Chaos Criterion in Quantum Field Theory

V.I. Kuvshinov\textsuperscript{1} and A.V. Kuzmin\textsuperscript{2}

\textsuperscript{1} Institute of Physics, NASB, 70 Scarina av., Minsk, 220056, BELARUS
E-mail: kuvshino@dragon.bas-net.by

\textsuperscript{2} Institute of Physics, NASB, 70 Scarina av., Minsk, 220056, BELARUS
E-mail: avkuzmin@dragon.bas-net.by

(Received 9 October 2001)

Chaos criterion for quantum field systems is proposed. Its accordance with classical chaos criterion is demonstrated in semiclassical limit of quantum mechanics.

Key words: Gauge theory, chaos, Green function, local instability

PACS numbers: 11.10.Lm

1 Introduction

Originally phenomenon of chaos was associated with problems of classical mechanics and statistical physics. Attempt of substantiation of statistical mechanics initiated intensive study of chaos and uncovered its basic properties mainly for classical mechanical systems \cite{1}. One of the main results in this direction was creation of KAM theory and understanding of the phase space structure of Hamiltonian systems \cite{2, 3}. It was clarified that the origin of chaos is local instability of dynamical system. Local instability leads to mixing of trajectories in phase space and thus to non-regular behavior of the system and chaos \cite{4, 5, 6}. Significant property of chaos is its prevalence in various natural phenomena. It explains the importance of study of chaos and a large number of works in this field.

Significant progress is achieved in understanding of chaos in semiclassical regime of quantum mechanics via analysis of the spectral properties of the system \cite{6, 7}. Semiclassical restrictions are important, because a large number of energy levels in small energy interval is needed to provide statistics \cite{8}. It was found in one-dimensional quantum mechanics that classical chaos can also strongly affect on quantum properties of the system such as the rate of quantum tunnelling \cite{9}.

Investigation of the stability of classical field solutions faces difficulties caused by the fact that they possess an infinite number of degrees of freedom. That is why authors often restrict their consideration by the investigation of some model field configurations \cite{10}.

There are papers devoted to chaos in quantum field theory \cite{11}. But there is no generally recognized definition of chaos for quantum systems beyond semiclassical approximation \cite{12}. This fact restricts use of chaos theory in the field of elementary particle physics. At the same time it is well known that the field equations of all four types of fundamental interactions possess chaotic solutions \cite{13} and that high energy physics in particular reveals the phenomenon of intermittency \cite{14}.

The aim of this work is to formulate some results depicted chaos in classical field theories and to propose chaos criterion for quantum ones. We also consider application of proposed chaos criterion for investigation of stability of classical solutions of field equations with respect to small perturbations of initial conditions. Proposed method is applicable to all types of field configurations possessing both finite and infinite number of freedoms.

To clear the role of chaos in particle physics one has to learn how chaos affects on observable quantities. Thus it is necessary to study chaos in quantum gauge field theories, modern theories of particle in-
interactions. But as it was already mentioned it is still unclear what is chaos in quantum field theory and what phenomena is it become apparent in?

One can not directly apply well developed methods of classical mechanics to investigate chaos in classical field theory, because field system possesses infinite number degrees of freedom. In such cases one reduces field system to the system with finite number degrees of freedom. For example in the theory of turbulence it is accepted that just a finite number of Fourier modes are strongly coupled and they determine the dynamics of the field [5]. In gauge theories another way is often used. One considers some type of exact solutions of field equations with finite number of freedoms and investigate the dynamics of gauge fields from the viewpoint of chaos within this manifold. Spatially homogeneous solutions are often used for these purposes [15]. This restriction reduces the number of degrees of freedom of field system to a finite one. Thus one can immediately use well developed methods of classical mechanics to study chaotic dynamics of fields. We have to mention that the role of spatially homogeneous solutions and their physical meaning are still unclear. Therefore we have to consider these field configurations as a convenient from the viewpoint of chaos theory model field systems.

