Minimal Attractor Embedding Estimation Based on Matrix Decomposition for Analysis of Dynamical Systems

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The theoretical ground of a local-topological method for defining a minimal attractor embedding dimension on the basis of matrix decomposition for different types of dynamical systems is proposed. The computer confirmation of the theoretical results is presented. The investigation of digital electrocardio signals using local-topological analysis of chaotic attractor trajectories is carried out.

Key words: dynamical systems, chaotic dynamics, state-space, attractor, minimal embedding dimension, nonlinear decomposition

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1 Introduction

The nonlinear dynamical systems (NDSs) with self-organization named as complex systems [1]-[4] are investigated with great activity in last decades. I. Prigogine and H. Haken have established [1], [2] that functioning a complex NDS is closely connected with the presence of chaos in their behavior. Complex NDSs operate under a wide range of conditions and are, therefore, adaptable and flexible. The NDSs behavior can be described on the basis of construction of an attractor in \( m \)-dimensional Euclidean state-space \( R^m \). It is necessary to select the state-space with the minimal dimension \( m_0 \) as the value \( m_0 \) is an upper limit of the degrees of freedom for a system and, hence, \( m_0 \) gives the minimal number of differential equations for the NDS modelling. The determining \( m_0 \) on the basis of various correlative topological methods requires large computer expenditures and significant volume of experimental data [4]. The method proposed in [5]-[7] allows to reduce the computational complexity and the amount of the required data \( N \) and thus to decrease the lacks of correlative topological methods. The basic idea of the method is the following. On the subset of chaotic attractor in Euclidean space \( R^m \) a function \( z[m] \) is constructed. It defines a measure of topological instability of the attractor when enlarging state space dimension \( R^m \to R^{m+1} \). The value of \( z[m] \) changes monotonously when enlarging \( m \), but if \( m \geq m_0 \), then \( z[m] = const \) and does not depend on \( m \) [5]-[7]. Thus, \( m_0 \) is minimal embedding dimension of investigating NDS attractor.

However, the proposed locally topological method has especially heuristic character as far as in [5] the problems of the theoretical verification (including the necessary and sufficient conditions) of the topological stabilization for parameters of a chaotic attractor were not investigated. The goal of this paper is derivation of the locally topological method for defining minimal attractor embedding dimension of a dynamical system. This paper proves theoretically on the basis of matrix decompositions [8]-[11] in state-space the heuristic fact of stabilization for characteristic parameters of NDS at-
tractor [4]-[7]. We also present the numerical confirmation of the obtained theoretical results and computer electrocardiosignal exploration [7],[12],[13].

2 The matrix decomposition for operators of nonlinear dynamical system into state-space

Further we will give a brief description of the theoretical ground for obtaining minimal attractor dimension of complex NDS as a specific case of NDS on the basis of local-topological method [5]-[7], therefore we use the state-space method for the NDS description. Form the point of view of behavior analysis, the discrete NDS is described in state-space by the relations:

\[ \vec{u}^{(m)}_{n+1} = \vec{f}_1 \left( \vec{u}^{(m)}_n, x_n \right), \quad (1a) \]
\[ y_n = \vec{f}_2 \left( \vec{u}^{(m)}_n, x_n \right), \quad (1b) \]

(where \( \vec{f}_1(\cdot), \vec{f}_2(\cdot) \) are some nonlinear vector functions, \( \vec{u}^{(m)}_n \) is a state-space vector belonging to the state-space \( U \), \( n \) denotes discrete time, \( m \) is dimension of \( U \), \( x_n \) and \( y_n \) are input and output signals respectively). In general, we suppose that \( x_n \neq 0 \), i.e., we consider the NDS with nonzero input signal. We study behavior of the solution for the relation (1a) near to a specific standard state \( \vec{u}^*_n \) being considered as the undisturbed one permanently disturbed by external actions or internal fluctuations on value \( \vec{v}_n \) [1]. For this NDS we will linearize the function \( \vec{f}_1(\cdot) \) near the state \( \vec{u}^{(m)}_n \). In this case we have to use the matrix nonlinear decomposition proposed in [8], [9], [10], [11] for expansion in matrix series of the vector function \( \vec{f}_1(\cdot) \) into state-space. According to [8]-[11] a change of vector-function into state-space can be decomposed into matrix series of the form

\[ \Delta \vec{f}(\vec{v}_n, \vec{u}^{(m)}_n), x_n) = \vec{f}_1(\vec{u}^{(m)}_n) + \vec{v}_n, x_n) \]
\[ -\vec{f}_1(\vec{u}^{(m)}_n), x_n) = L^{(1)}_m \vec{v}_n + \frac{1}{2!} L^{(2)}_{m \times m^2} (\vec{v}_n \otimes \vec{v}_n) \]
\[ + \frac{1}{3!} L^{(3)}_{m \times m^3} (\vec{v}_n \otimes \vec{v}_n \otimes \vec{v}_n) + \ldots, \]

