Broken Solitons Skyrmions

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The problem of constructing internally rotating solitons of fixed angular frequency $\omega$ in
the family of Skyrme models is reformulated as a variational problem for an energy-like
functional, called pseudoenergy, which depends parametrically on $\omega$. Our results confirm the
existence of two types of instabilities determined by the relation between the mass parameter
$\mu$ of the potential and the angular frequency $\omega$.

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Many field theories of interest in
fundamental physics support topological solitons – spatially localized, stable lumps of energy
whose strongly particle-like characteristics make them natural theoretical models of elementary
particles. Perhaps the best developed model from this viewpoint is the Skyrme model, whose
solitons are posited to model atomic nuclei.

Together with the original Skyrme model in $d = 3 + 1$ [1] and the Faddeev–Skyrme model [2],
the baby Skyrme model [3, 4] can be considered as a member of the Skyrme family, the Lagrangian
of all these models has a similar structure, it includes the usual $O(3)$ sigma model kinetic term,
the Skyrme term, which is quartic in derivatives, and the potential term which does not contain the
derivatives.

It is important that the soliton configurations possess both rotational and
isorotational degrees of freedom. Traditional
approach to study the spinning solitons is related with rigid body approximation [5, 6].
The assumption is that the solitons could rotate without changing its shape. This restriction
is not very satisfactory, a consistent approach is to solve a full system of field equations
without imposing any spatial symmetries on the isospinning solitons. Furthermore, almost all
previous studies of spinning solitons (see e.g. [8, 9]) were concerned with minimization of the
total energy functional $E_J[\varphi]$ for a fixed value of the isospin $J$. However if we do not assume
the spinning soliton will have precisely the same shape as the static one, this approach becomes
rather involved, it is related with a numerical solution of a complicated differential-integral
equation.

Recently the isospinning solitons were
considered in the Faddeev–Skyrme model [8, 10]
and in the baby Skyrme model [11] beyond the
rigid body approximation. The approach of the
papers [10, 11] is to consider the static pseudo-
energy minimization problem, where the pseudo-
energy functional $E_\omega[\varphi]$ depends parametrically
on the angular frequency $\omega$. The important
conclusion which is general for all models of
the Skyrme family, is that there is a new
type of instability of the solitons due to extra
nonlinear velocity dependence generated by the
Skyrme term [10]. Interestingly, we observe that
the critical behavior of the isospinning baby
Skyrmions depends also on the structure of
the potential of the model, for example the
isospinning configurations of higher degree may
become unstable with respect to decay into
constituents.

Since isorotation involves only rotational

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symmetry of the target space $S^2$, it is convenient to consider the Faddeev–Skyrme model on a general oriented Riemannian manifold $M$. This allows one to treat in unified fashion the case of principal interest, $M = R^3$, and the cases of soliton chains or strings, $M = R^2 \times S^1$, sheets $M = R \times T^2$, or geometrically nontrivial domains (of potential interest for cosmological applications, for example). Given a time-dependent field $\varphi : R \times M \to S^2$, we have at each fixed time $t$ a mapping $\varphi(t, \cdot) : M \to S^2$ which we shall, in a slight abuse of notation, again denote $\varphi$, and a time derivative $\dot{\varphi}$, which is a section of the bundle $\varphi^{-1}TS^2$ over $M$. Using these, we define, at time $t$, the kinetic and potential energy functionals to be

$$T = \int_M \frac{1}{2} |\dot{\varphi}|^2 + \frac{1}{2} |\varphi^\ast(\iota_\varphi \Omega)|^2; \quad V = \int_M \frac{1}{2} |d\varphi|^2 + \frac{1}{2} |\varphi^\ast \Omega|^2 + U(\varphi), \quad (1)$$

where $\Omega$ is the area form on $S^2$, $\varphi^\ast \Omega$ its pullback to $M$, $\iota$ denotes interior product, and $U : S^2 \to [0, \infty)$ is a smooth potential function which we assume attains its minimum value $0$ at some point $\psi_{\infty} \in S^2$, and is invariant under rotations about $\psi_{\infty}$. If $M$ is noncompact, we assume that $\varphi(t, x) \to \psi_{\infty}$, as $x \to \partial_{\infty}M$, sufficiently fast for all integrals to converge.

A uniformly isorotating field $\varphi(t, x) = R(\omega t) \psi(x)$ is a critical point of the restricted action $S : X_{\omega}^3 \to R$ iff the static field $\psi : M \to S^2$ is a critical point of the pseudoenergy functional

$$F_{\omega}(\psi) = \int_M \left\{ \frac{1}{2} (|d\psi|^2 - \omega^2 |d(\psi_{\infty} \cdot \psi)|^2) + \frac{1}{2} |\psi^\ast \Omega|^2 + (U(\psi) - \frac{1}{2} \omega^2 |\psi_{\infty} \times \psi|^2) \right\}. \quad (2)$$

The first two terms of (2), taken together, can be interpreted as the Dirichlet energy of the map $\psi : M \to S^2$, where $S^2$ is given the deformed metric

$$\langle X, Y \rangle_{\omega} = X \cdot Y - \omega^2 (\psi_{\infty} \cdot X)(\psi_{\infty} \cdot Y) \quad (3)$$

for all $X, Y \in T_{\psi}S^2$. For $0 < \omega < 1$ this metric gives $S^2$ the geometry of an oblate sphere, squashed along the direction of $\psi_{\infty}$. For $\omega > 1$, the metric is singular, changing from Riemannian to Lorentzian in a strip around the equator (orthogonal to $\psi_{\infty}$). Consequently, the pseudoenergy $F_{\omega}$ is no longer bounded below for $\omega > \omega_1 = 1$ which has strong consequences. The fourth and fifth terms together can be interpreted as a deformed potential

$$U_{\omega}(\psi) = U(\psi) - \frac{1}{2} \omega^2 |\psi_{\infty} \times \psi|^2. \quad (4)$$

Since $U$ attains its minimum at $\psi_{\infty}$ and is rotationally invariant, its hessian about $\psi_{\infty}$ must be $\mu^2 \langle \cdot, \cdot \rangle$ for some constant $\mu \geq 0$, interpreted physically as the mass of mesons in the field theory. Hence, if $\omega > \omega_2 = \mu$, $F_{\omega}$ is again unbounded below.

Thus, there is no reason why isospinning solitons should persist for frequencies $\omega \in (1, \mu)$, since $F_{\omega}$ is unbounded below when $\omega > \min\{1, \mu\}$. Hence, we have the possibility that isospinning hopfions are destabilized by nonlinear velocity terms in the field equation before they reach the upper limit $\omega = \mu$. 

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Full scale numerical calculations performed in our works [10, 11] confirmed this conclusion, both in the Faddeev–Skyrme and the baby-Skyrme model.

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References