Chaotic dynamics of spatially homogeneous field configurations was studied in [16]. It was shown that in SU(2) gauge theory dynamics of spatially homogeneous fields is chaotic at any densities of energy of the system.

Behavior of spatially homogeneous model field configurations in the presence of classical vacuum Higgs field was studied in [17, 18]. New feature was the interaction between gauge fields in the presence of non-zero classical vacuum Higgs field appeared due to spontaneous symmetry breakdown [19]. It brought to the appearance of order-to-chaos transition with the rise of the energy density of gauge fields. Analytical Toda criterion was used to study behavior of model system from the viewpoint of chaos theory [18]. It was found that at low densities of energy, behavior of the system is regular, while at the densities of energy larger than some critical value $E_c$ dynamics of the system becomes locally unstable and chaotic. Critical energy density was calculated. These results are in accord with ones obtained in [17] where $SU(2) \otimes U(1)$ gauge field theory was considered. It describes the dynamics of the boson sector of electro-weak theory. Dynamics of spatially homogeneous field configurations was studied numerically. It was demonstrated that in the presence of non-zero classical vacuum Higgs field order-to-chaos transition occurred with the rise of the energy density of the system.

General conclusion followed from [16, 17, 18] is that in classical gauge field theory with single vacuum dynamics of gauge fields is chaotic at any densities of energy, while if gauge invariance of the vacuum state is spontaneously broken then order-to-chaos transition occurs with the rise of the energy density of the system.

Following previous authors [10, 15, 16] we primarily consider dynamics of some classical model field system and study the influence of quantum corrections on its chaotic behavior.

2 Quantum corrections

Result formulated in the previous section is authentic in classical field theory. Does it remain true if we take into account quantum properties of the fields, for example, quantum properties of Higgs vacuum? The answer is no [20].

2.1 Classical model field system

To demonstrate this we consider $SU(2) \otimes U(1)$ gauge field theory with the Lagrangian

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}H_{\mu\nu}H^{\mu\nu}$$

$$+ \frac{1}{8}g^2 \rho^2 \left( W^2_1 + W^2_2 + \frac{Z^2}{\cos^2 \theta_w} \right) + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho$$

$$- \frac{1}{2} m^2 \rho^2 - \frac{1}{4!} \lambda \rho^4.$$  

(1)

Here $\lambda$ denotes a coupling constant of Higgs fields, $\theta_w$ is Weinberg angle, $A_\mu$ corresponds to electromagnetic field, $W^1_\mu, W^2_\mu$ describe W-bosons, $Z_\mu$ describes neutral Z-boson, $G^a_{\mu\nu}$ is a field tensor of non-abelian gauge fields and $H_{\mu\nu}$ is a tensor of abelian

one. The field $A_\mu$ contributes to both $G^a_{\mu\nu}$ and $H_{\mu\nu}$. We work in the unitary gauge and Higgs doublet has the form

$$\varphi = \left( \begin{array}{c} 0 \\ \frac{\rho}{\sqrt{2}} \end{array} \right).$$  \hspace{1cm} (2)

Further we build spatially homogeneous model field system to demonstrate the role of quantum corrections. We consider W-bosons at classical level only and assume that classical fields of Z-bosons and photons vanish

$$Z_\mu = 0, \quad A_\mu = 0. \hspace{1cm} (3)$$

Nevertheless we shall take into account quantum properties of vacuum fields and study how do they change the dynamics of the model field system. For more details of the method see [18].

Expressions (9-10) show that dynamics of the model field configuration in the classical Higgs vacuum with zero vacuum expectation value of Higgs fields is always chaotic. It coincides with conclusion made in [16] where other methods of chaos detection were used. For future consideration it is useful to formulate this conclusion in the following way: the dynamics of the model field system is stochastic at any, even small densities of energy.

