On the Andronov–Hopf Bifurcation

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Two-dimensional quadratic dynamical systems are mainly considered. We study the Andronov–Hopf bifurcation by means of canonical systems with field-rotation parameters. Applying such systems, we construct, for example, a quadratic system with at least four limit cycles in (3 : 1) distribution and develop techniques of the functions of limit cycles for the investigation of various limit cycle bifurcations. All these results will be used further for the study of local bifurcation surfaces and global families of multiple limit cycles, and will be applied to the solution of Hilbert’s Sixteenth Problem on the maximum number and relative position of limit cycles of arbitrary polynomial systems.

Key words: Hilbert’s Sixteenth Problem, Andronov–Hopf bifurcation, field-rotation parameter, limit cycle
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1 Introduction

In this paper, the bifurcation of birth of limit cycles from a singular point of the focus or center type (the Andronov–Hopf bifurcation) is considered [1–16] and, first of all, with the help of this bifurcation, examples of quadratic systems with the maximum number of limit cycles are constructed.

In Section 2, Bautin’s theorem on the cyclicity of a singular point of the focus or center type in the quadratic case [7] and also some examples of quadratic systems with at least four limit cycles in (3 : 1) distribution with different number of singular points at infinity, which were obtained for the first time in the works [9, 10, 13], are given.

In Section 3, a canonical system with two field-rotation parameters is constructed and the explicit expression of coefficients of the constructed canonical system via the coefficients of an arbitrary quadratic system is given.

This system is especially convenient for the cases of two singular points in the finite plane, and it is considered in Section 4, where concrete results on limit cycle bifurcations are given. In particular, it is proved that a quadratic system can have at least four limit cycles, and also theorems on the existence of quadratic systems with various number and relative position of limit cycles are proved. A new numerical analytic approach to the construction of systems with a certain number and relative position of limit cycles is developed, and with the help of this approach, a quadratic system with two antisaddles in the finite plane and a saddle at infinity, which has at least four limit cycles in (3 : 1) distribution, is constructed. Unlike the first examples of quadratic systems with four limit cycles [9, 10], the techniques developed in our works [6, 12, 14] is more general and allows to find the whole classes of systems with various number and relative position of limit cycles including the case of four limit cycles. Besides, in this section we develop theory of functions of limit cycles and techniques for the control of limit cycle bifurcations by means of two field-rotation parameters. All these results will be used further for the
study of local bifurcation surfaces and global families of multiple limit cycles [15, 16].

In Section 5, where various examples on the construction of functions of limit cycles and the corresponding numerical results are given, the concrete values of parameters of a quadratic system with at least four limit cycles (to within its even numbers) in (3 : 1) distribution are presented.

2 Preliminaries

On the history of discovery of the bifurcation. The Andronov–Hopf bifurcation is a bifurcation of the birth of a limit cycle (the auto-oscillation) from a singular point of the focus or center type when the parameter of the system is passing through the critical value corresponding to the change of character of stability of the singular point.

Both the discovery of the bifurcation of the birth of a limit cycle from a singular point with only imaginary eigenvalues and the establishment of the connection between this bifurcation and Lyapunov’s focus quantities belong to A. A. Andronov. In his report “Mathematical problems of auto-oscillations” at the 1st All-Union conference on oscillations in 1931, A. A. Andronov told, without formulas, about a bifurcation of the birth of a cycle from a focus in connection with the appearance of auto-oscillations in the lamp generator (in the same report, he considered the bifurcation of a double limit cycle). In the first edition “Theory of oscillations” by A. A. Andronov and S. È. Khaikin (1937), the bifurcation of the birth of a limit cycle from a weak focus in the plane had already been described with mathematical proofs and examples. There were also recurrence differential equations for Lyapunov’s quantities. Later, this bifurcation was considered by Andronov’s collaborators in numerous works on the investigation of dynamical systems arising from applications, and it was considered both for systems of the second order and for higher-dimensional dynamical systems [1, 2, 6]. In 1942, E. Hopf generalized Andronov’s result to the higher-dimensional case, and now the bifurcation is often referred to as a “Hopf bifurcation”. Then, a bifurcation of the birth of several limit cycles from a singular point of the center type or from a weak focus of an arbitrary degree of nonroughness (of any codimension) was considered, this bifurcation is called a “generalized Hopf bifurcation” [6].

Bautin’s result. The main problem of the study of the Andronov–Hopf bifurcation is the problem on the maximum number of limit cycles which can appear from a singular point under the perturbation of the system. This problem has been solved completely only for the quadratic case of polynomial systems, and this is one of the best results in the local analysis of such systems. In 1939, N. N. Bautin announced and, in 1952, published the complete proof of the following theorem [7]:

**Theorem 1 (by N. N. Bautin)** The maximum number of limit cycles which can appear in a quadratic system from a singular point of the focus or center type is equal to three.

In [7], the quadratic system

\[
\frac{dx}{dt} = \sum_{i+j=1}^{2} a_{ij} x^i y^j, \quad \frac{dy}{dt} = \sum_{i+j=1}^{2} b_{ij} x^i y^j \tag{1}
\]

with a singular point of the focus or center type at the origin \(O(0,0)\) is considered, and a notation of the cyclicity of a singular point is introduced.

**Definition 1** A focus or a center \(O(0,0)\) of the systems (1) has the cyclicity of the order \(k\), if:

a) it is possible to find such numbers \(\varepsilon_0\) and \(\delta_0\) that there is no point in the \(\varepsilon_0\)-neighborhood of the point corresponding to \(O\) in the parameter space of the system (1), to which a system of the type (1) having more than \(k\) limit cycles in the \(\delta_0\)-neighborhood of the point \(O\) in the plane \(x, y\), would correspond;

b) for any positive numbers \(\varepsilon < \varepsilon_0\) and \(\delta < \delta_0\), it is always possible to find such a point in the \(\varepsilon\)-neighborhood of the point under consideration in
the parameter space corresponding to the given system (1), to which a system of the type (1) having exactly \( k \) limit cycles in the \( \delta \)-neighborhood of the point \( O \), corresponds.

The cyclicity problem of the point \( O(0,0) \) for the system (1) is solved in [7] with the help of the canonical system

\[
\begin{align*}
\dot{x} &= -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\
\dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2,
\end{align*}
\tag{2}
\]

to which an arbitrary quadratic system (1) with a focus or a center is reduced. This problem depends on the structure of coefficients of the return map in the neighborhood of the singular point and demands to know all center conditions.

