag{2}$$

где связь штрихованных величин с нештрихованными очевидна. Избавляясь от дилатаций полей и индукций, которые фактически сводятся к выбору единиц измерения, требуем в дальнейшем $ad - cb = 1$, тем самым редуцируем исходную группу до $SL(2.R)$.

Здесь следует подчеркнуть важное обстоятельство, что преобразования $SL(2.R)$ можно применять и при наличии источников, если трактовать их как преобразование констант – переход от однозарядового описания к описанию той же физической системы, но в терминах двух ненаблюдаемых по отдельности зарядов: $\begin{pmatrix} 0 \\ e \end{pmatrix} \to \begin{pmatrix} g' \\ e' \end{pmatrix}$. Наблюдаемый заряд

задаётся инвариантной длинной вектора в зарядовом пространстве – $e = \sqrt{e'^2 + g'^2}$. Такая операция допустима, только если и всё остальные наблюдаемые не будут изменяться при переходе к штрихованным полям и зарядам.

Впервые, частный случай подобных преобразований – $U(1)$-дуальные преобразования был рассотрен в микроскопической электродинамике, смотри историю вопроса в [5]. Как уже упоминалось, использование трёх систем гиперкомплексных чисел позволяет наряду с дуальными преобразованиями в пространстве Минковского единообразно описать [4] их аналоги в расслоеном пространстве-времени Галилея-Ньютона и в R^4. Поэтому в дальнейшем ограничимся рассмотрением однопараметрических подгрупп $SL(2.R)$.

Поскольку первые два уравнения в (1) не зависят от метрики, можно ожидать, что они одновременно инвариантны относительно, по крайней мере, двух из упомянутых дуальных симметрий, выживающих при ограничении на псевдориманову или риманову метрику, соответственно. Доказательство этого предположения, начнём с выявления в полевых уравнениях комплексной структуры, тесно связанной с искусственно введённым знаком минус перед 1-формой H. Благодаря этому, уравнения легко представимы в виде:

$$d\left[j_- \begin{pmatrix} H \\ E \end{pmatrix} \wedge dt + \begin{pmatrix} B \\ D \end{pmatrix}\right] = \begin{pmatrix} 0 \\ J \end{pmatrix}, \tag{3}$$

где $j- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j_-^2 = -1$. Уравнение (6), без источников сохраняет свой вид при умножении на $e^{j_-\vartheta} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$, т.е. при дуальных преобразованиях из группы $U(1) \approx SO(2)$. Эта, присущая уравнениям максвелловской электродинамики в пространстве Минковского инвариантность, легко обобщается на случай наличия внешних источников, если трактовать (2) в терминах эффективных, ещё раз подчеркнём, не наблюдаемых по раздельности, зарядов:

$$d\left[j_- \begin{pmatrix} H \\ E \end{pmatrix} \wedge dt + \begin{pmatrix} B \\ D \end{pmatrix}\right] = \begin{pmatrix} J_0^g = J\sin(\vartheta_0 = 0) \\ J_0^e = J\cos(\vartheta_0 = 0) \end{pmatrix}. \tag{4}$$

Таким образом, $U(1)$-дуальные преобразования имеют универсальный вид для любых псевдоримановых пространств, вне зависимости от конкретного выбора метрики. Вообще говоря, это не удивительно, поскольку стандартным образом комплексная структура вводится с помощью операции звёздочка $(*)$, двойное применение которой в n-мерном пространстве к кососимметричным тензором T типа $\begin{pmatrix} 0 \\ k \end{pmatrix}$ даёт

$$*(*T) = (-1)^{k(n-k)}\mathrm{sgn}\,[\det(g_{ij})]T. \tag{5}$$

В отсутствие метрики на полевых уравнениях реализуется и вторая возможность, которая остаётся единственной, когда $\det(g_{ij}) > 0$, то есть в произвольном римановом пространстве четырёх измерений:

$$d\left[j_+ \begin{pmatrix} \tilde{H} = -H \\ E \end{pmatrix} \wedge dt + \begin{pmatrix} B \\ D \end{pmatrix}\right] = \begin{pmatrix} \tilde{J}_0^g = Jsh(\vartheta_0 = 0) \\ \tilde{J}_0^e = Jch(\vartheta_0 = 0) \end{pmatrix}, \tag{6}$$

где $j_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\quad j_+^2 = 1$. Очевидна инвариантность (6) относительно гиперболических поворотов из группы $SO(1.1)$ – $e^{j_+\vartheta} = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix}$. Таким образом, преметри-

ческие уравнения электродинамики (1) действительно дважды дуально инвариантны.