2.2 Effective potential

Now we take into account quantum properties of vacuum fields and study how do they change the dynamics of the model field system. Method of effective potential is used [22, 23]. One loop effective potential generated by the Lagrangian (1) has the form (see also [23])

$$U (\rho) = \frac{1}{4!} \lambda \rho^4 + \frac{3g^4 \rho^4}{128 \pi^2} \left( -\frac{1}{2} \right)$$

$$+ \ln \left( \frac{g^2 \rho^2}{2 \mu^2} \right) + \frac{3g^4 \rho^4}{256 \pi^2 \cos^4 \theta_w} \left( -\frac{1}{2} \right)$$

$$+ \ln \left( \frac{g^2 \rho^2}{2 \mu^2 \cos^2 \theta_w} \right).$$  \hspace{1cm} (11)

Where $\mu^2$ is the renormalization constant. Here we took into account contributions of all Feynman diagrams with one loop of any ($W_1, W_2, Z$ or $A$) gauge field and external lines of Higgs field. Diagrams
with the loop of Higgs scalars are considered as corrections of the higher order and neglected. This potential leads to spontaneous symmetry breaking and non-zero vacuum expectation value of scalar $\rho$–field appears. Classical vacuum of scalar field is $\rho = 0$, but it is not so in quantum case, because of Coleman-Weinberg effect [22].

It is easy to calculate that the squared vacuum expectation value of Higgs field reads

$$\nu^2 = \frac{2\mu^2}{g^2} \times \exp \left[ \frac{(18g^4 \ln \cos \theta_w - 32\pi^2 \lambda \cos^4 \theta_w)}{9g^4 (3 - 8 \sin^2 \theta_w)} \right].$$

(12)

In the next section it will be demonstrated that its existence changes qualitatively the chaotic behavior of spatially homogeneous solutions in quantum vacuum of scalar field compared to pure classical consideration.

### 2.3 Role of quantum corrections

The Lagrangian describing the dynamics of the model field configuration is modified due to quantum corrections and has the following form

$$L = \frac{1}{2} \left( \dot{q}_1^2 + \dot{q}_2^2 \right) - \frac{1}{8} g^2 \nu^2 \left( q_1^2 + q_2^2 \right) - \frac{1}{2} g^2 q_1^2 q_2^2 \sin^2 \xi.$$  

(13)

To study dynamics of the model system we use Toda criterion [18, 21]. In the case of the Lagrangian (13) we have

$$B = \left( \frac{1}{2} g^2 \nu^2 + g^2 \left( q_1^2 + q_2^2 \right) \sin^2 \xi \right) > 0,$$

(14)

$$C = \frac{1}{16} g^4 \nu^4 + \frac{1}{4} g^4 \nu^2 \left( q_1^2 + q_2^2 \right) \sin^2 \xi - 3 g^4 q_1^2 q_2^2 \sin^4 \xi.$$  

(15)

These expressions show that at small densities of energy, when the system exists near the minimum of potential, parameter $C$ is larger than zero and the motion is stable. But at large enough densities of energy the parameter $C$ becomes negative and the motion becomes unstable and non-regular. Thus if the energy density of the model spatially homogeneous field system increases, we obtain the order-to-chaos transition. Critical density of energy $E_c$ corresponding the order-to-chaos transition equals minimal value of the potential of the system on the line defined by the condition $C = 0$. For the system described by the Lagrangian (13) the energy density $E_c$ is given by the following expression

$$E_c = \frac{3}{32} g^2 \nu^4 \sin^2 \xi.$$  

(16)

Thus at the energy densities less than $E_c$ dynamics of the model system is regular and at the energy densities larger than $E_c$ in the phase space of the system we obtain regions of unstable motion and chaos.

It was demonstrated that the same result could be obtained if we use another method of detection of chaos [20].

The main conclusion we can make is that vacuum quantum fluctuations of fields regularize classical dynamics of the model spatially homogeneous field system. Many authors, who studied classical systems of gauge and Higgs fields with just few degrees of freedom, made the conclusion that classical Higgs fields tend classical gauge fields to order (for example see [10],[18]). In this work we have demonstrated analytically that the same conclusion can be made if we take into account quantum properties of gauge fields. We confirmed it only for the particular model system. To clear the role of Higgs fields further investigations are needed. The next problem is to increase the number of degrees of freedom of model system.