(2)

where

\[ L^{(1)}_m = L^{(1)}_{m \times m} = \left( \frac{\partial}{\partial \vec{u}^*_n} \otimes \vec{f}_1 \right) \vec{v}_n \]
\[ = \left[ \left( \frac{\partial}{\partial \vec{u}^*_1} \vec{f}_1 \right) \ldots \left( \frac{\partial}{\partial \vec{u}^*_m} \vec{f}_1 \right) \right]_0, \]
\[ L^{(2)}_{m \times m^2} = \left( \frac{\partial}{\partial \vec{v}^*_n} \otimes \left( \frac{\partial}{\partial \vec{u}^*_n} \otimes \vec{f}_1 \right) \right) \vec{v}_n - \ldots, \]
\[ L^{(3)}_{m \times m^3} = \left( \frac{\partial}{\partial \vec{v}^*_n} \otimes \left( \frac{\partial}{\partial \vec{v}^*_n} \otimes \vec{f}_1 \right) \right) \vec{v}_n, \]

\[ \vec{f}_1 = \begin{bmatrix} f_{11} \\ \vdots \\ f_{m1} \end{bmatrix} , \]

and \( \otimes \) denotes the symbol of the Kronecker matrix product [8]-[11]. As a result, instead of \( \vec{u}^*_n \) there is a new solution \( \vec{u}^{(m)}_n = \vec{u}^*_n + \vec{v}_n, ||\vec{v}_n|| \ll 1 \), where \( ||.|| \) denotes a norm of a vector. In view of this, one rewrites the relation (2) as follows:

\[ \vec{f}_1(\vec{u}^{(m)}_n, x_n) = \vec{f}_1(\vec{u}^{* (m)}_n) + L^{(1)}_{m} (\vec{u}^{(m)}_n - \vec{u}^{* (m)}_n) \]
\[ + \frac{1}{2!} L^{(2)}_{m \times m^2} (\vec{u}^{(m)}_n - \vec{u}^*(m)_n) \otimes (\vec{u}^*(m)_n - \vec{u}^{* (m)}_n) \]
\[ + \frac{1}{3!} L^{(3)}_{m \times m^3} (\vec{u}^{(m)}_n - \vec{u}^*(m)_n) \otimes (\vec{u}^*(m)_n - \vec{u}^{* (m)}_n) \otimes (\vec{u}^{* (m)}_n - \vec{u}^*(m)_n) + \ldots. \]

(3)

We also suppose zero state \( \vec{u}^{* (m)}_n = 0 \) for the NDS under investigation. Taking into account this condition, the decomposition (3) becomes:

\[ \vec{f}_1(\vec{u}^{* (m)}_n, x_n) = \vec{f}_1(0, x_n) + L^{(1)}_{m} \vec{u}^{* (m)}_n \]
\[ + \frac{1}{2!} L^{(2)}_{m \times m^2} \vec{u}^{* (m)}_n \otimes \vec{u}^{* (m)}_n \]
\[ + \frac{1}{3!} L^{(3)}_{m \times m^3} \vec{u}^{* (m)}_n \otimes \vec{u}^{* (m)}_n \otimes \vec{u}^{* (m)}_n + \ldots. \]

(4)
3 Theoretical estimation of the minimal attractor embedding dimension for a discrete linearized dynamical system

According to (4) linearized in state-space LDS can be described by means of the following first relation:

\[
\tilde{f}_1(\tilde{u}_n^{(m)}), x_n) = L_m^{(1)} \tilde{u}_n^{(m)} + \tilde{f}_1(0, x_n),
\]

where \( \tilde{f}_1(0, x_n) \) has to be a linear vector function relatively \( x_n \), i.e.

\[
\tilde{f}_1(0, x_n) = \tilde{a}^{(m)} x_n,
\]

here \( \tilde{a}^{(m)} \) is a vector-column of size \( m \). Taking into account (1a), (5), (6) we obtain:

\[
\tilde{u}_{n+1}^{(m)} = \tilde{f}_1 \left( \tilde{u}_n^{(m)}, 0 \right) = L_m^{(1)} \tilde{u}_n^{(m)} + \tilde{a}^{(m)} x_n.
\]

Analogously, the second relation (18) for a linearized NDS or, simply, a linear dynamical system (LDS) can be expressed in the form:

\[
y_n = \tilde{f}_2 \left( \tilde{u}_n^{(m)}, 0 \right) = \tilde{d}^{(m)T} \cdot \tilde{u}_n^{(m)},
\]

where \( \tilde{d}^{(m)T} \) is a vector-row of size \( m \), \( T \) is transposition symbol. It is well-known that by means of similarity transform \( T_m L_m^{(1)} T_m^{-1} = K_N \) when \( \det T_m \neq 0 \) the relations (7a) and (7b) can be represented in the form of Kalman’s equations [14]:

\[
\tilde{u}_{n+1}^{(m)} = K_m \tilde{u}_n^{(m)} + \tilde{b}^{(m)} x_n;
\]