Putting \( x = \rho \cos \varphi, \ y = \rho \sin \varphi \), we introduce the polar coordinates into the system (2) and we find the solution of the system as a series on degrees of the initial solution \( \rho_0 \). The segment of the straight line \( \varphi = 0 \) for the sufficiently small \( \rho \) is a segment without contact for trajectories of the system (see Figure 1). Letting \( \varphi = 2\pi \) in the found solution, we obtain the return map

\[
\rho = \rho_0 u_1(2\pi, \lambda_i) + \rho_0^2 u_2(2\pi, \lambda_i) + \ldots + \rho_0^k u_k(2\pi, \lambda_i) + \ldots
\tag{3}
\]
in the sufficiently small segment \( \varphi = 0, \ 0 \leq \rho_0 \leq \rho \).

![FIG. 1. The return map in the neighborhood of a singular point.](image)

The coefficients of the return map, as follows from their constructing, are the whole functions of the parameters \( \lambda_i \), and they become homogeneous polynomials for \( \lambda_1 = 0 \). For the system (2), all center cases are known [1–6], and they can be obtained from the conditions of vanishing the first seven coefficients of the return map (i.e., of three consecutive Lyapunov’s quantities):

\[
\begin{align*}
L_1 &\equiv \alpha_3 = -\frac{\pi}{4} \lambda_5 (\lambda_3 - \lambda_6), \\
L_2 &\equiv \alpha_5 = \frac{\pi}{24} \lambda_2 \lambda_4 (\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6), \\
L_3 &\equiv \alpha_7 = -\frac{5\pi}{32} (\lambda_3 - \lambda_6)^2 (\lambda_3 \lambda_6 - 2\lambda_6^2 - \lambda_2^2).
\end{align*}
\]

Since the functions \( u_k(2\pi, \lambda_i) \) with \( k > 7 \) must also vanish for the values \( \lambda_i \) satisfying the conditions \( \lambda_1 = \alpha_3 = \alpha_5 = \alpha_7 = 0 \), then \( u_k \) can be represented in the form

\[
u_k(2\pi, \lambda_i) = \lambda_1 \theta_k^{(1)} + \alpha_3 \theta_k^{(3)} + \alpha_5 \theta_k^{(5)} + \lambda_2 \lambda_4 (\lambda_3 - \lambda_6)(\lambda_3 \lambda_6 - 2\lambda_6^2 - \lambda_2^2)\theta_k.
\]

Note that the factor \( (\lambda_3 - \lambda_6) \) is contained in the expression for \( \alpha_7 \) with the second degree. It is possible to show that \( \theta_k \) also contains this factor for any \( k > 7 \), that allows to introduce the third Lyapunov’s quantity \( \alpha_7 \) into the expression of \( u_k(2\pi, \lambda_i) \) and to represent the return map (3) in the form

\[
\rho - \rho_0 = \rho_0 (2\pi \lambda_1 \psi_1 + \alpha_3 \psi_3 \rho_0^2 + \alpha_5 \psi_5 \rho_0^4 + \alpha_7 \psi_7 \rho_0^6),
\]

where \( \psi_j \) are series of \( \rho_0 \) degree with the coefficients as whole functions of parameters \( \lambda_i \) and such that

\[
\begin{align*}
\psi_1(\lambda_i, 0) &= 1 + \lambda_1 \varphi(\lambda_1), \\
\psi_j(\lambda_i, 0) &= 1, \ j \neq 1.
\end{align*}
\]

In the sufficiently small neighborhood of the origin, positive simple roots of the equation

\[
2\pi \lambda_1 + \alpha_3 \rho_0^2 + \alpha_5 \rho_0^4 + \alpha_7 \rho_0^6 = 0
\tag{4}
\]

approximate the roots of (3) for \( \rho = \rho_0 \) with any precision.

Considering (3) and (4), it is not so hard to see that at most three limit cycles depending on the sign of Lyapunov’s quantities can exist in a neighborhood of the origin, near the boundary of the stability domain in the parameter space, and in [7] is
shown how to obtain the system (2) with three limit cycles around the point $O(0,0)$.

The singularities in the behaviour of the system (2) near those points of the boundary of the stability domain, where the first and second Lyapunov’s quantities vanish, are determined by the sign of the third Lyapunov’s quantity $\alpha_7$. Considering the return map (3) and the equation (4), we can find all possibilities for the behaviour.

In Figure 2, there is a neighborhood of a point of the parameter space in the plane $\alpha_3, \alpha_5$, where the conditions $\lambda_1 = \alpha_3 = \alpha_5 = 0$ hold for $\alpha_7 > 0$. The plane $\alpha_3, \alpha_5$ is divided into the domains marked in the figure by 0, 1, 2, to which the same number of limit cycles in the neighborhood of the singular point $O(0,0)$ corresponds. These domains are determined by the conditions:

0) $\alpha_5 \geq 0$ or $\alpha_5 < 0$, $\alpha_3 > f(\alpha_5, \alpha_7)$;

1) $\alpha_3 < 0$ or $\alpha_3 = 0$, $\alpha_5 < 0$;

2) $\alpha_5 < 0$, $0 < \alpha_3 < f(\alpha_5, \alpha_7)$,

where the function $f(\alpha_5, \alpha_7)$ in the neighborhood of the value $\alpha_5 = 0$ has the asymptotic representation

$$(4\psi^7(0, \alpha_7))^{-1}\alpha_5^2 \equiv (4\alpha_7)^{-1}\alpha_5^2.$$ 

For varying parameter $\lambda_1$, one more limit cycle can appear around the origin. Thus, for a sufficiently small $\lambda_1 < 0$, when condition 2 holds, there are three limit cycles around the point $O$ (see Figure 2).

![Figure 2. Bautin’s result.](image)

In 1955, probably under the influence of Bautin’s result, I.G. Petrovskii and E.M. Landis [6] undertook an attempt to prove that a quadratic system can have at most three limit cycles and, in 1957, tried to generalize this result to the case of an arbitrary polynomial system, and even put forward their work for the Lenin price. However, there were some mistakes (found first by N.P. Erugin) in the work. The authors tried to correct them, but unsuccessfully. In spite of the fact that their work began the investigation on the whole of the behaviour of solutions of nonlinear differential equations in a complex domain and contained a number of fundamental results, it had not been completed [6]. And after that when in 1979 the Chinese mathematicians constructed concrete examples of quadratic systems with at least four limit cycles [9, 10], the conjecture of I.G.Petrovskii and E.M.Landis was completely contradicted.