### 3 Chaos criterion for quantum fields

#### 3.1 Generalized Toda criterion

Toda criterion of local instability for classical mechanical systems was at first formulated in [21]. It was reformulated for Hamiltonian systems with two degrees of freedom by Salasnich [18]. Agreement between Toda criterion and criterion of classical chaos...
based on KAM theory and conception of nonlinear resonance was checked on the particular example in [20]. Toda criterion was used for analysis of the influence of vacuum quantum fluctuations of Higgs fields on chaotic dynamics of Yang-Mills fields [24].

Let us consider classical Hamiltonian system with any finite number degrees of freedom

$$H = \frac{1}{2} \tilde{p}^2 + V(\tilde{q}),$$

(17)

here \(\tilde{p} = (p_1,...,p_N), \quad \tilde{q} = (q_1,...,q_N), \quad N > 1\). Behavior of the classical system is locally unstable if distance between two neighboring trajectories grows exponentially with time in some region of the phase space.

Consider small region \(\Omega\) of the phase space near the point \((\tilde{q}_0, \tilde{p}_0)\). Suppose that there are two classical trajectories \((\tilde{q}^{(1)}(t), \tilde{p}^{(1)}(t))\) and \((\tilde{q}^{(2)}(t), \tilde{p}^{(2)}(t))\) in \(\Omega\). Then the deviations of trajectories are \(\delta \tilde{q}(t) = \tilde{q}^{(1)}(t) - \tilde{q}^{(2)}(t)\) and \(\delta \tilde{p}(t) = \tilde{p}^{(1)}(t) - \tilde{p}^{(2)}(t)\). Their evolution in \(\Omega\) is governed by linearized Hamilton equations

$$\frac{d}{dt} \begin{pmatrix} \delta \tilde{q} \\delta \tilde{p} \end{pmatrix} = G \begin{pmatrix} \delta \tilde{q} \\delta \tilde{p} \end{pmatrix},$$

(18)

$$G \equiv \begin{pmatrix} 0 & I \\ -\Sigma & 0 \end{pmatrix}.$$  

Here \(I\) is the \(N \times N\) identity matrix, \(G\) is a stability matrix and matrix \(\Sigma\) is

$$\Sigma \equiv \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\tilde{q}_0}.$$

(19)

Matrix \(\Sigma\) and stability matrix \(G\) are functions of the point \(\tilde{q}_0\) of configuration space of the system. Solution of the equations (18) valid in \(\Omega\) has the following form

$$\begin{pmatrix} \delta \tilde{q}(t) \\ \delta \tilde{p}(t) \end{pmatrix} = \sum_{i=1}^{2N} C_i \exp \left\{ \lambda_i t \right\} \begin{pmatrix} \delta \tilde{q}(0) \\ \delta \tilde{p}(0) \end{pmatrix}.$$  

(20)

Here \(\lambda_i = \lambda_i(\tilde{q}_0)\) are eigenvalues of the stability matrix \(G\) and \(\{C_i\}\) is a full set of projectors. From (20) it is evident that:

a) If there is \(i\) such that \(\Re \lambda_i \neq 0\) then the distance between neighboring trajectories grows exponentially with time and the motion is locally unstable.

b) If for any \(i = \frac{1}{2} N \quad \Re \lambda_i = 0\) then there is no local instability and the motion is regular.

It is easy to notice that \(G^2 = \text{diag}(\Sigma, -\Sigma)\). Therefore if \((-\xi_i), \quad i = \frac{1}{1, N}\) are eigenvalues of the matrix \((-\Sigma)\) then

$$(-\xi_i) = \lambda_i^2, \quad \lambda_i^2 = \lambda_{i+N}^2, \quad i = \frac{1}{1, N}.$$  

(21)

Without loss of generality we can imply that \(\Re \lambda_i \geq 0\). Notice that

$$\xi_i = -\lambda_i^2 = (\Im \lambda_i)^2 - (\Re \lambda_i)^2 - 2i \Im \lambda_i \Re \lambda_i.$$  