\[
y_n = \tilde{c}^{(m)T} \tilde{u}_n^{(m)},
\]

where \( \tilde{u}_n^{(m)} \in U \) is the state-space for the LDS of the dimension \( m \), i.e. \( U = R_m \) is the Euclidean state-space, then \( K_m \) is a Frobenius \((m \times m)\)-matrix of the form:

\[
K_m = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_m & -\alpha_{m-1} & -\alpha_{m-2} & \ldots & -\alpha_1 \\
\end{bmatrix};
\]

by the way, \( T_m \) is a matrix of a similarity transform, \( b_m = \frac{1}{\alpha_0}, A_0 \) is the coefficient at the higher term for the equation describing a discrete stationary LDS [14]. Thus, we will consider the finite-dimensional stationary discrete LDS with one input \( x_n \) and output \( y_n \), in order to simplify our derivation as below [18]. In terms of state-space this system can be described on the basis of linear decomposition (8a), (8b). It was shown in [14],[15] that for any stationary LDS (both continuous and discrete) the state-space \( U \) is \( K_m \)-stationary (or it is invariant relatively to matrix operator \( K_m \) [15]-[17]). That is why the state-space \( U \) is the \( m \)-dimensional cyclic vector space \( C_K[\tilde{c}^T] \) whose basis consists of linearly independent vectors \( \{\tilde{c}^T, \tilde{c}^T K_m, \ldots, \tilde{c}^T K_m^{m-1}\} \) where \( \tilde{c}^T = \tilde{d}^T K_m^{-1} \) is a generative vector. For \( x_n = 0 \) the equations (8a)-(8b) describe free motion of a \( m \)-dimensional stationary discrete LDS:

\[
\tilde{u}_{n+1}^{(m)} = K_m \tilde{u}_n^{(m)}; \quad \text{(10a)}
\]

\[
y_n = \tilde{c}^{(m)T} \tilde{u}_n^{(m)}. \quad \text{(10b)}
\]

Let us calculate the distance between the neighboring vectors in state-space \( U = R_m \) using Euclidean metric \( l_2 \) [5], [6], [18]:

\[
d \left( \tilde{u}_n^{(m)}, \tilde{u}_{n+1}^{(m)} \right) = \| \Delta \tilde{u}_n^{(m)} \| = \| \tilde{u}_{n+1}^{(m)} - \tilde{u}_n^{(m)} \| = \| (K_m - E_m) \tilde{u}_n^{(m)} \| \leq \| K_m - E_m \| \cdot \| \tilde{u}_n^{(m)} \|, \quad \text{(11)}
\]
where $E_m$ is the identity $(m \times m)$-dimensional matrix. By analogy with (11) let us calculate:

$$d\left(\frac{\varepsilon^{(m)}(n)}{u_{n+1}}, \frac{\varepsilon^{(m)}(n)}{u_{n+2}}\right) = \left\| \frac{\varepsilon^{(m)}(n)}{u_{n+2}} - \frac{\varepsilon^{(m)}(n)}{u_{n+1}} \right\| \leq \left\| K_m - E_m \right\| \left\| \frac{\varepsilon^{(m)}(n)}{u_{n+1}} \right\|. \quad (12)$$

Starting from (8)-(12) let us estimate the relative distance [5], [6], [18] :

$$p_n^{(m)} = \frac{d(\varepsilon^{(m)}(n), u_{n+2})}{d(u_{n+1}, \varepsilon^{(m)}(n))} \approx \left\| \frac{\varepsilon^{(m)}(n)}{u_{n+1}} \right\| \leq \frac{\left\| K_m \right\| \left\| \frac{\varepsilon^{(m)}(n)}{u_{n+1}} \right\|}{\left\| \frac{\varepsilon^{(m)}(n)}{u_{n+1}} \right\|} \leq \left\| K_m \right\|. \quad (13)$$

Let us estimate Euclidean $l_2$-norm for Frobenius matrix $K_m$ in accordance with [19] :

$$\|K_m\| = \left( m - 1 + |\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_m|^2 \right)^{1/2} = \left( m - 1 + \sum_{i=1}^{m} |\alpha_i|^2 \right)^{1/2}. \quad (14)$$

With regard to (14) the relation (13) has the form :

$$p_n^{(m)} \leq \left( m - 1 + \sum_{i=1}^{m} |\alpha_i|^2 \right)^{1/2}, \quad (15)$$

where $\alpha_i = \frac{A_i}{A_0}$ are the parameters of LDS in $m$-dimensional state-space. As it follows from (15), in state-space $U = R^{m+1}$ the relative distance is estimated by means of [6],[7],[18]:

$$p_n^{(m+1)} \leq \left( m + \sum_{i=1}^{m+1} |\beta_i|^2 \right)^{1/2}, \quad (16)$$

where $\beta_i$ are the parameters of LDS in $(m + 1)$-dimensional state-space. We can estimate the dynamics of a variation for a topological structure of the attractor for $R^m \rightarrow R^{m+1}$ by means of the relation :

$$q_n^{(m,m+1)} = \frac{p_n^{(m+1)}}{p_n^{(m)}} \leq \frac{\left( m - 1 + \sum_{i=2}^{m+1} |\beta_i|^2 + |\beta_1|^2 + 1 \right)^{1/2}}{\left( m - 1 + \sum_{i=1}^{m} |\alpha_i|^2 \right)^{1/2}}. \quad (17)$$