**The example by Shi Sonling.** In the work [9], the system

$$\begin{align*}
\dot{x} &= -y + (25 + 9\delta - 8\varepsilon)xy - y^2, \\
\dot{y} &= x - \lambda y - x^2 - (5 + \delta)xy + y^2,
\end{align*}$$

(5)

with $\delta = 10^{-13}$; $\varepsilon = 10^{-52}$; $\lambda = 10^{-200}$, is considered, and Bendixon’s method of annual regions is used (for their construction, respectively Lyapunov’s method is applied).

As is known [1–6], solving the problem of distinguishing center and focus for the system

$$\begin{align*}
\frac{dx}{dt} &= -y - P(x,y), \\
\frac{dy}{dt} &= x + Q(x,y),
\end{align*}$$

(6)

where $P(x,y)$ and $Q(x,y)$, generally speaking, are analytic functions in a neighborhood of the origin, A.M. Lyapunov has proved the existence of a formal integral of this system in the form

$$U(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{k=3}^{\infty} F_k(x,y),$$

(7)

where $F_k(x,y)$ are homogeneous polynomials of degree $k$, for which by virtue of the system (6)

$$\frac{dU}{dt} = \sum_{k=2}^{\infty} G_{2k}(x^2 + y^2)^k.$$ 

(8)
With the help of the expansion (7), in [9] for the system (5) with \( \lambda = 0 \), the functions

\[
H_1(x, y) = \frac{1}{2}(x^2 + y^2) + F_3(x, y) + F_4(x, y), \quad (9)
\]
\[
H_2(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{k=3}^{6} F_k(x, y) \quad (10)
\]

are constructed and, using (8), it is proved that the corresponding closed curves

\[
H_1(x, y) = 10^{-121}, \quad (11)
\]
\[
H_2(x, y) = 10^{-39} \quad (12)
\]

are cycles without contact for the system (5), and along the cycles (11), (12), by virtue of (5), the following inequalities hold:

\[
\frac{dH_1}{dt} > 0, \quad \frac{dH_2}{dt} < 0. \quad (13)
\]

Since the system (5) has only two singular points in a finite part of the plane (a stable focus \((0,0)\) and an unstable focus \((1,0)\)) and only singular point (a saddle) at infinity, taking into account the inequalities (13) and the character of conductivity of the isocline of “infinity” \(1 - (25 + 9\delta - 8\epsilon)x - y = 0\) (see Figure 3), by the annual principle, it is valid the following conclusion [9].

\[\text{FIG. 3. The example by Shi Sonling.}\]

**Theorem 2 (by Shi Sonling)** The system (5) has at least four limit cycles in \((3 : 1)\) distribution.

A similar quadratic system is considered in [10]:

\[
\dot{x} = -y(1 - 3x + (2/9)y),
\]
\[
\dot{y} = x + \delta_1y - x^2 - (1 - \delta_2)xy + 3y^2, \quad (14)
\]

where it is proved that for some sufficiently small values of the parameters \(\delta_1, \delta_2\), the system (14) also has at least four limit cycles in \((3 : 1)\) distribution: at least three around one focus and one around another.

**The example by E. E. Andronova.** In both works [9, 10], the quadratic system has only singular point (a saddle) at infinity. The first example of a system with at least four limit cycles for three singular points at infinity (two saddles and one node) was constructed in the work [13]. Following this work, we will consider the system

\[
\dot{x} = -y + mxy + ny^2,
\]
\[
\dot{y} = x + \lambda y + ax^2 + bxy + cy^2. \quad (15)
\]

Let \(\lambda = b = n = 0\); then for the conditions \(c(a + m) > 0, a(a + m) < 0, a(m - c) > 0\), this system has two singular points of the center type ((0, 0) and \((-1/a, 0)\)) in the finite plane and three singular points (two saddles and one node) at infinity. The center domains, in this case, are bounded by two separatrix curves leaving to the saddles at infinity, and they are branches of the hyperbola

\[
F(x, y) = \frac{a}{m-c} \left( x + \frac{a+m-c}{a(m-2c)} \right)^2 - y^2
\]
\[
= \frac{(a+c)(a+m-c)}{ac(m-2c)}. \quad (16)
\]

Put additional conditions for the system (15) with \(\lambda = 0\):

\[
b(a + c) - n(2c + m) = 0, \quad b \neq 0, \quad n \neq 0 \quad (17)
\]

which correspond to vanishing the first Lyapunov’s quantity at the point \((0,0)\). Since under the condition

\[
(2c+m)(m-c)(6c^2+2a^2+4ac-m^2)
\]
\[
- a(a+c)^3 \geq 0, \quad (18)
\]

the hyperbola (16) is a curve without contact for the system (5) with

\[
a(m - 5a - 3c) > 0, \quad (19)
\]

then, as is shown in [13], the system (5) has at least two limit cycles: a cycle around the point (0, 0) and a cycle around the point \((-1/a, 0)\) (see Figure 4). And since for the new additional condition

\[ m - 5a - 3c = 0 \]

(20)

the point \(O(0,0)\) becomes a weak focus of the third order, the following theorem is true.

**Theorem 3 (by E. A. Andronova)** If for the system (15) with two saddles and one node at infinity the conditions (18), (19) hold, then there exist such small additions to its coefficients destroying the conditions (17), (20) that the perturbed system (15) has at least four limit cycles in (3 : 1) distribution.

![Figure 4](image)

**FIG. 4.** The example by E. A. Andronova.

**Remark 1** For \(m \to c\), the hyperbola (16) degenerates to the straight line without contact \(mx - 1 = 0\), and for the conditions \(c(a + m) \geq 0, a(a + m) < 0, a(m - c) \leq 0\), the system (15) has at least four limit cycles in \((3 : 1)\) distribution, with only saddle at infinity.

## 3 Construction of systems with field-rotation parameters

**Canonical systems with two singular points and two field-rotation parameters.** Consider the system

\[
\begin{align*}
\dot{X} &= d_{10}X + d_{01}Y + \sum_{i+j=2} d_{ij}X^iY^j, \\
\dot{Y} &= c_{10}X + c_{01}Y + \sum_{i+j=2} c_{ij}X^iY^j,
\end{align*}
\]

(21)

having focus at the origin \((i,j = 0, 1, 2)\). Using linear transformations, we will obtain a system with field-rotation parameters from it. On the example of this system in the case of two singular points in a finite part of the plane (a saddle and an antisaddle or two antisaddles) we will show how to study bifurcations of limit cycles. Applying numerical analytic methods, we will construct systems with various number of limit cycles.