(22)

Since matrix \(\Sigma\) is real and symmetric its eigenvalues \(\{\xi_i\}, \quad i = \frac{1}{1, N}\) are real. Therefore (22) leads to the restriction

$$\Im \lambda_i \Re \lambda_i = 0 \quad \forall i = \frac{1}{1, N}.$$  

(23)

Thus any eigenvalue of the stability matrix \(G\) is real or pure imaginary or equals zero. Therefore the generalized Toda criterion for classical Hamiltonian systems with any finite number of freedoms can be formulated as follows:

a) If \(\xi_i \geq 0, \quad \forall i = \frac{1}{1, N}\) then behavior of the system is regular near the point \(\tilde{q}_0\).

b) If \(\exists i = \frac{1}{1, N} : \xi_i < 0\) then behavior of the system is locally unstable near the point \(\tilde{q}_0\).

If one of these conditions holds in some region of the configuration space then the motion is stable or chaotic respectively in this region. These results in the case of the system with two degrees of freedom coincide with that obtained in [18].

### 3.2 Formulation of chaos criterion

Now we give some qualitative arguments which bring us to the formulation of the chaos criterion in quantum mechanics and quantum field theory. We shall use the language of path integrals to achieve mathematical description of both quantum mechanics and quantum field theory in the framework of the same formalism. It lets us to trace the way from classical chaos through quantum mechanics to
the chaos in quantum field theory which is our final goal. Quantum mechanics plays the role of the bridge. From statistical mechanics and ergodic theory it is known that chaos in classical systems is a consequence of the property of mixing [4, 5, 6]. Mixing means rapid (exponential) decrease of correlation function with time [6]. Thus rapid (exponential) decrease of correlation function is the signature of chaos [25]. In other words, if correlation function exponentially decreases than the corresponding motion is chaotic, if it oscillates or is constant then the motion is regular [25]. We expand criterion of this type for quantum field systems. All stated below remains valid for quantum mechanics, since mathematical description via path integrals is the same.

For field systems the analogue of the classical correlation function is two-point connected Green function

\[ G_{ik}(x,y) = -\frac{\delta^2 W[J]}{\delta J_i(x) \delta J_k(y)} |_{J=0}. \]  

(24)

Here \( W[J] \) is generating functional of connected Green functions, \( J \) are sources of the fields, \( x, y \) are 4-vectors of space-time coordinates.

We formulate chaos criterion for quantum mechanics and quantum field theory in the following form:

a) If two-point Green function (24) exponentially goes to zero when the distance between its arguments goes to infinity then the system is chaotic.

b) If it oscillates or remains constant in this limit then we have regular behavior of the quantum system.

3.3 Agreement between quantum and classical criteria

To check the agreement between generalized Toda criterion and formulated quantum chaos criterion in semi-classical limit we shall calculate two-point Green function in semi-classical approximation of quantum mechanics. Generating functional is

\[ Z[J] = \int D\bar{q} \exp \left\{ i \int_{-\infty}^{\infty} dt \left[ \frac{1}{2} \frac{d^2}{dt^2} - \bar{q} + J^T \bar{q} \right] \right\}. \]

(25)

Here index \( T \) denotes transposition. Consider certain solution of classical equations of motion \( \bar{q}_0(t) \).

Introduce new variable describing deviations from the classical trajectory \( \delta \bar{q}(t) = \bar{q} - \bar{q}_0(t) \), then under semi-classical approximation

\[ Z[J] = \exp \{ i S_0[J] \} \int D\delta \bar{q} \exp \left[ i \int_{-\infty}^{\infty} dt \left( \frac{1}{2} \frac{d^2}{dt^2} - \frac{1}{2} \delta \bar{q}^T \Sigma \delta \bar{q} + J^T \delta \bar{q} \right) \right]. \]

(26)

Here classical action

\[ S_0[J] = \int_{-\infty}^{\infty} dt \left[ \frac{1}{2} \bar{q}_0^2 - V(\bar{q}_0(t)) + J^T \bar{q}_0 \right], \]