One can see from (17) that if $m = m_0$ is the dimension of state-space $U$ for given LDS so under the transition to $(m+1)$-dimensional state-space the following relations take place:

$$f : R^m \rightarrow R^{m+1} \Rightarrow \beta_{i+1} \rightarrow \alpha_i, \ i = 1, \ldots, m; \ \beta_1 \rightarrow 1. \quad (18)$$

With regard to (17) we can denote :

$$\sigma_m = m - 1 + \sum_{i=1}^{m} |\alpha_i|^2. \quad (19)$$

Taking into account (18), (19) we can write the relation (17) in the following way :

$$q_n^{(m,m+1)} \leq \frac{(\sigma_m + \beta_1^2 + 1)^{1/2}}{\sigma_m^{1/2}} \rightarrow \left( 1 + \frac{2}{\sigma_m} \right)^{1/2}. \quad (20)$$

For large $\sigma_m$ the value $q_n^{(m,m+1)} \rightarrow 1 + 1/\sigma_m \rightarrow 1$. It should be noted that for not large $\sigma_m$, i.e. for low-dimensional LDS the value $q_n^{(m,m+1)}$ is stabilized to some $q_0^{(m,m+1)} \geq 1$. However, in the case of large values of parameters $|\alpha_i \rightarrow \infty$ even for low-dimensional LDS stabilization of $q_n^{(m,m+1)}$ to 1 takes place as well. For low-dimensional state-space LDS $(m$ is not too large) and small values of LDS parameters $|\alpha_i$ are small quantities) the stabilization $q_n^{(m,m+1)}$ tends to the value $q_0^{(m,m+1)} \neq 1$.

Thus, it is appropriate to introduce the value $q_n^{(m,m+1)}$ averaged over the point set of attractor, so called function of topological instability $z[m]$ [5],[6],[18] :

$$z[m] = \frac{1}{|S|} \sum_{n \in S} q_n^{(m,m+1)}, \quad (21a)$$
where $S$ is some subset of time indexes $n$ belonging to reconstructed attractor area, i.e. $S \subset \mathbb{Z}_N = \{0, 1, \ldots, N - 1\}$ is a residue-class ring of integer number modulo $N$ [16]. Taking into account (20), we can see that the function of topological instability $z[m]$ should be constant: $z_{st} \rightarrow 1$. Then the coefficient of topological stabilization can be evaluated as follows:

$$\rho[m] = \frac{z[m+1]}{z[m]}$$  \hspace{1cm} (21b)

An moment of stabilization of $\rho[m]$ indicates the minimal attractor embedding dimension $m_0$ for dynamical system.

This theoretical result was confirmed by numerical experiment. Pseudorandom process $y_n$ generated on the basis of LDS was used for investigating function of topological instability $z[m]$. The real-valued sequence $\{y_n\}$ with the length $N = 2^\alpha$ was calculated by means of the Rice model [20]. The coefficients $\{Y_k\}$ of the discrete Fourier transform (DFT) were obtained on the basis of given points of the sample spectrum as follows:

$$S_y[k] = \begin{cases} 
A \cdot k, & k = 0, \ldots, l - 1; \\
A \cdot l \cdot C_k, & k = l, \ldots, N - l - 1; \\
A \cdot (N - k - 1), & k = N - l, \ldots, N - 1, 
\end{cases}$$

where $A = \text{const}$, $l \leq N/2 - 2$, $C_k = \cos \frac{2\pi}{N} \varepsilon_k$, $\varepsilon_k$ is the random variable belonging to the interval $[0,1]$. Then the inverse DFT was applied for calculating $N$-point real-valued sequence $y_n$ on the basis of first $(N/2 + 1)$ DFT coefficients $Y_k$ (formed by means of $S_y[k], k = 0, \ldots, N/2$) and highly efficient Hermitian conjugate split-radix FFT developed in [20] (program FFTESR). The example of the real-valued sequence for $N = 64$, $A = 62$ and $l = 17$ generated by the Rice model is presented in Figure 1a. The form of estimated function $\rho[m]$ (Figure 1b) confirms the obtained theoretical results [18],[21].