We will present also some numerical results. In particular, we will determine a way of the construction of quadratic systems having at least four limit cycles in \((3 : 1)\) distribution and will give concrete values of the coefficients of the system with such number of limit cycles, when there are two foci in the finite plane and a saddle at infinity.

Let system (21) have two singular points in the finite plane. Then the polynomial

\[ S(k) = A_3k^3 + A_2k^2 + A_1k + A_0, \]

(22)

with

\[
\begin{align*}
A_0 &= c_{10}d_{20} - c_{20}d_{10}, \\
A_1 &= c_{10}d_{11} + c_{01}d_{20} - c_{11}d_{10} - c_{20}d_{01}, \\
A_2 &= c_{10}d_{02} + c_{01}d_{11} - c_{02}d_{10} - c_{11}d_{01}, \\
A_3 &= c_{01}d_{02} - c_{02}d_{01},
\end{align*}
\]

satisfies one of the following conditions:

1) \(A_3 \neq 0, A_3^2 - 3A_1A_3 > 0, S(k_1)S(k_2) > 0;\)
2) \(A_3 \neq 0, A_3^2 - 3A_1A_3 \leq 0;\)
3) \(A_3 = 0, A_2 \neq 0, A_2^2 - 4A_0A_2 = 0;\)
4) \(A_3 = A_2 = 0, A_1 \neq 0,\)

where \(k_1, k_2\) are the roots of the equation \(S'(k) = 0\). If \(O(0,0)\) is a weak focus, the system (21)–(23) is reduced to the form [11):

\[
\begin{align*}
\dot{x} &= -y + b_{11}xy + b_{02}y^2 - \gamma y^2, \\
\dot{y} &= x - x^2 + a_{02}y^2 + \gamma xy
\end{align*}
\]

(24)

under the condition of

\[
\begin{align*}
b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, b_{11} < 1, \\
g_3^0 = b_{02}(b_{11} + 2a_{02}) \neq 0
\end{align*}
\]

(25)
or
\[ b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, \quad b_{11} > 1, \quad g_0^0 \neq 0. \] (26)

In the system (24), as will be shown below, the parameter \( \gamma \) rotates the field. To obtain a general form of the system with limit cycles, it is sufficient to introduce another rotation parameter.

**Theorem 4** A quadratic system (21) with a focus at the origin, and no other singular points in a finite part of the plane except a simple saddle point, can be reduced by nonsingular linear phase-variable transformations to the form
\[
\begin{align*}
\dot{x} &= (\alpha x - y)(1 + \gamma y) - \alpha x^2 + b_{11}xy \\
&\quad + (b_{02} + \alpha a_{02})y^2 \equiv P, \\
\dot{y} &= (x + \alpha y)(1 + \gamma y) - x^2 - \alpha b_{11}xy \\
&\quad + (a_{02} - \alpha b_{02})y^2 \equiv Q,
\end{align*}
\] (27)

where
1) \( b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, \) \( b_{11} < 1; \)
2) \( \alpha \neq 0 \lor \alpha = 0: \) \( g_3 \neq 0 \lor g_3 = 0, \) \( g_5 \neq 0 \lor g_3 = g_5 = 0, \) \( g_7 \neq 0 \)

(here \( g_3, g_5, g_7 \) are respectively the first, second, and third focus quantities of the focus \( O(0,0) \) of (27) for \( \alpha = 0 \)).

**Remark 2** The first and second focus quantities, which we will need for the further work, will be obtained (to within a constant positive cofactor) from the formulas given in [6]:
\[
\begin{align*}
g_3 &= b_{02}(b_{11} + 2a_{02}) - \gamma(3a_{02} + b_{11} - 1); \\
g_5 &= -6a_{02}\gamma^3 + (6b_{02} + 5a_{02}b_{02})\gamma^2 \\
&\quad + (3a_{02}^2 - 5b_{02}^2 - b_{11}^2 - 3a_{02}b_{11} \\
&\quad + a_{02}b_{11}^2 + 7a_{02}b_{11} - 8a_{02} - 4b_{11} + 5)\gamma, \\
\end{align*}
\] (29)

where
\[ \gamma = \gamma^* = \frac{b_{02}(b_{11} + 2a_{02})}{3a_{02} + b_{11} - 1}. \]

**Theorem 5** A quadratic system (21), with a focus at the origin and no other points in a finite part of the plane except a simple antisaddle-point, can be transformed by nonsingular linear phase-variable transformations into the form (27), where
\[
\begin{align*}
1) &\quad b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, \quad b_{11} > 1; \\
2) &\quad \alpha \neq 0 \lor \alpha = 0: \quad g_3 \neq 0 \lor g_3 = 0, \quad g_5 \neq 0 \lor g_3 = g_5 = 0, \quad g_7 \neq 0.
\end{align*}
\] (30)

Properties of the canonical systems. Let us mention some properties of the canonical systems.

**Lemma 1** The parameters \( \alpha, \gamma \) rotate the vector field of the system (27).

**Proof** By virtue of the definition of a field-rotation parameter [8]:
\[
PQ'_\gamma - QP'_\gamma = P(x + \alpha y)y - Q(\alpha x - y)y \\
= (\alpha^2 + 1)y^2((b_{11} - 1)x^2 + b_{02}xy + a_{02}y^2).
\]

Since \( b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, \) the sign of the obtained expression depends only on whether the condition (28) or the condition (30) is used with the system (27). Therefore the parameter \( \gamma \) rotates the vector field of this system. On the other hand, the system (27) can be expressed as
\[
\begin{align*}
\dot{x} &= P + \alpha Q, \\
\dot{y} &= Q - \alpha P.
\end{align*}
\]
Then
\[
(P + \alpha Q)(-P) - (Q - \alpha P)Q = (P^2 + Q^2) \leq 0,
\]
and so the parameter \( \alpha \) also rotates the field of (27).

**Lemma 2** If the system (27) satisfies to conditions of Theorem 4 or of Theorem 5, then its limit cycles surrounding the focus \( O(0,0) \) intersect the \( x \)-axis at points tending to \( O \) for \( \gamma \to +\infty \) (\( \alpha < 0 \)) and \( \gamma \to -\infty \) (\( \alpha > 0 \)).