(27)

and matrix \( \Sigma \) defined in (19) is real and symmetric. Therefore there exists orthogonal transformation \( O \) reducing \( \Sigma \) to diagonal form. For simplicity we suppose that \( \xi_i = \xi_i(\bar{q}_0(t)), \quad i = 1, N \) do not depend on time and remain constant on the classical trajectory. After orthogonal transformation given by \( \delta \bar{q} = O\bar{x}, \quad J^T = \bar{\eta}^T O^T, \quad O^T \Sigma O = \text{diag}(\xi_1, \ldots, \xi_N); \quad O^T O = O O^T = I \) we obtain

\[ Z[\bar{\eta}] = L \exp \{ i S_0[\bar{\eta}] \} \int D\bar{x} \exp \left[ i \int_{-\infty}^{\infty} dt \left( \frac{1}{2} \bar{x}^2 - \frac{1}{2} \bar{x}^T \text{diag}(\xi_1, \ldots, \xi_N) \bar{x} + \bar{\eta}^T \bar{x} \right) \right], \]

(28)

here \( L \) denotes Jacobian of the orthogonal transformation. After analytical extension of generating functional (28) into Euclidian space and path integration we get

\[ Z_E[\bar{\eta}] = N \exp \left( -S_0^E[\bar{\eta}] \right) \prod_{i=1}^{N} \exp \frac{1}{2} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \times \eta_i(\tau_1) \left[ \delta(\tau_1 - \tau_2) \left( -\frac{d^2}{dt^2} + \xi_i \right) \right]^{-1} \eta_i(\tau_2). \]

(29)

Here and further there is no sum over \( i \), \( N \) is a normalization factor and \( S_0^E \) is an analytical extension of (27) into Euclidian space. Classical action \( S_0^E[\bar{\eta}] \) is a linear functional of the sources \( \bar{\eta} \). We are interested in consideration of quantum propagator for finite (small) time intervals that correspond to classical motion in small region \( \Omega \) (see sec. 3.1) and implies local approximation of potential energy surface by N-dimensional surface of the second order.
Thus we can consider sources \{\eta_i(\tau)\} which are zero outside some finite time interval. This provides the existence of the generating functional (29). Inverse operator
\[\Delta_i(\tau_1, \tau_2) = \left[\delta(\tau_1 - \tau_2) \left(-\frac{d^2}{d\tau_2^2} + \xi_i\right)\right]^{-1}\] (30)
has to satisfy the following equation
\[\left(-\frac{d^2}{d\tau_1^2} + \xi_i\right) \Delta_i(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2).\] (31)
The solution is
\[\Delta_i(\tau_1, \tau_2) = \frac{1}{2\pi} \int d\omega \frac{e^{i\omega(\tau_1-\tau_2)}}{\omega^2 + \xi_i}.\] (32)
Euclidian connected two-point Green function equals \(\Delta_i(\tau_1, \tau_2)\). Its analytical extension to real (physical) time is
\[G_i(t_1, t_2) = \frac{i}{2\pi} \int d\omega \frac{e^{i\omega(t_1-t_2)}}{\omega^2 + \lambda_i^2}.\] (33)
here \(\tau \to it, \omega \to -i\omega\). Green function is defined up to any solution of corresponding homogeneous equation. We use this freedom to make Green function finite in the limit \((t_1 - t_2) \to +\infty\) for real \(\lambda_i\) and to obtain single formula for any \(\lambda_i\) (both real and imaginary). Thus two-point connected Green function (33) can be represented in the form
\[G_i(t_1, t_2) = \frac{i}{2} Re \left(\frac{e^{-\lambda_i(t_1-t_2)}}{\lambda_i}\right), \quad t_1 > t_2.\] (34)
From the expression (34) it is seen
a) If classical motion is locally unstable (chaotic) then according to Toda criterion there is real eigenvalue \(\lambda_i\). Therefore Green function (34) exponentially goes to zero for some \(i\) when \((t_1 - t_2) \to +\infty\). Opposite is also true. If Green function (34) exponentially goes to zero under the condition \((t_1 - t_2) \to +\infty\) for some \(i\), then there exists real eigenvalue of the stability matrix and thus classical motion is locally unstable.