![FIG. 1. a) The plot of real valued sequence of the length $N = 64$ generated by means of the Rice model. b) The plot of values of the coefficient of topological stabilization $\rho[m]$ for the sequence presented in Fig.1 a.](image)

4 Minimal attractor embedding dimension estimation for a discrete linear dynamical system with an input action

Now let us consider a finite-dimensional stationary discrete LDS with a nonzero input action $x_n$. Let us estimate the distance between the neighbouring vectors in state-space $U = \mathbb{R}^m$. Taking into account
(8a) let us write:
\[
d\left(\bar{u}_n^{(m)}, \bar{u}_{n+1}^{(m)}\right) = \left\|\left(K_m - E_m\right) \bar{u}_n^{(m)} + \bar{u}_n^{(m)} x_n\right\|
\leq \left\|K_m - E_m\right\| \cdot \left\|\bar{u}_n^{(m)}\right\| + |b_m| |x_n|
\] (22)

where \(b_m = 1/\lambda_0\), \(A_0\) is the parameter of \(m\)-dimensional stationary discrete LDS, i.e. the coefficient at higher term for the difference or for differential equation (in the case of continuous LDS) [14]. By analogy with (22) one can see that [6],[7],[22]:
\[
d\left(\bar{u}_n^{(m)}, \bar{u}_{n+1}^{(m)}\right) \leq \left\|K_m - E_m\right\| \cdot \left\|\bar{u}_n^{(m+1)}\right\| + |b_m| |x_n|
\] (23)

With regard to (13), (19), (22), (23) we can estimate the relative distance [6],[7],[22] by means of the following relation:
\[
p_n^{(m)} \approx \left\|K_m - E_m\right\| \cdot \left\|\bar{u}_n^{(m)}\right\| + |b_m| |x_n|
\leq \left\|K_m\right\| \cdot \left\|\bar{u}_n^{(m)}\right\| + |b_m| |x_n| + \frac{|b_m| |x_n|}{\left\|K_m - E_m\right\|}
\] (24)

Calculating Euclidean \(l_2\)-norm [19] for the matrices \(\left\|K_m\right\|\) and \(\left\|K_m - E_m\right\|\) we can write the relation (24) as follows:
\[
p_n^{(m)} \leq \left(m - 1 + \sum_{i=1}^{m} |\alpha_i|^2\right)^{\frac{1}{2}} \left\|\bar{u}_n^{(m)}\right\| + |b_m| |x_n|(1 + \delta_m)
\] (25a)

where
\[
\delta_m = \left[2(m - 1) + |\alpha_1 - 1|^2 + \sum_{i=2}^{m} |\alpha_i|^2\right]^{-1/2}
\] (25b)

according to (19), and \(\alpha_i = \frac{1}{\lambda_i}\) are the parameters of matrix of LDS in the state space \(U = R^m\). It is clear that if the external action \(x_n\) is a jump function with a large amplitude (near the point \(n\)), then \(p_n^{(m)}\) becomes ambiguous value, and so, the stabilization of \(q_n^{(m,m+1)}\) will be broken. Taking into account the notations (19) and (25b) we can write the relation (25a) in the following way:
\[
p_n^{(m)} \leq \left(\frac{1}{\sigma_m} \cdot \left\|\bar{u}_n^{(m)}\right\| + |b_m| |x_n| + \frac{|b_m| |x_n|}{\left|\sigma_m\right|} + 1\right)^{1/2}
\] (26)

On the other hand, we can estimate the relative distance in the state-space \(U = R^{m+1}\) by means of
\[
p_n^{(m+1)} \leq \sum_{i=1}^{m+1} |\beta_i|^2 \quad \Delta_m = \left[2(m + |\beta_1 - 1|^2 + \sum_{i=2}^{m+1} |\beta_i|^2\right]^{-1/2}
\] (27a)

where
\[
\Sigma_m = m + \sum_{i=1}^{m+1} |\beta_i|^2
\] (27b)
\[
\Delta_m = \left[2m + |\beta_1 - 1|^2 + \sum_{i=2}^{m+1} |\beta_i|^2\right]^{-1/2}
\] (27c)

and \(\beta_i\) are the LDS parameters in \((m + 1)\)-dimensional state-space. As it follows from (25a),(26), (27a), we can estimate the dynamics of changing the topological structure of attractor for \(R^m \rightarrow R^{m+1}\) by means of
\[
q_n^{(m,m+1)} = \frac{p_n^{(m+1)}}{p_n^{(m)}} \leq \left(\frac{\Sigma_m^{1/2} \cdot \left\|\bar{u}_n^{(m+1)}\right\| + |b_m+1| \cdot |x_n|(1 + \Delta_m)}{\left\|\bar{u}_n^{(m)}\right\| + |b_m| \cdot |x_n|(1 + \delta_m)}\right)^{1/2}
\] (28)

If \(m = m_0\) is the dimension of state-space \(U\) for LDS under consideration, its mapping into \((m + 1)\)-dimensional space makes correct the relation (18) as well as the following relation:
\[
f : R^m \rightarrow R^{m+1} \Rightarrow b_{m+1} \rightarrow 0.
\] (29)