**Proof** We will carry out the proof for the system (27) under the conditions of Theorem 4 and for \( \gamma \to +\infty \) with \( \alpha < 0 \). The reasoning is similar for the other cases. Thus, after making the substitution
\[
\begin{align*}
x &= \mu X, \\
y &= \mu Y, \\
\mu &= 1/\gamma
\end{align*}
\]

and returning to the original notation, we will obtain
\[
\begin{align*}
\dot{x} &= (ax-y)(1+y) + \mu(-ax^2 + b_{11}xy) \\
&\quad + (b_{02} + \alpha a_0 y)^2), \\
\dot{y} &= (x+\alpha y)(1+y) + \mu(-x^2 - \alpha b_{11}xy) \\
&\quad + (a_{02} - \alpha b_{02})y^2).
\end{align*}
\] (31)

For the system (31) there is an upper horizontal straight line tangent to the isocline of “zero”, with the equation
\[
y = \overline{y} = -(1 - \mu(2q + \alpha b_{11}))^{-1}
\sim -1 - \mu(2q + \alpha b_{11}), \quad \mu \to 0,
\]

where
\[
q = -\alpha - \sqrt{\alpha^2(1 - b_{11} + \alpha b_{02} - a_{02})}.
\]

If \(y = y(x)\) is the equation of a trajectory of the system (31) through a point on the line \(y = \overline{y}\), that on this line
\[
y' = K_0 + o(1), \quad \mu \to 0,
\]
\[
y'' = \frac{1}{\mu}
\times \frac{K_0(1+\alpha^2)((1-b_{11})x^2 + b_{02}x - a_{02})}{(-\alpha x + 1)(2q + \alpha b_{11}) - \alpha x^2 - b_{11}x + b_{02} + a_{02})^2}
+ o(1),
\]

where
\[
K_0 = -\frac{(x-q)^2}{(ax+1)(2q + \alpha b_{11}) - \alpha x^2 - b_{11}x + b_{02} + \alpha a_{02}}.
\]

It follows that \(y' < 0, y'' < 0\) on the line \(y = \overline{y}\) for sufficiently small \(\mu > 0\) and sufficiently large negative \(x\). For \(x \to -\infty\) there is an asymptotic formula holding uniformly with respect to \(y\) and \(\mu\) with \(\overline{y} < y < \overline{y} + \mu\) and \(0 < \mu < \mu_0\), where \(\mu_0\) is a fixed number:
\[
y' = \frac{1}{\alpha} + o(1), \quad x \to -\infty.
\]

Consider three distinct cases:
1) \(c \equiv \alpha^2 b_{11} - \alpha b_{02} + a_{02} < 0\); 
2) \(c = 0\); 
3) \(c > 0\).

These cases correspond to the inequalities
\[
\overline{y} < -1, \quad \overline{y} = -1, \quad \overline{y} < -1.
\]

Let us compare the vector field of the system (31) for \(\mu > 0\) with the vector field \(f_0\) of the same system for \(\mu = 0\). Since trajectories of the field \(f_0\) differ from trajectories of the field \(f_\mu(\alpha x - y, x + \alpha y)\) only by nonisolated equilibrium points \(y = -1\), then we can consider the field \(\mathcal{F}_0\) instead of \(f_\mu\).

Find the determinant
\[
\det [f_\mu, f_0] = \mu y(1 + \alpha^2)((b_{11} - 1)x^2 + b_{02}xy + a_{02}y^2) \equiv D.
\]

For \(y < 0\) we have \(D > 0\), i.e., for increasing \(t\), trajectories of the field \(f_\mu\) intersect trajectories of the field \(f_0\) from the right to the left. To localize the limit cycles of the system (31) we will use the fact that they do not intersect the straight line \(y = \overline{y}\), which is transversal to the field \(f_\mu\).

Consider the first case. Let the point \(\mathcal{M}(\overline{x}, \overline{y})\), \(\overline{x} < \alpha\), be arbitrary. Then, in the strip \(\overline{y} < y < 0\), the part of the trajectory of the system (29) passing through the point \(\mathcal{M}\) will be to the right of the part of the trajectory \(S\) of the field \(\mathcal{F}_0\) through the same point. Let \(S\), for decreasing \(t\), intersects the \(x\)-axis at the point \(M_1\) the first time and at the point \(M_2\) the second time. Then limit cycles of the system (31) will be in the domain bounded by the arch \(\mathcal{M}M_1\) of the trajectory \(S\) and the arch of the trajectory \(S_1\) of (31) passing through the point \(M_1\). For sufficiently small \(\mu > 0\) the second arch will be arbitrary close to the arch \(M_1M_2\) of the trajectory \(S\); hence, for sufficiently small \(\mu > 0\), limit cycles of the field \(f_\mu\) reach the domain bounded by the straight line \(y = -1\) and an arch of the trajectory of the field \(\mathcal{F}_0\) passing through the point \((\overline{x}, -1)\), the other end of which is also on the line \(y = -1\).

The reasoning is similar for the second case.
In the third case we consider the trajectory of the field $f_\mu$ passing through the point $\mathcal{M}(x, y)$, $x < 0$, with $|x|$ sufficiently large, for example, such that $|y| > |\alpha|/2$. Then it will intersect the straight line $y = -1$ at a point $(x_1, -1)$, and $|x_1 - x_0| < 2|\alpha||y+1|$ for sufficiently small $\mu$. The rest of the reasoning is as in the first case.

Hence, in all three cases, for sufficiently small $\mu > 0$ the limit cycles of the field $f_\mu$ reach a closed bounded domain. Thus for $\gamma \to +\infty$ the limit cycles of the system (27) contract to the point $O(0, 0)$, and this proves the lemma.

Consider the system (27) with conditions (28) or conditions (30). Since limit cycles of the system, if they exist, do not intersect when only one of the parameters $\alpha$ or $\gamma$ varies (because the parameters rotate the field), there is a function $\gamma = \varphi(\alpha, x, a_{02}, b_{11}, b_{02})$ equal to the value of $\gamma$ for which the system (27) has a limit cycle passing through the point $(x, y)$ for given values of the parameters $\alpha$, $a_{02}$, $b_{11}$, $b_{02}$. The set of points $(x, y, \gamma)$, where $(x, y)$ is a point of the limit cycle and $\gamma$ is the corresponding value of $\varphi$ for fixed $\alpha$, $a_{02}$, $b_{11}$, $b_{02}$, forms an Andronov–Hopf manifold of the system (27). To study bifurcations of limit cycles, it is sufficient to consider the function $\varphi_0$ equal to $\varphi$ for $y = 0$. Since the coefficients $a_{02}$, $b_{11}$, $b_{02}$ are usually fixed, we will use the abbreviated notation of the function $\varphi$: $\gamma = \varphi_0(\alpha, x)$.

