Time-Delay Systems: An Overview

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Time-delay naturally arises in many real-world systems, due to the fact that the instantaneous rate of change of such systems does not only depend on their current time but rather on their previous history as well. Hence, time-delays are ubiquitous, their introduction often leads to suppression of oscillations, multistability and chaotic motion in the dynamical systems. This review presents some models with different kinds of time-delays such as discrete, distributed and combination of both discrete and distributed time-delays with special emphasis on the reason of incorporating such delays into the system.

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1. Introduction

The introduction of time-delays into the mathematical models of various biological, physical and engineering problems has long been proved vital in order to correctly represent the processing of real-world systems. For example, in population biology, they account for age structure or maturation time period, while in epidemiological settings they represent, for instance, latency or immunity. In neural networks, time-delay inevitably incorporates into the system, since the transmission of information between the neurons is not instantaneous. Uncontrolled delays may degrade network performance and interfere with information processing by making the equilibria unstable [1]. In this respect, time-delays are usually a non-negligible component of the process and lead to qualitatively new behaviour in the dynamical system not observed in the same system without the time-delays [2]. Therefore, the dynamics of time-delay systems are not just of fundamental importance in applications but still poses a crucial challenge for theoretical and mathematical studies.

This review provides an overview and categorisation of the different time-delays such as constant time-delay [3], distributed time-delay [4], and the presence of both discrete and distributed delay together [5]. Besides that, the paper presents some major definitions and

applications to various types of time-delay.

2. Types of time-delay systems

The general class of delay differential equations (DDE) take of the form of a continuous time dynamical system

$$\dot{x}(t) = F(x(t), x_t), \quad (1)$$

where $x(t)$ represents the state of the system parameterized by time t . The function $F : R \times R \rightarrow R$ depends on the current state $x(t)$ and on a delay function x_t . The definition of distinct time-delays specify as follows.

2.1. Single discrete time-delay

The simplest delay function is called a *single discrete time-delay*, for which x_t in (1) only depends on a single past state, i.e. $x_t = x(t - \tau)$ where $\tau > 0$ [6], as follows:

$$\dot{x}(t) = F(x(t), x(t - \tau)). \quad (2)$$

Examples for this type of DDE include the modeling of traffic dynamics of car-following models [7], optical laser [8] and word recognition with time-delayed neural networks [9]. A typical and famous example of single time-delay is a shower problem given by [1]

$$\dot{x} = -\kappa(x(t - \tau) - x_d), \quad (3)$$

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FIG. 1. Solution of the model (3). Parameter values are $\kappa = \tau = x_d = 1$.

where x is a water temperature at the mixer output, τ is a time that hot water takes it to reach the shower-head, a constant κ is the rate of person's reaction during the shower, and x_d is the desired water temperature to reach. In general, a time-delay model (3) describes a temperature of the water as a function of time, as shown in Figure 1, the oscillation may occur in the beginning and it will reach the desired temperature after some time.

2.2. Multiple discrete time-delays

The delay function $x_t = (x(t - \tau_1), x(t - \tau_2), \dots)$ in (1) depends on several corresponding past states called *multiple discrete time-delays* where $\tau_1 > 0, \tau_2 > 0, \dots$. Multiple discrete time-delays are realized in optoelectronic oscillators [8], and also in heterogeneous networks [10].

2.3. Distribution of time-delays

A more common way to incorporate time-delays is to assume that the dynamics is influenced solely by a weighted average of past states, which is called *distributed time-delays*, i.e. x_t in (1) is defined as follows:

$$x_t = \int_0^\infty g(\tau)x(t - \tau)d\tau. \quad (4)$$

One should note that in many biological [11] and social [12] systems the time-delays are not constant; they may change over time and/or depend on the system parameters [2]. In the 1970s, some seminal papers, such as those by May [13] and Cushing [14] showed that in applications to biology, models with distributed time-delays

are often more tractable and also more realistic than models with discrete delays. Since then, different types of distributed delays, typically represented by some distribution kernels such as uniform or gamma distribution, have been studied in the context of modelling neural networks. Here, the presence of many parallel pathways with different axon sizes and lengths results in different distributions of transmission velocities, which can be studied using distributed time-delays models [15–18]. The instantiation of a single discrete time-delay has been proposed in [19]. The authors incorporate time-delay τ into the model to break a life cycle of *Anopheles* mosquito at the larva stage that accounts for the period of growth or development to pupa. The model has shown the local stability of the system's equilibrium and found the critical values for Hopf-bifurcation to occur. When the time-delay τ crosses the imaginary axis from the left half plane to the right half plane, the system's equilibrium undergoes stability switching from stable to periodic to unstable and vice versa. The authors in [3] have studied multiple discrete time-delay for the Subthalamic Nucleus (STN) – Globus Pallidus (GP) network. In their analysis, time-shifting transformation has been used to reduce the degree of freedom of the model to a system with two discrete delays, for which the existence of a unique steady state is established. Conditions for stability of the steady-state are derived in terms of system parameters and the time-delays. Numerical stability analysis is performed using traceDDE and DDE-BIFTOOL in Matlab to investigate different dynamical regimes in the STN-GP model, and obtained critical stability boundaries separating stable (healthy) and oscillatory (Parkinsonian-like) neural firing. The result showed that the strength of the synaptic connection from GP to STN population plays an important role in determining the stability of the steady state. In fact, when the time-delay is only present in the self-interaction of the GP population, the stability region (healthy firing of neurons) increases with increasing strength of the synaptic connection from GP to STN population. In paper [4], the authors have investigated a globally coupled network of active (oscillatory) and inactive (non-oscillatory) oscillators with distributed-delay coupling. Their reason of incorporating distributed delay in such a system is as follows: The interactions between nodes are often non-instantaneous, and one has