If the relations (18),(29) are fulfilled and according to (19), (25a) – (28) the value \( q_n^{(m,m+1)} \) can be estimated as follows

\[
q_n^{(m,m+1)} \leq \frac{(\sigma_m + 2)^{1/2} \left( \| u_n^{(m)} \| + \frac{|b_m| |x_n|}{|m-2\alpha_1 + \sigma_m|^{1/2}} \right)}{\sigma_m^{1/2} \| u_n^{(m)} \| + |b_m| |x_n|} < 1 \tag{30}
\]

As we can see from (30) the term \( |b_m| |x_n| \) can have large value for the external action as a jump function. Really, for large \( \sigma_m \) the value \( q_n^{(m,m+1)} \) will tend to 1 as it follows from the arguments with respect to (20). But if the action \( x_n \) has a high amplitude in the moment \( n \), the stabilization \( q_n^{(m,m+1)} \) can be broken in view of the fact that

\[
q_n^{(m,m+1)} \leq \frac{(\sigma_m + 2)^{1/2} \left( \| u_n^{(m)} \| \right)}{\sigma_m^{1/2} \| u_n^{(m)} \| + |b_m| |x_n|} < 1 \tag{31}
\]

for the large \( \sigma_m \) and sharp increase of the action \( x_n \) at the moment \( n \). According to (21a) and (21b) in this case the averaged upon the point set of attractor the value \( q_n^{(m,m+1)} \) is presented in Figure 2b, and etirely, is confirmed by mentioned above theoretical derivations.

The computer experiment was performed for the case [7] where a real-valued non-stationary signal \( y_n \) for \( N = 128 \) was generated in time domain by the pointwise multiplication of initial signal \( y_0 = \sin \frac{2\pi}{N} (kn + \varepsilon_n) \), \( 0 \leq k \leq N/2 - 1; n = 0, \ldots, N-1; \varepsilon_n \in [0,1] \) in some time intervals using input signal \( x_n \).

FIG. 2. a) The plot of real-valued 128-point output signal \( y_n \), generated by switching on the initial output signal \( y_0 = \sin \frac{2\pi}{N} (kn + \varepsilon_n) \), \( 0 \leq k \leq N/2 - 1; n = 0, \ldots, N-1; \varepsilon_n \in [0,1] \) in some time intervals using input signal \( x_n \). b) The plot of values of the coefficient of topological stabilization \( \rho[m] \) for the sequence presented in Fig. 2a.

5 Minimal attractor embedding dimension for a multi-channel linear dynamical system

If discrete multi-channel LDS has \( l \) input, described by vector of input actions \( \vec{x}^T = (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(l)}) \), and \( k \) output, described by vector of output signals \( \vec{y}^T = (y_2^{(1)}, y_2^{(2)}, \ldots, y_2^{(k)}) \), then in \( m \)-dimensional state-space this system is described by the relations...
analogous (8a),(8b):  

$$\bar{u}_{n+1}^{(m)} = K_m \bar{u}_n^{(m)} + B_{m \times l} \bar{v}; \quad (32a)$$  

$$\bar{y} = C_{k \times m} \bar{u}_n^{(m)}. \quad (32b)$$  

In this case we can carry out the derivations analogously (22)-(24) from which one can see that [6]  

$$p_n^{(m)} \leq \left| \frac{\|K_m\| \cdot \|\bar{u}_n^{(m)}\| + \|B_{m \times l}\| \cdot \|ar{v}\| + \|B_{m \times l}\| \cdot \|ar{v}\|}{\|ar{u}_n^{(m)}\| + \|B_{m \times l}\| \cdot \|ar{v}\|} \right| \quad (33)$$  

As it is seen from the comparison (24) and (33), multi-channel LDS in contrast to one-channel LDS has term $\|B_{m \times l}\| \cdot \|ar{v}\|$ instead of $|b_m| \cdot |x_n|$. Thus, the estimate of value $q_n^{(m,m+1)}$ when $R^m \to R^{m+1}$ requires more common conditions than (18) and (29) which not necessary lead to the relation (31). Hence, for the system with some inputs it is more difficult to observe the violation of stabilization for values $q_n^{(m,m+1)}$, $z[m]$ or $\rho[m]$ when amplitude $x_n^{(i)}$ at certain input channel $i$ sharply increases.

It is worth noting that the main conclusions obtained on the basis of Euclidean matrix $l_2$-norm are true for other types of matrix norms ($l_1$-$l_\infty$-norms, maximum row or column norm, spectral norm and so on). For example, the spectral norm  

$$\|K_m\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ is the eigen value for matrix } K_m^T K_m\}$$  

is determined by Carmichael and Mason’s bound [19]:  

$$\|K_m\|_2 \leq \left(1 + \sum_{i=1}^{m} |\alpha_i|^2\right)^{1/2} \leq \left(1 + \sum_{i=1}^{m} |\alpha_i|^2\right)^{1/2}.$$  

This estimation is a few lower than the estimation (14) for the Euclidean norm. It is obvious that the application of spectral norm instead of Euclidean one in the reasoning relative to (15)-(20), (24)-(31),(33) does not change the main conclusions obtained above.