**Lemma 3** The function $\gamma = \varphi_0(\alpha, x)$ satisfies the following conditions:

1) $\varphi_0 \to +\infty$, $x \to +0$, $\alpha < 0$ ($\alpha > 0$);
2) $\varphi_0 \to b_{02}(b_{11}+2a_{02})/(3a_{02}+b_{11}-1)$, $x \to +0$, $\alpha = 0$.

**Proof** The first condition follows from Lemma 2. For definiteness let $\alpha < 0$. Then Lemma 2 implies that

$$
\lim_{x \to +0} \varphi_0(\alpha, x) = +\infty.
$$

For fixed $\gamma$, the system (27) has a finite number of limit cycles in a fixed neighborhood of the point $O$; hence

$$
\lim_{x \to +0} \varphi_0(\alpha, x) = +\infty.
$$

The reasoning is similar for $\alpha > 0$. The second condition follows from [11], and this complete the proof of the lemma.

## 4 Bifurcations of limit cycles in two-parameter families of vector fields

**Bifurcations of algebraic limit cycles.** Consider the system

\[
\begin{align*}
\dot{x} &= -y(ax + by + 1), \\
\dot{y} &= x(ax + by + 1) + \beta(x^2 + y^2 - c)
\end{align*}
\]

which, for $0 < c(a^2 + b^2) < 1$ and $\beta \neq 0$, has an algebraic limit cycle in the form of a circle and two singular points: the focus $(x_1, 0)$ and the saddle $(x_2, 0)$, where

\[
\begin{align*}
x_1 &= -1 + \sqrt{1 + 4\beta c(a + \beta)/(2(a + \beta))}, \\
x_2 &= -1 - \sqrt{1 + 4\beta c(a + \beta)/(2(a + \beta))}.
\end{align*}
\]

Transforming the system (32) to the form (27) and retaining the original notation, we obtain:

\[
\begin{align*}
\alpha &= a_1/b_1, \\
\gamma &= (a_1(b_1 + 2) - (x_1 - x_2))/b_1, \\
b_{11} &= \alpha(x_1 - x_2)/(\alpha x_1 + 1), \\
b_{02} &= (b_2(a + \beta)x_1 + 1) - aa_1)(x_1 - x_2)/(b_1(ax_1 + 1)) + \gamma, \\
a_{02} &= (a_1b - b(2(a + \beta)x_1 + 1))(x_1 - x_2) + a_1(b_1(b_02 - \gamma) - a_1))/b_1^2,
\end{align*}
\]

where

\[
a_1 = bx_1/2, \quad b_1 = \sqrt{(ax_1+1)(2(a+\beta)x_1+1) - a_1^2}.
\]

The system (27) with coefficients (35) clearly satisfies the condition (26) and has an ellipsoidal algebraic limit cycle. We use the following notation for
the coefficients (33):
\[ \alpha', \gamma', a'_{02}, b'_{11}, b'_{02}. \]  

(34)

**Theorem 6**  If (27), (28) with coefficients (34) is a system obtained from (31), then, when \( \alpha \in [0, \alpha'] \) and \( x \in [0,1] \), there is a value of \( \gamma \) satisfying the condition \( \varphi_0(\alpha', x) < \gamma < \varphi_0(0, x) \) such that the system (27), (28) with the coefficients
\[ \alpha, \gamma, a_{02}, b_{11}, b_{02} \]  

(35)

has a limit cycle passing through the point \( (x, 0) \).

When the parameters \( \alpha, \gamma \) vary along a certain simple arch, under the rest of the coefficients are fixed, the algebraic limit cycle of the system (27), (28), (35) can be converted into a double limit cycle.

**Remark 3**  The first part of Theorem 6 is stated for \( \alpha' > 0 \); a similar result holds when \( \alpha' < 0 \) (see Figure 5).

![FIG. 5. The bifurcation of an algebraic limit cycle.](image)

**Proof**  Since the system (27), (28), (35) has a limit cycle (an ellipse) when \( \alpha = \alpha', \gamma = \gamma' \), and the parameter \( \gamma \) rotates the field, this system determines a function \( \gamma = \varphi_0(\alpha', x) \) for \( x \in [0,1] \). This follows from Lemma 3 and the fact that there is a value of \( \gamma \) for which the system (27) has a separatrix loop of the saddle \((1,0)\). It follows from Theorem 7 in [11] that the function \( \gamma = \varphi_0(0, x) \) for the system (27), (28), (34) is determined as well. Fix an arbitrary \( x_0 \in ]0,1[. \) There are values of \( \gamma_1 = \varphi_0(\alpha', x_0) \) and \( \gamma_2 = \varphi_0(0, x_0) \) corresponding to \( x_0 \), for which the system (27), (28), (35) has a limit cycle passing through this point. In fact, since \( \alpha' > \alpha_0 > 0 \), we conclude that when \( \alpha \) passes from the value \( \alpha' \) to \( \alpha_0 \) the vector field of the system (27), (28), (35) with \( \gamma = \gamma_1 \) rotates counter-clockwise, and the limit cycle of the system changes size: it contracts if it is stable, and expands if it is unstable. For the cycle to return to its former dimensions, the field of the system must be rotated in the opposite direction. This can be done by increasing \( \gamma \) from the value \( \gamma_1 \) to some value \( \gamma_0 \) for \( \alpha = \alpha_0 \). When \( \alpha \) varies from zero to \( \alpha_0 \), the field of the system (27), (28), (35) with \( \gamma = \gamma_2 \) rotates clockwise; hence \( \gamma_0 \) must be smaller than \( \gamma_2 \). Since \( x_0 \) and \( \alpha_0 \) are arbitrary, this proves the first part of the theorem.