to include the associated time-delays in the analysis of network dynamics. Conditions for ageing transition, associated with suppression of oscillations, are derived for gamma and uniform delay distributions in terms of coupling parameters and the proportion of inactive oscillators. In their analysis, the dependence on increasing the width of the distribution has been studied. Also, for the same mean delay the model allows for the aging transition to happen for smaller coupling strength and a lower proportion of inactive elements. For gamma distribution with sufficiently large mean time-delay, it may be possible to achieve ageing transition for an arbitrary fraction of inactive oscillators, as long as the coupling strength lies in a certain range. Their result suggested that the loss of collective oscillations is affected not only by the mean time-delay in the connections between different oscillators but also by other features of the delay distribution, which is very important for applications. In paper [5], the dynamics of a neural network with both discrete and distributed delay has been considered. In their analysis, a coupled model of the original two sub-systems of a neural network has been considered, where one sub-system receives a delayed input from another sub-system. The exciting feature of their model is the effect of both discrete and distributed time-delays together. The distributed time-delays describe the feedbacks of the neural between the two subsystems while the discrete time-delays represent neural networks interactions within each of the two sub-systems. Their justification for having both types of delay together is due to the close proximity of neurons inside each sub-network. It is reasonable to assume that the variation of the time-delays in the coupling between them is negligibly small compared to the variation of the time-delays in sub-networks. This leads to more flexibility in the analytical explorations and in choosing the discrete time-delay within the sub-networks and distributed time-delays in the interactions between them. In their result, the characteristic equation for determining the stability of the trivial steady-state, for any general distribution kernel, has been obtained analytically. They have used three commonly distributed kernels such as Dirac delta, uniform and gamma distributions in order to further analysis in the dynamics of such systems. For each of these distributions, they have investigated analytical conditions for stability

of null-solution in terms of time-delays and system parameters. Their result suggested that stability of the trivial steady-state depends on the synaptic weights, the strength of the connection between the two sub-networks and time-delays in the connection. In delta distribution kernel case (discrete time-delay), the region of stability of trivial steady-state becomes bigger with increasing of the synaptic weights. In the case of the uniform distributed kernel, the stability area depends on the width of the distribution. As the width of the distribution gets bigger, the region of stability shrinks and becomes an isolated area in the region. For the weak gamma distribution, reducing the coupling strengths leads to a smaller size of the stability region, in which the trivial steady-state is stable, and no oscillations are possible. In their observation, the results provide important circumstances with such neural systems with both discrete and distributed time-delays, which can support successful memory retrieval from associative memory models. In paper [20], dynamics of a predator-prey discrete and distributed time-delay has been investigated. They have shown that the importance of both delays in such systems, as in a variety of contexts, it is not only appropriate but also instructive to include time-delays in such models to correctly account for certain specific biological properties of the dynamics. They have studied a predator-prey model with discrete time-delay representing prey handling time and assumed equal to the predator maturation period, and a distributed time-delay describing intra-species interactions. By using a particular case of a weak gamma-distributed time-delay, they have shown that it is not possible for both prey and predators to become extinct. At the same time, the single-species steady state is stable independent of time-delays whenever the co-existence steady state is not feasible. For the co-existence steady-state, they have derived conditions for linear stability and Hopf bifurcation in terms of system parameters. Using normal form theory and the centre manifold reduction, they have found the direction of Hopf-bifurcation and derived the conditions for stability of bifurcating periodic solutions. Numerical simulations have been performed to illustrate different types of dynamics in the system, and they show perfect agreement with the analytical analysis.

3. Conclusions

In summary, this paper reviews three contributions of time-delays into the system; namely, discrete (single and multiple) time-delays, distributed time-delays and the combination of both types together. A selection of time-delays depends on a system's need. For example, single and multiple discrete time-delays are experimentally observed in optical laser and coupled optoelectronic oscillators, respectively

[8]. In optical systems there is a little variation of time-delay, however, social and biological systems need more variation of time-delays, to accurately define the system, that can be drawn from time-delay distribution [11]. Recently, researchers trend goes toward analyzing systems with multiple and distributed time-delays. In many applications in nature and engineering the time-delays are not constant but rather variable [21].

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