6 Minimal attractor embedding dimension for a nonlinear dynamical system

Let us estimate minimal attractor embedding dimension for NDS [13], [22]. First of all, we calculate the distance (11) and (12) between the neighboring vectors in state-space $U$ using linearization of function $f_1(\bar{u}_n^{(m)}, x_n)$ according to matrix decomposition (4):

$$d(\bar{u}_n^{(m)}, \bar{u}_n^{(m)}) = \|\bar{u}_n^{(m)} - \bar{u}_n^{(m)}\|$$

$$= \left| \bar{f}_1(\bar{u}_n^{(m)}, x_n) - \bar{u}_n^{(m)} \right|$$

$$\approx \left| \bar{f}_1(0, x_n) + L^{(1)}_m \bar{u}_n^{(m)} + \frac{1}{2} \lambda^{(2)}_m \bar{u}_n^{(m)} \otimes \bar{u}_n^{(m)} \right|$$

$$- \bar{u}_n^{(m)} \right| \leq \left| L^{(1)}_m - E_m \right| \cdot \|\bar{u}_n^{(m)}\| + \left| \bar{f}_1(0, x_n) \right|$$

$$+ \frac{1}{2} \left| L^{(2)}_m \cdot \|\bar{u}_n^{(m)}\| \cdot \|\bar{u}_n^{(m)}\| \right| \quad (34)$$  

$$d(\bar{u}_{n+1}^{(m)}, \bar{u}_{n+2}^{(m)}) = \left| \bar{u}_{n+1}^{(m)} - \bar{u}_{n+2}^{(m)} \right|$$

$$\left| \bar{f}_1(\bar{u}_{n+1}^{(m)}, x_n) - \bar{u}_{n+1}^{(m)} \right| \leq \left| L^{(1)}_m - E_m \right| \cdot \|\bar{u}_{n+1}^{(m)}\|$$

$$+ \left| \bar{f}_1(0, x_n) \right| + \frac{1}{2} \left| L^{(2)}_m \cdot \|\bar{u}_{n+1}^{(m)}\| \otimes \bar{u}_{n+1}^{(m)} \right| \quad (35)$$

Taking into account (34) and (35) we can calculate the relative distance (13) by means of:

$$p_n^{(m)} \approx \left| \bar{f}_1(0, x_n) \right| + \left| \bar{f}_1(\bar{u}_{n+1}^{(m)}, x_n) \right|$$

$$\left| \bar{u}_{n+1}^{(m)} \right| + \left| \bar{f}_1(0, x_n) \right| + \left| \bar{f}_1(\bar{u}_{n+1}^{(m)}, x_n) \right|$$

$$\left| \bar{u}_{n+1}^{(m)} \right| + \left| \bar{f}_1(0, x_n) \right| + \left| \bar{f}_1(\bar{u}_{n+1}^{(m)}, x_n) \right|,$$  

where, according to (1a) and (4),

$$\left| \bar{u}_{n+1}^{(m)} \right| = \left| \bar{f}_1(0, x_n) + L^{(1)}_m \bar{u}_n^{(m)} + \frac{1}{2} L^{(2)}_m \bar{u}_n^{(m)} \otimes \bar{u}_n^{(m)} \right|$$
7 Electrocardiosignal analysis based on minimal attractor embedding dimension estimating. Concluding remarks

Chaotic time series represent the basic information about the complex NDS, i.e. NDS with chaotic behavior. We propose a method of chaotic biomedical signal processing on the basis of the obtained theoretical results [7], [13]. The behavior of a NDS can be completely described by means of reconstruction of the chaotic attractor $R_m^A$ in state-space. For the attractor construction we have to use the minimal value of $m$, i.e. $m_0$, because the minimal embedding dimension $m_0$ characterizes the minimal number of differential equations demanded for mathematical modeling of NDS. For computing $m_0$ many correlative-topological methods are used, among them the Grassberger-Procaccia algorithm [4] is most conventional. Such methods have large computational complexity and demand long time series ($N \approx 10^4 - 10^5$) for their implementation. According to (11)-(17), (21a,b) we propose the method for $m_0$ determination based on topological structure analysis of $R_m^A$. This method requires much less experimental data and is stable to changing $m_0$ [5],[7],[12]. If the topological stabilization of $R_m^A$ occurs (i.e. the topological structure of the chaotic attractor is invariant to transformation $R_m^A \rightarrow R_{m+1}^A$), we have that $m \geq m_0$, $z[m] = z_{st}$, where $z_{st}$ does not depend on $m$. Consequently if $m \geq m_0$, then according to (21a) and (21b) $\rho[m] = \frac{z[m+1]}{z_{st}} = 1$ in this case, which is confirmed by the results of the above-mentioned numerical experiments (see Figure 1 and Figure 3).