To prove the second part, we consider the system (27), (28), (35) for \( \alpha = 0 \). It follows from Theorem 7 in [11] that the function \( \gamma = \varphi_0(0, x) \) corresponding to this system can be monotonic or can have one extremum if there are not more than two limit cycles around the focus. Since we are considering (27), (28) under extra restrictions on the coefficients inherited from the original system (34), the function \( \gamma = \varphi_0(0, x) \) for the system (27), (28), (35) will be assumed to be in general monotonic. Suppose for definiteness that \( g_0^0 \equiv b_{02}(b_{11} + 2a_{02}) < 0 \). If \( \alpha' > 0 \), and the function \( \gamma = \varphi_0(\alpha', x) \) has an extremum, then the corresponding value of \( \gamma \) together with \( \alpha = \alpha' \) form the required pair of parameters. If there is no extremum for \( \alpha' > 0 \), then the monotonicity of the function \( \varphi_0 \) implies that an extremum can be obtained by letting \( \alpha \) tend to zero and applying the first part of the theorem. Since the function \( \varphi_0 \) is monotonic, there is a neighborhood \( |\alpha| < \varepsilon \) of \( \alpha = 0 \) in which the system (27), (28), (35) with \( \gamma = \varphi_0(\alpha, x) \) has at least one limit cycle. Then, considering the case \( \alpha' < 0 \), we can find a sufficiently small \( \alpha > 0 \) and a corresponding value of \( \gamma \) for which we have a double limit cycle.
The proof is completed.

**Limit cycles in the case of a saddle and an antisaddle.** We will consider bifurcations of limit cycles in the case when there are only two singular points in the finite plane: a saddle and an antisaddle.

**Theorem 7** Any limit cycle of an arbitrary system (27), (28), by varying the coefficients in the parameter space along a certain simple arch, can be converted into a triple limit cycle.

**Proof** It is easy proved that (27), (28) can be reduced to the system with a weak focus of the third order. On the other hand, there is a triple limit cycle in such system [11]. Hence we have to prove the existence of a certain arch in the parameter space of the system connecting the point corresponding to an arbitrary limit cycle with the found triple limit cycles.

Consider the parameters space of the system (27):

\[ \Omega = P_1 \times P_2, \]

where \( P_1 \) and \( P_2 \) are the sets of points \( p_1(\alpha, \gamma) \) and \( p_2(a_{02}, b_{11}, b_{02}) \), respectively. The inequalities \( b_{02}^2 - 4(b_{11} - 1)a_{02} < 0 \) and \( b_{11} < 1 \) in (28) specify, in \( P_2 \), the interior of a “cup” of a cone. The domain \( \Omega_1 \subset \Omega \) in which these inequalities are satisfied is simply connected. It is clear that condition 2 in (2.28) does not violate the connectedness of \( \Omega_1 \), since the condition for a center implies that this is a manifold \( \Omega_2 \) in \( \Omega \) with codimension at least two. Hence the domain \( \Omega_0 = \Omega_1 \setminus \Omega_2 \), corresponding to the condition (28), is simply connected. Thus the system (27), (28) has a simply connected parameter domain, where there is a point corresponding to a triple limit cycle. Connecting this point by a simple arch with the point corresponding to an arbitrary limit cycle, we obtain the conclusion of the theorem.

**A quadratic system with four limit cycles.** Let us find a system for which it is possible to obtain at least four limit cycles. Suppose that the system (21) has two antisaddles in the finite plane: \((0,0)\) and \((1,0)\). Then Theorem 5 implies that it is reduced to a system of the form (27) with the condition (32), and the following result holds.

**Theorem 8** Any system (27), (30), by varying the coefficients in the parameter space along a certain simple arch, can be reduced to a system having at least four limit cycles.

**Proof** The domain of the coefficients of the system (27) determined by the condition (30) is clearly simply connected. Hence, by letting \( a_{02}, b_{11}, b_{02} \) vary along some simple arch, we obtain the conditions

\[ g_3^0 > 0, \ g_5 < 0, \ b_{02}^2 - 4(b_{11} - 1)a_{02} < 0, \ b_{02} > 0, \ \alpha = 0. \]

For \( 0 < \gamma < b_{02} \), for example, Theorem 9 in [11] implies that the graph of the function \( \gamma = \varphi_0(\alpha, x) \) is as shown in Figure 6 (the dashed curve).

![FIG. 6. The function of limit cycles in the case of four limit cycles.](image)

For \( \alpha \neq 0 \) we have a curve \( \gamma = \varphi_0(\alpha, x) \), such that \( \gamma \to \infty \) for \( x \to +0 \) (the continuous curve). Hence, for some sufficiently small \( \alpha > 0 \) and the corresponding \( \gamma \), the system (25), (30) has at least four limit cycles: at least three around of focus \((0,0)\) and one around the focus \((1,0)\) (Figure 7). This proves the theorem.
For the system
\[ \dot{x} = -y(1 + \gamma y) + b_{02}y^2, \]
\[ \dot{y} = x(1 + \gamma y) - x^2 + a_{02}y^2, \] (36)
where
\[ b_{02}^2 + 4a_{02} < 0; \]
\[ \alpha \neq 0 \lor \alpha = 0: g_3 \neq 0 \lor g_3 = 0, \] (37)
\[ g_5 \neq 0 \lor g_3 = g_5 = 0, \]
the inequality \( b_{02}^2 + 4a_{02} < 0 \) determines an unbounded domain in the space of the coefficients \( a_{02}, b_{02} \) of the system (38). Let \( a_{02} \to -\infty \), fix \( b_{02} \), and take \( \gamma = \gamma_0 \); this corresponds to a separatrix loop of the saddle \((1,0)\). The following lemma is valid in this case.

Lemma 4 For the system (36), (37) with \( a_{02} \to -\infty \), we have the following properties:
1) \( \gamma/\sqrt{-a_{02}} \to 0; \)
2) the separatrix loop contracts to the segment \([0,1]\).

Proof For definiteness we assume that \( b_{02} > 0 \) and \( g_3^0 < 0 \). We first clarify when the system (36), (37) has no limit cycles in the domain bounded by the separatrix loop. Consider the integral of the divergence of the field \( f(x,y) \) of the systems (36) along a closed curve with period \( T \) in the domain specified:
\[ \int_0^T \text{div} f(x,y) \, dt \]
\[ = \int_0^T (\gamma x + 2a_{02}y) \, dt = \int_0^T (\gamma \dot{y} - 2a_{02}\dot{x}) \, dt \]
\[ + \int_0^T (a_{02}(2b_{02} - 3\gamma)y^2 - \gamma^2 xy + \gamma x^2) \, dt \]
\[ = \int_0^T F(x,y) \, dt, \]
where
\[ F(x,y) = a_{02}(2b_{02} - 3\gamma)y^2 - \gamma^2 xy + \gamma x^2. \]
The discriminant of the quadratic form \( F(x,y) \) is
\[ D = \gamma^4 - 4a_{02}\gamma(2b_{02} - 3\gamma). \]
Thus the system (36), (37) has no limit cycles only if \( \gamma > 2b_{02}/3 \). For \( \gamma = 2\sqrt{-a_{02}} \) and \( x \in [0,1] \) it has no limit cycles, since
\[ \gamma^* = 2a_{02}b_{02}/(3a_{02} - 1) < 2b_{02}/3 < \sqrt{-a_{02}} \]
and \( g_3 \) has constant sign for \( \gamma > \gamma^* \) (by virtue of (28) in [11]), it has no cycles for \( \gamma > 2\sqrt{-a_{02}} \). It is thus easily established that the system has cycles only for \( \gamma^* < \gamma < 2b_{02}/3 + \varepsilon(a_{02}) \), where \( \varepsilon(a_{02}) \to 0 \) and \( a_{02} \to -\infty \). Then \( \gamma_0 \) is bounded for \( a_{02} \to -\infty \), and so \( \gamma/\sqrt{-a_{02}} \to 0 \) for \( a_{02} \to -\infty \).