b) If all eigenvalues of the stability matrix \(G\) are pure imaginary, that corresponds classically stable motion, then in the limit \((t_1 - t_2) \to +\infty\) Green function (34) oscillates as a sine. Opposite is also true. If for any \(i\) Green functions oscillate in the limit \((t_1 - t_2) \to +\infty\) then \(\{\lambda_i\}\) are pure imaginary for any \(i\) and classical motion is stable and regular.

Thus we have demonstrated that proposed quantum chaos criterion coincides with Toda criterion in the semi-classical limit (corresponding principle).

### 3.4 Example from field theory

One of possible applications of proposed chaos criterion in field theory is an investigation of the stability of classical solutions with respect to small perturbations of initial conditions. To study the stability of certain classical solution of field equations one has to calculate (for instance using one loop approximation) two-point Green function in the vicinity of considered classical solution.

To demonstrate this, let us consider real scalar \(\varphi^4\)-field with classical Lagrangian
\[L = \frac{1}{2} (\partial_{\mu} \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4\] (35)
Here \(\lambda > 0\) is a coupling constant, \(m^2\) is some parameter which can be larger or less then zero. In both cases \(\varphi = 0\) is a solution of field equations. Asymptotic of two-point Green function calculated in the vicinity of the classical solution \(\varphi = 0\) in the zero order of perturbation theory is
\[G(x, y)_{\rho \to \infty} \rho^{-\frac{3}{2}} e^{im\sqrt{x^2 + y^2}}\] (36)
Here \(\rho = (x - y)^2\) and we accept that 4-vector \(x - y\) is inside the light cone \((x^0 - y^0)^2 > 0\), in other words \(\rho > 0\). We can study the stability of considered solution with respect to small perturbations. Expression (36) shows that we have two different cases

a) Green function oscillates and slowly (non-exponentially) goes to zero when \(\rho \to \infty\). According proposed chaos criterion considered solution is stable. Indeed, from (36) it follows that parameter \(m\) is real in this case. Therefore \(\varphi = 0\) is a stable vacuum state.

b) Green function exponentially goes to zero in the limit \(\rho \to \infty\). From proposed chaos criterion it follows that \(\varphi = 0\) is an unstable solution. That is
true since from (36) one can see that parameter \(m\) has to be pure imaginary. It is known that in this case state \(\varphi = 0\) becomes unstable, two new stable vacuums are appeared and we obtain spontaneous symmetry breakdown [19].

4 Conclusions

In this work we demonstrated that taking into account quantum properties of Higgs vacuum can lead to non-trivial results such as order-to-chaos transition which escapes one’s notion under pure classical consideration. We have also formulated generalized Toda criterion for Hamiltonian systems with any finite number degrees of freedom. Basing on the formal analogy between statistical mechanics and quantum field theory we proposed chaos criterion for quantum mechanical and quantum field systems. Consideration of quantum mechanics is needed to provide a bridge from classical chaos to chaos in quantum field theory. We have demonstrated that proposed chaos criterion corresponds to generalized Toda criterion in semi-classical limit of quantum mechanics in the case when Lyapunov exponents do not depend on time. We proposed method to investigate stability of classical field solutions with both finite and infinite number degrees of freedom. For real scalar \(\varphi^4\)-field we analyzed the stability of vacuum state and showed that spontaneous symmetry breakdown and degeneration of vacuum state correspond to signatures of quantum chaos.

References

T. Kawabe and S. Ohta, Phys. Rev. 44D 1274 (1991);
G.Z. Baseyan, S.G. Matinyan, Pis’ma Zh. Eksp. Teor. Fiz. 31 76 (1980);
[23] E. Huang, Quarks, leptons and gauge fields (World Scientific 1982).