We propose a method of estimating complexity degrees of heart dynamics by means of electrocardiogram (ECG) time series exploration based on the local topological analysis (11)-(17), (21a), (21b) of attractor reconstructed from electrocardiosignals. In this section the local topological method is used.
for cardiosignal analysis and for cardiac dynamics complexity degree estimation. Basic information obtained by experimental investigation of ECG can be extracted from digital electrocardiosignals: $y_n = y[n\Delta t]$, where $n = 1, 2, \ldots, N, \Delta t$ is the time interval of measurement [12]. According to the Takens method [23], for $m \leq 2d + 1$ the points of the chaotic attractor $R^m_A \subset R^m$ in matrix notation are given by:

$$\vec{u}_n^{(m)T} = \begin{bmatrix} y[np] y[(n+1)p] \ldots y[(n+m-1)p] \end{bmatrix},$$

(37)

where $n = 1, 2, \ldots, L^{(m)}; L^{(m)} = N_p - m + 1; m$ is the embedding dimension, $N_p \approx N/p$. Further, we investigate the topological stabilization of the attractor constructed from ECG [7], [12], [21]. For attractor reconstruction Takens method [23] has been used. In these numerical experiments $N_p = 200$, $1 < p < 4$, $N = 200 - 800$, $\Delta t = 2ms$. In common ECG length equals to 2500 points and we used only the first part of it (Figure 4a). According to (11)-(21a,b) or (34)-(36) as well as (37) numerical implementation of local-topological analysis has been made using calculations $p_n^{(m)}$, $q_n^{(m,m+1)}$, $z[m]$ and $\rho[m]$. Taking into account (37) one can see:

$$u_l = y[lp], \quad l = 1, 2, \ldots N_p, \quad N_p \approx N/p.$$  

(38)

Distances (11) increase when $R^m \rightarrow R^{m+1}$, because from (37), (38) it is easy see [5]:

$$d(\vec{u}_n^{(m)}, \vec{u}_{n+1}^{(m)}) = \left[ \sum_{i=0}^{m-1} (u_{n+i} - u_{n+i+1})^2 \right]^{\frac{1}{2}}.$$  

(39)

and

$$d(\vec{u}_{n+1}^{(m)}, \vec{u}_{n+2}^{(m)}) = \left[ \sum_{i=0}^{m} (u_{n+i} - u_{n+i+1})^2 \right]^{\frac{1}{2}} \geq d(\vec{u}_n^{(m)}, \vec{u}_{n+1}^{(m)}).$$  

(40)

According to (13) we suggest to describe changes of topological structure of $R^m_A$ by relative distances between attractor points:

$$p_n^{(m)} = \frac{d(\vec{u}_n^{(m)}, \vec{u}_{n+2}^{(m)})}{d(\vec{u}_n^{(m)}, \vec{u}_{n+1}^{(m)}), \quad n = 1, 2, \ldots, L^{(m)} - 2.}$$  

(41)

Consequently, dynamics of changes in $R^m_A$ topological structure when $R^m \rightarrow R^{m+1}$, with using (17) can be represented by the sequence $\{q_n^{(m,m+1)}\}$, the terms of that being ratios of relative distances (41) in $R^m$ and $R^{m+1}$ respectively:

$$q_n^{(m,m+1)} = \frac{p_n^{(m+1)}}{p_n^{(m)}}, \quad n = 1, 2, \ldots, L^{(m+1)} - 2. \quad (42)$$

Further, according to (21a) we average the terms of the sequence $\{q_n^{(m,m+1)}\}$ over all $n$ and obtain the function of instability $z[m]$ from the formula:

$$z[m] = \frac{1}{L^{(m+1)} - 2} \sum_{n=1}^{L^{(m+1)}-2} q_n^{(m,m+1)}.$$  

(43)

The value $|z[m] - z_{sl}|$ is a measure of topological instability of $R^m_A$ when transformation $R^m \rightarrow R^{m+1}$ occurs. At least, the coefficient of topological stabilization $\rho[m]$ in $R^m_A$ is calculated by means of (21b). The plot of values for coefficient of topological stabilization $ro(m)$ is represented in Figure 4b for the case of electrocardiosignal.

Thus, the numerical results confirm the convergence of the used algorithm. As it follows from Figure 4b the topological stabilization takes place (i.e. $m = m_0$) when $|\rho[m] - 1| < \varepsilon (\varepsilon = 0.01)$ [12]. From the calculated dependence $\rho[m]$ one can see that $m_0 = 5$. This result is in a just good agreement with those obtained by other authors with GPA employment [24]. At the same time, sequence length in these investigations $N = 200 - 800$, and in [24] $N = 16000$, that is more than an order longer. Consequently, this algorithm of the minimal embedding dimension determination can increase the efficiency of ECG investigation.
8 Conclusion

The paper shows practical usage of matrix series proposed in [10] for expansion of vector functions into the state-space for dynamical systems with the aim of minimal attractor embedding dimension estimating on the basis of local topological analysis algorithms.

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References