We now prove the second part of the lemma. It is clear that the ordinates of points of a limit cycle of the equation
\[ y' = \frac{x(1 + \gamma y) - x^2 + a_{02}y^2}{-y(1 + \gamma y) + b_{02}y^2}, \]
corresponding to the system (36), cannot exceed in absolute value the ordinates of points of the isocline of “zero” of this equation, which represents an ellipse
\[ x(1 + \gamma y) - x^2 + a_{02}y^2 = 0. \]
The extremal points of this ellipse satisfy the system
\[ x - x^2 + \gamma xy + a_{02}y^2 = 0, \quad \gamma y - 2x + 1 = 0 \]
and have the coordinates
\[ x_{1,2} = \sqrt{-a_{02}/(2\sqrt{-a_{02}})}, \quad y_{1,2} = 1/(\gamma \pm 2\sqrt{-a_{02}}). \]
Hence the first part of the lemma implies that, for \( a_{02} \to -\infty \), the absolute value of \( y_{1,2} \) is smaller than arbitrary given \( \varepsilon_0 \).

We will construct a closed contour formed of segments of the straight lines
\[ y = \pm \varepsilon_0, \quad y = \pm (x + \varepsilon_0), \quad x = 1. \]
The following inequalities hold when the absolute value of \( a_{02} \) is large enough:
\[ x(1+\gamma y) - x^2 + a_{02}y^2 < 0, \quad -\varepsilon_0 \leq x < 0, \]
\[ (x \pm y)(1+\gamma y) - x^2 + (a_{02} \pm b_{02})y^2 < 0, \quad 0 \leq x < 1, \]
\[ -y + (b_{02} - \gamma) y^2 > 0 (\leq 0), \quad -\varepsilon_0 \leq y < 0 (0 < y \leq \varepsilon_0). \]

As a result, Figure 8 gives a picture of conductivity for the given contour. Since \( \varepsilon_0 \) is sufficiently small, the cycle occupies an arbitrarily narrow domain and, for \( a_{02} \to -\infty \), it contracts to the segment \([0, 1]\). This proves the lemma.

5 Numerical results

These results consist essentially of the construction of the function \( \varphi_0 \) for the system (27) with conditions (28) or conditions (30) using the Runge–Kutta method and other numerical methods.

Example 1. Consider the system (27), (28) with coefficients as in (32). The example of finding \( \varphi_0 \) was constructed with
\[ a = 1; \quad b = 0, 4; \quad c = 0, 2; \quad \beta = 0, 25. \]

In this example can compare the exact value of \( \gamma \) for which a limit cycle of the system (27), (28) with coefficients as in (34) passes through the point \((x_0, 0)\), where
\[ x_0 = (x_1 - \sqrt{c})(2(a + \beta)x_1 + 1), \]
with the approximation \( \gamma = \varphi_0(x_0) \). Thus, for the limit cycle passing through the point \((0, 553; 0)\) with \( \gamma = -0, 3057 \), we have the value \( \varphi_0 = -0, 3056 \). The evolution of the limit cycle of the given system for varying \( \alpha \) is shown in Figure 5. For \( \alpha^* = 0, 0087 \), \( \alpha_1 = 10^{-4} \), \( \alpha_0 = 0 \), \( \alpha_4 = 10^{-2} \) we have only a simple cycle for all \( \gamma \); for \( \alpha_2 = -10^{-8} \) and \( \alpha_3 = -10^{-3} \) we can obtain a double cycle. Concerning the generation of a double cycle, we can also consider the sign change of the divergence of the field \( f(x, y) \) of the system (27) at the saddle \((1, 0)\). In our case \( \text{div} f(0, 95; 0) = 0, 0838 \) for \( \alpha = -10^{-2} \) and \( \text{div} f(0, 95; 0) = -0, 0064 \) for \( \alpha = -10^{-3} \).

Example 2. Let the system (27), (28) be arbitrary. In this example we seek values of the function \( \varphi_0 \) for the following coefficient values:
\[ a_{02} = -1; \quad b_{11} = -16; \quad b_{02} = 3, 5; \quad \alpha = 0, \pm 10^{-4}. \]

For \( \alpha = 0; 10^{-4} \), we have one extremum; for \( \alpha = -10^{-4} \) there are two extrema, i.e., for some \( \alpha < -10^{-4} \) and the corresponding \( \gamma \), the system (27), (28) has a triple limit cycle.

Example 3. Consider the system (27) with the condition (28). In this example we let
\[ a_{02} = 10; \quad b_{11} = 14; \quad b_{02} = 3; \quad \alpha = 0, \pm 10^{-6}. \]
The graph of the function $\varphi_0$ for $\alpha = 0$ is shown in Figure 6. For $\alpha = 10^{-6}$ the system has at least four limit cycles (to within their even number): at least three around the focus $(0, 0)$ and one around the focus $(1, 0)$.

In conclusion, we note that possibilities of the application of numerical analytic methods in the qualitative theory of the differential equations can be various:

1) the calculation and investigation of Lyapunov’s focus quantities;
2) finding parameter values corresponding to the moment of formation of a separatrix cycle, and determining the character of stability of the formed separatrix cycle;
3) the control of semistable limit cycles;
4) solving the problems of coexistence of limit cycles around different singular points;
5) obtaining bifurcation curves and surfaces in the parameter space of the system;
6) the construction of concrete systems with a certain number and distribution of limit cycles;
7) the localization and construction of limit cycles.

References