Однако, в работе [4] было показано, что сузествует универсальная бикватернионная запись уравнений Максвелла для трёх типов пространств. Кроме рассмотренных выше $R^{1.3}$ и R^4, были определены дуальные преобразования уравнений электродинамики в неевклидовом пространстве Галилея-Ньютона. При этом впервые было показано, что имеет место неразрывная связь между всеми тремя дуальными симметриями. Так в терминах вектора Римана-Эльберштейна $(\bar{B} - j_{\mp,0}\bar{E})$ дуальные преобразования сводятся к его умножению на $exp(j_{\mp,0}\vartheta)$, где свойства $j_{\mp}$ описаны выше, а $j_0^2 = 0$. Всеё это позволяет обобщить уравнения (1) так, чтобы они охватывали и нетривиальный случай расщепления $(1+3)$, соответствующий геометрии Гелилея-Ньютона:

$$d\left[j_{\mp,0}\vartheta\begin{pmatrix}(\pm,+)H\\E\end{pmatrix}\wedge dt+\begin{pmatrix}B\\D\end{pmatrix}\right]=\begin{pmatrix}0\\J\end{pmatrix}. \tag{7}$$

Но, в отличие от $j_{\mp}$, мнимую дуальную единицу j_0 можно выбрать двумя способами:

$$j_0^m=\begin{pmatrix}0&0\\-1&0\end{pmatrix},\quad\begin{pmatrix}0&1\\0&0\end{pmatrix}. \tag{8}$$

Подстановка (7) в (8) приводит к двум системам уравнений, первая из которых:

$$d\left(-H\wedge dt+D\right)=J,\quad db=0 \tag{9}$$

является формулировкой на языке дифференциальных форм уравнений "магнитной"или "домаксвелловской" электродинамики, отличающейся от уравнений Максвелла отсутствием тока смещения, а вторую:

$$dD=J,\quad d\left(E\wedge dt+B\right)=0 \tag{10}$$

принято называть "электрической". Напомним, что связь между пространственно-временными и дуальными преобразованиями двух типов галилей-инвариантных уравнений электродинамики в вакууме подробно исследована в [7].

В заключении лишь отметим, что выбор дуальных единиц (8) в виде разбиения мнимой единицы $j_- = j_0^m + j_0^e$ обеспечивает после введения метрики правильный нерелятивистский предел от лоренц-инвариантных уравнений электродинамики к галилей-инвариантным.

Список литературы

[1] Тамм И. Е. Собрание научных трудов. Т. 1. М., 1975

[2] F. W. Hehl, Yu. N. Obukhov, G. F. Rubilar, M. Blagojevic / arXiv:gr-qc/0506042 v1 P. 1 – 15.

[3] Толкачёв Е. А. // ДНАН Беларуси. 2014. Т. 58, №6, С.41-46.

[4] Курочкин Ю. А., Толкачёв Е. А. // ДАН БССР. 1990. Т.34. №8. С. 695 – 697.

[5] Стражев В. И., Томильчик Л. М. Электродинамика с магнитным зарядом. Мн., 1975.

[6] Б. А. Дубровин, С. П, Новиков, А. Т. Фоменко Современная геометрия. М., 1979.

[7] Толкачёв Е. А. //ДНАН Беларуси. 2011. Т. 55, №5, с. 44-48.

Организаторы выражают благодарность за спонсорскую помощь:
Национальной академии наук Беларуси
Белорусскому Республиканскому фонду фундаментальных исследований
ЗАО «Солар ЛС»
ООО «Магия света»
ЗАО «Голографическая индустрия»

National Academy of Sciences of Belarus
B.I. Stepanov Institute of Physics
Belarussian Republican Foundation for Fundamental Research
Belarussian State University
Belarussian Physical Society
Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine

V CONGRESS OF PHYSICISTS OF BELARUS:

Proceedings of the IX International Conference

"METHODS OF NON-EUCLIDEAN GEOMETRY IN PHYSICS AND MATHEMATICS"

Bolyai–Gauss–Lobachevsky-9 (BGL-9)

Minsk, 27 – 30 November 2015

Edited by Yu. Kurochkin and V. Red'kov

Подписано к печати Формат
Бумага офсетная Усл. изд. л.
Тираж экз.

Институт физики имени Б.И. Степанова НАН Беларуси
220072 Минск, ГСП. пр. Независимости, 68